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Gronwall-Bellman and Pachpatte type double integral inequalities with applications

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Abstract. Some new generalized two-dimensional non-linear double integral inequalities of Gronwall-Bellman and Pachpatte type have been discussed which are handy tools in the study of partial, integral, and integro-differential equations. Some applications are also presented to illustrate our results.

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1. Introduction

The most desirable thing for researchers is to solve the problems explicitly. If it is not possible then we try to reduce the complexity of the problem. The theory of inequalities defines explicit bounds for some specific integral inequalities. Integral inequalities that provide explicit bounds on unknown functions are very useful in the analysis of quantitative as well as qualitative behavior of solutions of differential and integral equations. One of the most popular inequalities in this direction is the Gronwall inequality [6]. Gronwall inequality is an important tool for obtaining various estimates in the theory of ordinary and stochastic differential equations. It also provides a comparison theorem that can be used to prove the uniqueness of a solution to an initial value problem. The differential form of this inequality was proven by Gronwall in 1919. The integral form was proven by Richard Bellman in 1943 [2]. A nonlinear generalization of Gronwall-Bellman inequality is known as Behari-Lasalle inequality. Other variants and generalizations can be found in B. G. Pachpatte [10]. Gronwall-Bellman integral inequalities have a lot of contributions to analyzing the behavior of solutions of many differential and integral equations, for details see the papers [3, 4, 5, 8, 9, 11, 13, 17, 18, 19, 20]. Additionally, it can be used to analyze the numerical modeling of biological systems like immune system and tumors, as well as HIV infection of $CD4^+$ T cells [16] . In recent years, an increasing number of Gronwall-Bellman type integral inequalities, and generalizations have been discovered to address difficulties encountered in differential equations. A lot of work has been done in this direction [1, 12, 14, 15]. To discuss the abstract analysis of the solutions of certain types of differential equations, Grownwall-Bellman and Pachpatte-type inequalities can play a significant role. Such inequalities are deficient in analyzing the abstract analysis of the solutions of some more types of differential equations. This gives us the motivation in this direction.

2. MAIN RESULTS:

In what follows, $\mathbf{R}$ represents the set of real numbers; $\mathbf{R}_+$ the set of all non negative real numbers; $\mathbf{M}_1 := [r_0, r_6)$ and $\mathbf{M}_2 := [t_0, t_6)$ are the subsets of $\mathbf{R}$; $\Delta := \mathbf{M}_1 \times \mathbf{M}_2$. $\mathbf{C}(X, Y)$, the class of continuous functions defined on X to Y. $D_i A(y_1, y_2, \cdots, y_n)$ $:= A_{y_i}(y_1, y_2, \cdots, y_n)$, where $1 \leq i \leq n$, is the partial derivative of A with respect to $i-$th variable.

Theorem 2.1. *Let $a, b, l, e_i, f_i, g_i \in C(\Delta, \mathbf{R}_+)$ be such that a and b are non decreasing functions in each variable; let $\gamma_i \in C^1(\mathbf{M}_1, \mathbf{M}_1)$ and $\delta_i \in C^1(\mathbf{M}_2, \mathbf{M}_2)$ be non decreasing functions such that $\gamma_i(\varpi) \leq \varpi$ and $\delta_i(\psi) \leq \psi$; let $\eta, \xi \in C(\mathbf{R}_+, \mathbf{R}_+)$ be non decreasing functions such that $\eta(\varpi) > 0,\ \xi(\varpi) > 0$ for $\varpi > 0$ and*

$$\eta(l(\varpi,\psi)) \leq a(\varpi,\psi) + b(\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\eta(l(\alpha,\beta))\Big\{e_i(\alpha,\beta)\Big(\xi(l(\alpha,\beta)) + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(l(p,q))dqdp\Big) + g_i(\alpha,\beta)\Big\}d\beta d\alpha \tag{2. 1}$$

for $(\varpi, \psi) \in \Delta$, then

$$l(\varpi,\psi) \leq \eta^{-1}\Big[a(\varpi,\psi)\exp\Big\{J^{-1}\Big(J(k(\varpi_0,\psi)) + b(\varpi,\psi)B(\varpi,\psi)\Big)\Big\}\Big] \tag{2. 2}$$

for $(\varpi, \psi) \in [\varpi_0, \varpi_5) \times [\psi_0, \psi_5)$ where J^{-1}, η^{-1} are inverses of J, η respectively. $(\varpi_5, \psi_5) \in \Delta$ is chosen arbitrarily on the boundary of planar region Δ, provided that:

$$J(k(\varpi_0,\psi)) + b(\varpi,\psi)B(\varpi,\psi) \in Dom(J^{-1});\ \ J(u) := \int_{u_0}^{u}\frac{ds}{\xi(\eta^{-1}(a\exp(s)))}, u_0 \geq 0$$

$$a(\varpi,\psi)\exp\Big\{J^{-1}\Big(J(k(\varpi_0,\psi)) + b(\varpi,\psi)B(\varpi,\psi)\Big)\Big\} \in Dom(\eta^{-1})$$

$$k(\varpi_0,\psi) := b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\bar{\varpi})}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}g_i(\alpha,\beta)d\beta d\alpha$$

$$B(\varpi,\psi) \ := \ \sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}e_i(\alpha,\beta)\Big(1 + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)dqdp\Big)d\beta d\alpha.$$

Proof. In the light of monotonicity of a and b, the inequality (2.1) can be written as

$$\eta(l(\varpi,\psi)) \leq a(\bar{\varpi},\psi) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\eta(l(\alpha,\beta))\Big\{e_i(\alpha,\beta)\Big(\xi(l(\alpha,\beta)) + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(l(p,q))dqdp\Big) + g_i(\alpha,\beta)\Big\}d\beta d\alpha, \tag{2. 3}$$

for all $(\varpi,\psi) \in [\varpi_0,\bar{\varpi}] \times \mathbf{M}_2$, $\bar{\varpi} \leq \varpi_5$. Right hand side of inequality (2.3) can be denoted by $z_5(\varpi,\psi)$. Then, obviously $z_5(\varpi,\psi) > 0$ and non decreasing function in each variable such that $z_5(\varpi_0,\psi) = a(\bar{\varpi},\psi)$. Then, (2.3) can be written as

$$l(\varpi,\psi) \leq \eta^{-1}(z_5(\varpi,\psi)). \tag{2. 4}$$

$z_5(\varpi,\psi)$ is non decreasing continuous function for all $(\varpi,\psi) \in [\varpi_0,\bar{\varpi}] \times \mathbf{M}_2$ and

$$z_{5\varpi}(\varpi,\psi) = b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\eta(l(\gamma_i(\varpi),\beta))\Big\{e_i(\gamma_i(\varpi),\beta)\Big(\xi(l(\gamma_i(\varpi),\beta)) + \int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(l(p,q))dqdp\Big) + g_i(\gamma_i(\varpi),\beta)\Big\}d\beta\gamma_i^{'}(\varpi).$$

Monotonicity of η, l, $\gamma_i(\varpi) \leq \varpi$ and inequality (2.4), yield

$$z_{5\varpi}(\varpi,\psi) \leq b(\bar{\varpi},\psi)z_5(\varpi,\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\gamma_i(\varpi),\beta)\Big(\xi(\eta^{-1}(z_5(\gamma_i(\varpi),\beta))) + \int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_5(p,q)))dqdp\Big) + g_i(\gamma_i(\varpi),\beta)\Big\}d\beta\gamma_i^{'}(\varpi).$$

$$\Rightarrow \frac{z_{5\varpi}(\varpi,\psi)}{z_5(\varpi,\psi)} \leq b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\gamma_i(\varpi),\beta)\Big(\xi(\eta^{-1}(z_5(\gamma_i(\varpi),\beta))) + \int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_5(p,q)))dqdp\Big) + g_i(\gamma_i(\varpi),\beta)\Big\}d\beta\gamma_i^{'}(\varpi). \tag{2. 5}$$

Taking ψ as fixed in (2.5) for all $(\varpi,\psi) \in [\varpi_0,\bar{\varpi}] \times \mathbf{M}_2$, replace ϖ by s then integrating from ϖ_0 to ϖ with respect to s and making change of variable techniques, we have

$$\log(z_5(\varpi,\psi)) \leq \log(a(\bar{\varpi},\psi)) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta)\Big(\xi(\eta^{-1}(z_5(\alpha,\beta))) + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_5(p,q)))dqdp\Big) + g_i(\alpha,\beta)\Big\}d\beta d\alpha.$$

$$\Rightarrow z_5(\varpi,\psi) \leq a(\bar{\varpi},\psi)\exp\Big[b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta)\Big(\xi(\eta^{-1}(z_5(\alpha,\beta))) + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_5(p,q)))dqdp\Big) + g_i(\alpha,\beta)\Big\}d\beta d\alpha\Big]. \tag{2. 6}$$

Consider

$$k(\varpi,\psi) := b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\bar{\varpi})}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} g_i(\alpha,\beta)d\beta d\alpha + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta)$$
$$\times\Big(\xi(\eta^{-1}(z_5(\alpha,\beta))) + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(\eta^{-1}(z_5(p,q)))dqdp\Big)d\beta d\alpha$$

Then, obviously $k(\varpi,\psi)$ is continuous and non decreasing for all $(\varpi,\psi) \in [\varpi_0,\bar{\varpi}] \times [\psi_0,\psi_5)$,

$$\Rightarrow z_5(\varpi,\psi) \le a(\bar{\varpi},\psi)\exp(k(\varpi,\psi)), \tag{2. 7}$$

$$\begin{aligned} k_{\varpi}(\varpi,\psi) &= b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\gamma_i(\varpi),\beta)\Big(\xi(\eta^{-1}(z_5(\gamma_i(\varpi),\beta))) \\ &\quad + \int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(\eta^{-1}(z_5(p,q)))dqdp\Big)d\beta\gamma_i^{'}(\varpi). \end{aligned}$$

Monotonicity of ξ, η^{-1}, z_5; $\gamma_i(\varpi) \le \varpi$ and $\delta_i(\psi) \le \psi$, yield

$$\begin{aligned} k_{\varpi}(\varpi,\psi) &\le b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\gamma_i(\varpi),\beta)\xi(\eta^{-1}(z_5(\gamma_i(\varpi),\psi))) \\ &\quad \times\Big(1 + \int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta\gamma_i^{'}(\varpi). \end{aligned}$$

Equivalently,

$$\begin{aligned} \frac{k_{\varpi}(\varpi,\psi)}{\xi(\eta^{-1}(a(\bar{\varpi},\psi)\exp k(\varpi,\psi)))} &\le b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\gamma_i(\varpi),\beta) \\ &\quad \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta\gamma_i^{'}(\varpi). \end{aligned} \tag{2. 8}$$

Keeping ψ fixed in (2.8), setting $\varpi \to s$ and integrating with respect to s over $[\varpi_0,\varpi]$, making change of variable technique and by using the definition of J, we have

$$\begin{aligned} J(k(\varpi,\psi)) &\le J(k(\varpi_0,\psi)) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta) \\ &\quad \times\Big(1 + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta d\alpha. \end{aligned}$$

$$\begin{aligned} \Rightarrow k(\varpi,\psi) &\le J^{-1}\Big\{J(k(\varpi_0,\psi)) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta) \\ &\quad \times\Big(1 + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta d\alpha\Big\}. \end{aligned} \tag{2. 9}$$

A combination of inequalities (2.4), (2.7) and (2.9), yields

$$l(\varpi,\psi) \leq \eta^{-1}\Big[a(\bar{\varpi},\psi)\exp\Big\{J^{-1}\Big(J(k(\varpi_0,\psi)) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta d\alpha\Big)\Big\}\Big]. \quad (2.\ 10)$$

Since $\bar{\varpi}$ is arbitrary therefore this completes the proof. □

Theorem 2.2. *Let $a, b, l, e_i, f_i, g_i, \gamma_i, \delta_i, \eta$ and ξ be as defined in Theorem 2.1; let $\theta_\lambda \in C(\mathbf{R}_+, \mathbf{R}_+)$, $1 \leq \lambda \leq 2$, be non decreasing functions with $\theta_\lambda(s) > 0$ for $s > 0$ and*

$$\eta(l(\varpi,\psi)) \leq a(\varpi,\psi) + b(\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} \eta(l(\alpha,\beta))\Big\{e_i(\alpha,\beta)\theta_1(l(\alpha,\beta)) \times\Big(\xi(l(\alpha,\beta)) + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(l(p,q))dqdp\Big) + g_i(\alpha,\beta)\theta_2(\log(l(\alpha,\beta)))\Big\}d\beta d\alpha,\ \forall\,(\varpi,\psi)\in\Delta. \quad (2.\ 11)$$

- *If $\theta_1(u) \geq \theta_2(\log(u))$, then*

$$l(\varpi,\psi) \leq \eta^{-1}\Big[a(\varpi,\psi)\exp\Big\{J_1^{-1}\Big(G_1^{-1}\Big(G_1(J_1(k_1(\varpi_0,\psi))) + b(\varpi,\psi)B(\varpi,\psi)\Big)\Big)\Big\}\Big] \quad (2.\ 12)$$

  for $(\varpi,\psi) \in [\varpi_0,\varpi_1) \times [\psi_0,\psi_1)$.
- *If $\theta_1(u) < \theta_2(\log(u))$, then*

$$l(\varpi,\psi) \leq \eta^{-1}\Big[a(\varpi,\psi)\exp\Big\{J_2^{-1}\Big(G_2^{-1}\Big(G_2(J_2(k_2(\varpi_0,\psi))) + b(\varpi,\psi)B(\varpi,\psi)\Big)\Big)\Big\}\Big] \quad (2.\ 13)$$

for $(\varpi,\psi) \in [\varpi_0,\varpi_2) \times [\psi_0,\psi_2)$, provided that J_λ^{-1}, G_λ^{-1} and η^{-1} are the inverses of J_λ, G_λ and η respectively; $(\varpi_\lambda,\psi_\lambda) \in \Delta$ are chosen arbitrarily on the boundary of planar region Δ, provided that:

$$G_\lambda(J_\lambda(k_\lambda(\varpi_0,\psi))) + b(\varpi,\psi)B(\varpi,\psi) \in Dom(G_\lambda^{-1})$$

$$G_\lambda^{-1}\Big(G_\lambda(J_\lambda(k_\lambda(\varpi_0,\psi))) + b(\varpi,\psi)B(\varpi,\psi)\Big) \in Dom(J_\lambda^{-1})$$

$$a(\varpi,\psi)\exp\Big\{J_\lambda^{-1}\Big(G_\lambda^{-1}\Big(G_\lambda(J_\tau(k_\lambda(\varpi_0,\psi))) + b(\varpi,\psi)B(\varpi,\psi)\Big)\Big)\Big\} \in Dom(\eta^{-1}).$$

$$\begin{aligned} k_\lambda(\varpi_0,\psi) &:= b(\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\theta_\lambda\left(\eta^{-1}\left[a(\alpha,\beta)\right.\right.\\ &+b(\alpha,\beta)\sum_{i=1}^{n}\int_{\gamma_i(\alpha_0)}^{\gamma_i(\alpha)}\int_{\delta_i(\beta_0)}^{\delta_i(\beta)}\eta(l(p,q))\Big\{e_i(p,q)\theta_1(l(p,q))\\ &\times\Big(\xi(l(p,q))+\int_{\gamma_i(\alpha_0)}^{p}\int_{\delta_i(\beta_0)}^{q}f_i(m,n)\xi(l(m,n))dndm\Big)\\ &\left.\left.+g_i(p,q)\theta_2(\log(l(p,q)))\Big\}dqdp\right]\right)g_i(\alpha,\beta)d\beta d\alpha. \end{aligned}$$

$$J_\lambda(u):=\int_{u_0}^{u}\frac{ds}{\theta_\lambda(\eta^{-1}(a\exp(s)))};\ \ G_\lambda(u):=\int_{u_0}^{u}\frac{ds}{\xi(\eta^{-1}(a\exp(J_\lambda^{-1}(s))))},\ u_0\geq 0.$$

Proof. In the light of monotonicity of a and b, the inequality (2.11) can be written as:

$$\begin{aligned} \eta(l(\varpi,\psi)) &\leq a(\bar{\varpi},\psi)+b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\eta(l(\alpha,\beta))\Big\{e_i(\alpha,\beta)\theta_1(l(\alpha,\beta))\\ &\times\Big(\xi(l(\alpha,\beta))+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(l(p,q))dqdp\Big)\\ &+g_i(\alpha,\beta)\theta_2(\log(l(\alpha,\beta)))\Big\}d\beta d\alpha \qquad (2.\ 14) \end{aligned}$$

for all $(\varpi,\psi)\in[\varpi_0,\bar{\varpi}]\times\mathbf{M}_2$, $\bar{\varpi}\leq\varpi_1$. Right hand side of inequality (2.14) can be denoted by $z_1(\varpi,\psi)$. Then, obviously $z_1(\varpi,\psi)>0$ and non decreasing function in each variable such that $z_1(\varpi_0,\psi)=a(\bar{\varpi},\psi)$. Then (2.14) can be written as

$$l(\varpi,\psi)\leq\eta^{-1}(z_1(\varpi,\psi)). \qquad (2.\ 15)$$

$z_1(\varpi,\psi)$ is continuous non decreasing function $\forall(\varpi,\psi)\in[\varpi_0,\bar{\varpi}]\times\mathbf{M}_2$ and

$$\begin{aligned} z_{1\varpi}(\varpi,\psi) &= b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\eta(l(\gamma_i(\varpi),\beta))\Big\{e_i(\gamma_i(\varpi),\beta)\theta_1(l(\gamma_i(\varpi),\beta))\\ &\times\Big(\xi(l(\gamma_i(\varpi),\beta))+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(l(p,q))dqdp\Big)\\ &+g_i(\gamma_i(\varpi),\beta)\theta_2(\log(l(\gamma_i(\varpi),\beta)))\Big\}d\beta\gamma_i^{'}(\varpi). \end{aligned}$$

Monotonicity of η, l, $\gamma_i(\varpi)\leq\varpi$ and inequality (2.15), yield

$$\begin{aligned} \frac{z_{1\varpi}(\varpi,\psi)}{z_1(\varpi,\psi)} &\leq b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\gamma_i(\varpi),\beta)\theta_1(\eta^{-1}(z_1(\gamma_i(\varpi),\beta)))\\ &\times\Big(\xi(\eta^{-1}(z_1(\gamma_i(\varpi),\beta)))+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_1(p,q)))dqdp\Big)\\ &+g_i(\gamma_i(\varpi),\beta)\theta_2(\log(\eta^{-1}(z_1(\gamma_i(\varpi),\beta))))\Big\}d\beta\gamma_i^{'}(\varpi). \qquad (2.\ 16) \end{aligned}$$

ψ can be kept as fixed in (2.16) for all $(\varpi,\psi)\in[\varpi_0,\bar{\varpi}]\times\mathbf{M}_2$, replace ϖ by s then integrating from ϖ_0 to ϖ with respect to s and making change of variable techniques, we have

$$\log(z_1(\varpi,\psi)) \le \log(a(\bar{\varpi},\psi))+b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta)\theta_1(\eta^{-1}(z_1(\alpha,\beta)))$$
$$\times\Big(\xi(\eta^{-1}(z_1(\alpha,\beta)))+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_1(p,q)))dqdp\Big)$$
$$+g_i(\alpha,\beta)\theta_2(\log(\eta^{-1}(z_1(\alpha,\beta))))\Big\}d\beta d\alpha.$$

$$\Rightarrow z_1(\varpi,\psi) \le a(\bar{\varpi},\psi)\exp\Big[b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta)\theta_1(\eta^{-1}(z_1(\alpha,\beta)))$$
$$\times\Big(\xi(\eta^{-1}(z_1(\alpha,\beta)))+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_1(p,q)))dqdp\Big)$$
$$+g_i(\alpha,\beta)\theta_2(\log(\eta^{-1}(z_1(\alpha,\beta))))\Big\}d\beta d\alpha\Big]. \quad (2.\ 17)$$

When $\theta_1(u) \ge \theta_2(\log(u))$, inequality (2.17) takes the form

$$z_1(\varpi,\psi) \le a(\bar{\varpi},\psi)\exp\Big[b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\theta_1(\eta^{-1}(z_1(\alpha,\beta)))$$
$$\times\Big\{e_i(\alpha,\beta)\Big(\xi(\eta^{-1}(z_1(\alpha,\beta)))$$
$$+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_1(p,q)))dqdp\Big)+g_i(\alpha,\beta)\Big\}d\beta d\alpha\Big]. \quad (2.\ 18)$$

Consider,

$$k_1(\varpi,\psi) := b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\theta_1(\eta^{-1}(z_1(\alpha,\beta)))e_i(\alpha,\beta)$$
$$\times\Big(\xi(\eta^{-1}(z_1(\alpha,\beta)))+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_1(p,q)))dqdp\Big)d\beta d\alpha$$
$$+b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\bar{\varpi})}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\theta_1(\eta^{-1}(z_1(\alpha,\beta)))g_i(\alpha,\beta)d\beta d\alpha.$$

Obviously $k_1(\varpi,\psi)$ is continuous and non decreasing for all $(\varpi,\psi)\in[\varpi_0,\bar{\varpi}]\times[\psi_0,\psi_1]$,

$$\Rightarrow z_1(\varpi,\psi) \le a(\bar{\varpi},\psi)\exp(k_1(\varpi,\psi)). \quad (2.\ 19)$$

$$k_{1\varpi}(\varpi,\psi) = b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\theta_1(\eta^{-1}(z_1(\gamma_i(\varpi),\beta)))e_i(\gamma_i(\varpi),\beta)$$
$$\times\Big(\xi(\eta^{-1}(z_1(\gamma_i(\varpi),\beta)))+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_1(p,q)))dqdp\Big)d\beta\gamma_i^{'}(\varpi).$$

Monotonicity of θ_1, η^{-1}, z_1 and $\gamma_i(\varpi) \leq \varpi$ yield

$$\frac{k_{1\varpi}(\varpi,\psi)}{\theta_1(\eta^{-1}(z_1(\varpi,\psi)))} \leq b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\gamma_i(\varpi),\beta)\xi(\eta^{-1}(z_1(\gamma_i(\varpi),\beta))) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta\gamma_i^{'}(\varpi).$$

Equivalently,

$$\frac{k_{1\varpi}(\varpi,\psi)}{\theta_1(\eta^{-1}(a(\bar{\varpi},\psi)\exp k_1(\varpi,\psi)))} \leq b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\gamma_i(\varpi),\beta)\xi(\eta^{-1}(z_1(\gamma_i(\varpi),\beta))) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta\gamma_i^{'}(\varpi). \tag{2. 20}$$

Keeping ψ fixed in (2.20) , setting $\varpi \to s$ and integrating with respect to s over $[\varpi_0,\varpi]$, making change of variable technique and by using the definition of J_1, we have

$$J_1(k_1(\varpi,\psi)) \leq J_1(k_1(\varpi_0,\psi)) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta)\xi(\eta^{-1}(z_1(\alpha,\beta))) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta d\alpha. \tag{2. 21}$$

Denote the right hand side of inequality (2.21) by $z_2(\varpi,\psi)$. Then, obviously $z_2(\varpi,\psi) > 0$ and non decreasing function in each variable such that $z_2(\varpi_0,\psi) = J_1(k_1(\varpi_0,\psi))$. Then, (2.21) is equivalent to

$$k_1(\varpi,\psi) \leq J_1^{-1}(z_2(\varpi,\psi)). \tag{2. 22}$$

Since $z_2(\varpi,\psi)$ is continuous non decreasing function, so

$$z_{2\varpi}(\varpi,\psi) = b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\gamma_i(\varpi),\beta)\xi(\eta^{-1}(z_1(\gamma_i(\varpi),\beta))) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta\gamma_i^{'}(\varpi).$$

Equivalently,

$$\frac{z_{2\varpi}(\varpi,\psi)}{\xi(\eta^{-1}(a(\bar{\varpi},\psi)\exp(J_1^{-1}(z_2(\varpi,\psi)))))} \leq b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\gamma_i(\varpi),\beta) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta\gamma_i^{'}(\varpi). \tag{2. 23}$$

Keeping ψ fixed in (2.23) for all $(\varpi,\psi) \in [\varpi_0,\bar{\varpi}] \times \mathbf{M}_2$, replace ϖ by s and integrating with respect to s over $[\varpi_0,\varpi]$, making change of variable technique and using the definition of G_1, we have

$$\begin{aligned} G_1(z_2(\varpi,\psi)) &\leq G_1(z_2(\varpi_0,\psi)) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta) \\ &\quad \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta d\alpha \\ &\leq G_1(J_1(k_1(\varpi_0,\psi))) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta) \\ &\quad \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta d\alpha. \end{aligned}$$

$$\begin{aligned} \Rightarrow z_2(\varpi,\psi) &\leq G_1^{-1}\Big\{G_1(J_1(k_1(\varpi_0,\psi))) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta) \\ &\quad \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta d\alpha\Big\}. \end{aligned} \tag{2. 24}$$

A combination of inequalities (2.15),(2.19),(2.22) and (2.24) yields:

$$\begin{aligned} &l(\varpi,\psi) \\ &\leq \eta^{-1}\Big[a(\bar{\varpi},\psi)\exp\Big\{J_1^{-1}\Big(G_1^{-1}\Big(G_1(J_1(k_1(\varpi_0,\psi)))+b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}e_i(\alpha,\beta) \\ &\quad \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta d\alpha\Big)\Big)\Big\}\Big]. \end{aligned} \tag{2. 25}$$

Since $\bar{\varpi}$ is arbitrary, this completes the proof.
When $\theta_1(u) < \theta_2(\log(u))$, inequality (2.17) becomes

$$\begin{aligned} z_1(\varpi,\psi) \;\leq\; & a(\bar{\varpi},\psi)\exp\left[b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} \theta_2(log(\eta^{-1}(z_1(\alpha,\beta))))\,\{e_i(\alpha,\beta)\right. \\ & \times\Big(\xi(\eta^{-1}(z_1(\alpha,\beta))) + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(\eta^{-1}(z_1(p,q)))dqdp\Big) \\ & \left.+g_i(\alpha,\beta)\}\,d\beta d\alpha\right]. \end{aligned} \tag{2. 26}$$

Consider,

$$\begin{aligned} k_2(\varpi,\psi) &= b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} \theta_2(log(\eta^{-1}(z_1(\alpha,\beta))))e_i(\alpha,\beta) \\ &\quad \times\Big(\xi(\eta^{-1}(z_1(\alpha,\beta))) + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(\eta^{-1}(z_1(p,q)))dqdp\Big)d\beta d\alpha \\ &\quad +b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\bar{\varpi})}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} g_i(\alpha,\beta)\theta_2(log(\eta^{-1}(z_1(\alpha,\beta))))d\beta d\alpha. \end{aligned}$$

Then,

$$k_2(\varpi_0,\psi) \quad := \quad b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\bar{\varpi})}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} g_i(\alpha,\beta)\theta_2(log(\eta^{-1}(z_1(\alpha,\beta))))d\beta d\alpha.$$

Obviously $k_2(\varpi,\psi)$ is continuous and non decreasing for all $(\varpi,\psi)\in[\varpi_0,\bar{\varpi}]\times[\psi_0,\psi_2)$,

$$\Rightarrow z_1(\varpi,\psi)\le a(\bar{\varpi},\psi)\exp\left(k_2(\varpi,\psi)\right). \qquad (2.\ 27)$$

$$k_{2\varpi}(\varpi,\psi) \;=\; b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\theta_2(\log(\eta^{-1}(z_1(\gamma_i(\varpi),\beta))))e_i(\gamma_i(\varpi),\beta)$$
$$\times\Big(\xi(\eta^{-1}(z_1(\gamma_i(\varpi),\beta)))+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(\eta^{-1}(z_1(p,q)))dqdp\Big)d\beta\gamma_i^{'}(\varpi).$$

Monotonicity of θ_2, η^{-1}, z_1 and $\gamma_i(\varpi)\le\varpi$, yield

$$\frac{k_{2\varpi}(\varpi,\psi)}{\theta_2(\eta^{-1}(a(\bar{\varpi},\psi)\exp k_2(\varpi,\psi)))} \le b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\gamma_i(\varpi),\beta)\xi(\eta^{-1}(z_1(\gamma_i(\varpi),\beta)))$$
$$\times\Big(1+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta\gamma_i^{'}(\varpi). \quad (2.\ 28)$$

Similarly to the above process from (2.21) to (2.25) $\forall\,(\varpi,\psi)\in[\varpi_0,\varpi_2)\times[\psi_0,\psi_2)$ where $(\varpi_2,\psi_2)\in\Delta$ are chosen arbitrary, we get the required result (2.13). □

Theorem 2.3. *Let $a, b, l, e_i, f_i, g_i, \gamma_i, \delta_i, \eta$ and ξ be defined as in Theorem* 2.1; *let $L, M \in C(\Delta\times R_+, R_+)$ and $v > w$ be such that:*

$$0\le L(\varpi,\psi,v)-L(\varpi,\psi,w)\le M(\varpi,\psi,w)(v-w).$$

$$\eta(l(\varpi,\psi)) \quad\le\quad a(\varpi,\psi)+b(\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\eta(l(\alpha,\beta))\Big\{e_i(\alpha,\beta)$$
$$\times\Big(\xi(l(\alpha,\beta))+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(l(p,q))dqdp\Big)$$
$$+g_i(\alpha,\beta)L(\alpha,\beta,\xi(l(\alpha,\beta)))\Big\}d\beta d\alpha \qquad (2.\ 29)$$

for $(\varpi,\psi)\in\Delta$, then

$$l(\varpi,\psi)\le\eta^{-1}\Big[a(\varpi,\psi)\exp\Big\{J^{-1}\Big(J(k_3(\varpi_0,\psi))+b(\varpi,\psi)\Big(B(\varpi,\psi)+D(\varpi,\psi)\Big)\Big)\Big\}\Big] \qquad (2.\ 30)$$

for all $(\varpi,\psi)\in[\varpi_0,\varpi_3)\times[\psi_0,\psi_3)$ provided that J^{-1}, η^{-1} are inverse of J, η respectively. $(\varpi_3,\psi_3)\in\Delta$ is chosen arbitrarily on the boundary of planar region Δ provided that:

$$J(k_3(\varpi_0,\psi))+b(\varpi,\psi)\Big(B(\varpi,\psi)+D(\varpi,\psi)\Big)\in Dom(J^{-1})$$

$$a(\varpi,\psi)\exp\left\{J^{-1}\Big(J(k_3(\varpi_0,\psi))+b(\varpi,\psi)\Big(B(\varpi,\psi)+D(\varpi,\psi)\Big)\Big)\right\}\in Dom(\eta^{-1}).$$

$$k_3(\varpi_0,\psi) \;\;:=\;\; b(\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} g_i(\alpha,\beta)L(\alpha,\beta,0)d\beta d\alpha.$$

$$D(\varpi,\psi) := \sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} g_i(\alpha,\beta)M(\alpha,\beta,0)d\beta d\alpha.$$

Proof. In the light of monotonicity of a and b, the inequality (2.29) can be written as

$$\begin{aligned}\eta(l(\varpi,\psi)) \;\;\leq\;\; & a(\bar{\varpi},\psi) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} \eta(l(\alpha,\beta))\Big\{e_i(\alpha,\beta)\\ & \times\Big(\xi(l(\alpha,\beta)) + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(l(p,q))dqdp\Big)\\ & + g_i(\alpha,\beta)L(\alpha,\beta,\xi(l(\alpha,\beta)))\Big\}d\beta d\alpha \qquad (2.\ 31)\end{aligned}$$

for all $(\varpi,\psi) \in [\varpi_0,\bar{\varpi}] \times \mathbf{M}_2$, $\bar{\varpi} \leq \varpi_3$. Right hand side of inequality (2.31) can be denoted by $z_3(\varpi,\psi)$. Then, obviously $z_3(\varpi,\psi) > 0$ and non decreasing function in each variable such that $z_3(\varpi_0,\psi) = a(\bar{\varpi},\psi)$. Then (2.31) can be written as

$$l(\varpi,\psi) \leq \eta^{-1}(z_3(\varpi,\psi)). \qquad (2.\ 32)$$

$z_3(\varpi,\psi)$ is continuous non decreasing function $\forall(\varpi,\psi) \in [\varpi_0,\bar{\varpi}] \times \mathbf{M}_2$ and

$$\begin{aligned}z_{3\varpi}(\varpi,\psi) \;\;=\;\; & b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} \eta(l(\gamma_i(\varpi),\beta))\Big\{e_i(\gamma_i(\varpi),\beta)\\ & \times\Big(\xi(l(\gamma_i(\varpi),\beta)) + \int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(l(p,q))dqdp\Big)\\ & + g_i(\gamma_i(\varpi),\beta)L(\gamma_i(\varpi),\beta,\xi(l(\gamma_i(\varpi),\beta)))\Big\}d\beta\gamma_i^{'}(\varpi).\end{aligned}$$

Then, by monotonicity of η and l, and inequality (2.32)

$$\begin{aligned}\frac{z_{3\varpi}(\varpi,\psi)}{z_3(\varpi,\psi)} \;\;\leq\;\; & b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} \Big\{e_i(\gamma_i(\varpi),\beta)\\ \times\Big(\xi(\eta^{-1}(z_3(\gamma_i(\varpi),\beta))) & + \int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(\eta^{-1}(z_3(p,q)))dqdp\Big)\\ + g_i(\gamma_i(\varpi),\beta)L(\gamma_i(\varpi),\beta, & \xi(\eta^{-1}(z_3(\gamma_i(\varpi),\beta))))\Big\}d\beta\gamma_i^{'}(\varpi). \qquad (2.\ 33)\end{aligned}$$

ψ can be kept as fixed in (2.33) for all $(\varpi,\psi) \in [\varpi_0,\bar{\varpi}] \times \mathbf{M}_2$, replace ϖ by s then integrating from ϖ_0 to ϖ with respect to s and making change of variable techniques, we have

$$\log(z_3(\varpi,\psi)) \leq \log(a(\bar{\varpi},\psi))+b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta) \times\Big(\xi(\eta^{-1}(z_3(\alpha,\beta)))+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_3(p,q)))dqdp\Big) +g_i(\alpha,\beta)L(\alpha,\beta,\xi(\eta^{-1}(z_3(\alpha,\beta))))\Big\}d\beta d\alpha.$$

$$\Rightarrow z_3(\varpi,\psi) \leq a(\bar{\varpi},\psi)\exp\Big[b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta) \times\Big(\xi(\eta^{-1}(z_3(\alpha,\beta)))+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_3(p,q)))dqdp\Big) +g_i(\alpha,\beta)L(\alpha,\beta,\xi(\eta^{-1}(z_3(\alpha,\beta))))\Big\}d\beta d\alpha\Big].$$

Here ξ,η and z_3 are non decreasing functions, also by using the condition of continuity of L

$$z_3(\varpi,\psi) \leq a(\bar{\varpi},\psi)\exp\Big[b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta) \times\Big(\xi(\eta^{-1}(z_3(\alpha,\beta)))+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_3(p,q)))dqdp\Big) +g_i(\alpha,\beta)(M(\alpha,\beta,0)\xi(\eta^{-1}(z_3(\alpha,\beta)))+L(\alpha,\beta,0))\Big\}d\beta d\alpha\Big]. \quad (2.\ 34)$$

Consider,

$$k_3(\varpi,\psi) = b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta) \times\Big(\xi(\eta^{-1}(z_3(\alpha,\beta)))+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_3(p,q)))dqdp\Big) +g_i(\alpha,\beta)M(\alpha,\beta,0)\xi(\eta^{-1}(z_3(\alpha,\beta)))\Big\}d\beta d\alpha +b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i\bar{\varpi})}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}g_i(\alpha,\beta)L(\alpha,\beta,0)d\beta d\alpha.$$

Obviously $k_3(\varpi,\psi)$ is continuous and non decreasing for all $(\varpi,\psi)\in[\varpi_0,\bar{\varpi}]\times[\psi_0,\psi_3)$,

$$z_3(\varpi,\psi)\leq a(\bar{\varpi},\psi)\exp\left(k_3(\varpi,\psi)\right). \quad (2.\ 35)$$

$$k_{3\varpi}(\varpi,\psi) \leq b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\gamma_i(\varpi),\beta) \times\Big(\xi(\eta^{-1}(z_3(\gamma_i(\varpi),\beta)))+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(\eta^{-1}(z_3(p,q)))dqdp\Big) +g_i(\gamma_i(\varpi),\beta)M(\gamma_i(\varpi),\beta,0)\xi(\eta^{-1}(z_3(\gamma_i(\varpi),\beta)))\Big\}d\beta\gamma_i^{'}(\varpi).$$

Monotonicity of ξ, η^{-1}, z_3, inequality (2.35) and $\gamma_i(\varpi)\leq\varpi$, yield

$$\frac{k_{3\varpi}(\varpi,\psi)}{\xi(\eta^{-1}(a(\bar{\varpi},\psi)\exp k_3(\varpi,\psi)))} \leq b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\gamma_i(\varpi),\beta) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)dqdp\Big)+g_i(\gamma_i(\varpi),\beta)M(\gamma_i(\varpi),\beta,0)\Big\}d\beta\gamma_i^{'}(\varpi). \quad (2.\ 36)$$

Keeping ψ fixed in (2.36), setting ϖ by s and integrating with respect to s over $[\varpi_0,\varpi]$, making change of variable technique and by using the definition of J, we have

$$J(k_3(\varpi,\psi))\leq J(k_3(\varpi_0,\psi))+b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)dqdp\Big)+g_i(\alpha,\beta)M(\alpha,\beta,0)\Big\}d\beta d\alpha.$$

$$\Rightarrow k_3(\varpi,\psi)\leq J^{-1}\Big[J(k_3(\varpi_0,\psi))+b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)dqdp\Big)+g_i(\alpha,\beta)M(\alpha,\beta,0)\Big\}d\beta d\alpha\Big]. \quad (2.\ 37)$$

A combination of inequalities (2.32),(2.35) and (2.37) yields:

$$l(\varpi,\psi)\leq\eta^{-1}\Big[a(\bar{\varpi},\psi)\exp\Big\{J^{-1}\Big(J(k_3(\varpi_0,\psi))+b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\{e_i(\alpha,\beta) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)dqdp\Big)+g_i(\alpha,\beta)M(\alpha,\beta,0)\Big\}d\beta d\alpha\Big)\Big\}\Big]. \quad (2.\ 38)$$

Since $\bar{\varpi}$ is arbitrary therefore this completes the proof. □

Corollary 2.4. *Let $a,b,l,e_i,f_i,g_i,\gamma_i,\delta_i$ and ξ be defined as in Theorem* 2.1. *Moreover, if m is a positive constant such that:*

$$l^m(\varpi,\psi)\leq a(\varpi,\psi)+b(\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}l^m(\alpha,\beta)\Big\{e_i(\alpha,\beta) \times\Big(\xi(l(\alpha,\beta))+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(l(p,q))dqdp\Big)+g_i(\alpha,\beta)\Big\}d\beta d\alpha \quad (2.\ 39)$$

for $(\varpi,\psi)\in\Delta$. *Then,*

$$l(\varpi,\psi)\le a^{\frac{1}{m}}(\varpi,\psi)\exp\Big[\frac{1}{m}J_4^{-1}\Big\{J_4(k_4(\varpi_0,\psi))+b(\varpi,\psi)B(\varpi,\psi)\Big\}\Big] \qquad (2.\ 40)$$

for $(\varpi,\psi)\in[\varpi_0,\varpi_4)\times[\psi_0,\psi_4)$ *provided that* J_4^{-1} *is the inverses of* J_4. $(\varpi_4,\psi_4)\in\Delta$ *is chosen arbitrarily on the boundary of planar region* Δ *such that:*

$$J_4(k_4(\varpi_0,\psi))+b(\varpi,\psi)B(\varpi,\psi)\in Dom(J_4^{-1}).$$

$$k_4(\varpi_0,\psi):=b(\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}g_i(\alpha,\beta)d\beta d\alpha;\ J_4(u):=\int_{u_0}^{u}\frac{ds}{\xi(a^{\frac{1}{m}}\exp(\frac{1}{m}s))},u_0\ge 0.$$

Proof. By inequality (2.39), we have

$$\begin{aligned} l^m(\varpi,\psi) \quad &\le \quad a(\bar\varpi,\psi)+b(\bar\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}l^m(\alpha,\beta)\Big\{e_i(\alpha,\beta)\\ &\times\Big(\xi(l(\alpha,\beta))+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(l(p,q))dqdp\Big)+g_i(\alpha,\beta)\Big\}d\beta d\alpha \end{aligned} \qquad (2.\ 41)$$

for all $(\varpi,\psi)\in[\varpi_0,\bar\varpi]\times\mathbf{M}_2$, $\bar\varpi\le\varpi_4$. Right hand side of inequality (2.41) can be denoted by $z_4(\varpi,\psi)$. Then, obviously $z_4(\varpi,\psi)>0$ and non decreasing function in each variable such that $z_4(\varpi_0,\psi)=a(\bar\varpi,\psi)$. Then (2.41) can be written as

$$l(\varpi,\psi)\le z_4^{\frac{1}{m}}(\varpi,\psi)). \qquad (2.\ 42)$$

$z_4(\varpi,\psi)$ is a continuous non decreasing function $\forall\ (\varpi,\psi)\in[\varpi_0,\bar\varpi]\times\mathbf{M}_2$ and

$$\begin{aligned} z_{4\varpi}(\varpi,\psi)&=b(\bar\varpi,\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}l^m(\gamma_i(\varpi),\beta)\Big\{e_i(\gamma_i(\varpi),\beta)\Big(\xi(l(\gamma_i(\varpi),\beta))\\ &+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(l(p,q))dqdp\Big)+g_i(\gamma_i(\varpi),\beta)\Big\}d\beta\gamma_i'(\varpi). \end{aligned}$$

Then, by monotonicity of l, condition $\gamma_i(\varpi)\le\varpi$ and inequality (2.42)

$$\begin{aligned} \frac{z_{4\varpi}(\varpi,\psi)}{z_4(\varpi,\psi)}&\le b(\bar\varpi,\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\gamma_i(\varpi),\beta)\Big(\xi(z_4^{\frac{1}{m}}(\gamma_i(\varpi),\beta))\\ &+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(z_4^{\frac{1}{m}}(p,q))dqdp\Big)+g_i(\gamma_i(\varpi),\beta)\Big\}d\beta\gamma_i'(\varpi). \end{aligned} \qquad (2.\ 43)$$

Taking ψ as fixed in (2.43) for all $(\varpi,\psi)\in[\varpi_0,\bar\varpi]\times\mathbf{M}_2$, replace ϖ by s then integrating from ϖ_0 to ϖ with respect to s and making change of variable techniques, we have

$$\begin{aligned} \log(z_4(\varpi,\psi))-\log(z_4(\varpi_0,\psi))&\le b(\bar\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta)\Big(\xi(z_4^{\frac{1}{m}}(\alpha,\beta))\\ &+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta}f_i(p,q)\xi(z_4^{\frac{1}{m}}(p,q))dqdp\Big)+g_i(\alpha,\beta)\Big\}d\beta d\alpha. \end{aligned}$$

$$\Rightarrow z_4(\varpi,\psi) \leq a(\bar{\varpi},\psi)\exp\Big[b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\Big\{e_i(\alpha,\beta)\Big(\xi(z_4^{\frac{1}{m}}(\alpha,\beta)) + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(z_4^{\frac{1}{m}}(p,q))dqdp\Big) + g_i(\alpha,\beta)\Big\}d\beta d\alpha\Big]. \quad (2.\ 44)$$

Consider,

$$k_4(\varpi,\psi) := b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\bar{\varpi})}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} g_i(\alpha,\beta)d\beta d\alpha + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta) \times\Big(\xi(z_4^{\frac{1}{m}}(\alpha,\beta)) + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(z_4^{\frac{1}{m}}(p,q))dqdp\Big)d\beta d\alpha.$$

Obviously $k_4(\varpi,\psi)$ is continuous and non decreasing function for all $(\varpi,\psi)\in[\varpi_0,\bar{\varpi}]\times[\psi_0,\psi_4)$, inequality (2.44) can be written as

$$z_4^{\frac{1}{m}}(\varpi,\psi) \leq a^{\frac{1}{m}}(\bar{\varpi},\psi)\exp\Big(\frac{1}{m}k_4(\varpi,\psi)\Big). \quad (2.\ 45)$$

$$k_{4\varpi}(\varpi,\psi) = b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\gamma_i(\varpi),\beta)\Big(\xi(z_4^{\frac{1}{m}}(\gamma_i(\varpi),\beta)) + \int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)\xi(z_4^{\frac{1}{m}}(p,q))dqdp\Big)d\beta\gamma_i^{'}(\varpi).$$

Monotonicity of ξ and z_4 yield

$$k_{4\varpi}(\varpi,\psi) \leq b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\gamma_i(\varpi),\beta)\xi(z_4^{\frac{1}{m}}(\gamma_i(\varpi),\beta)) \times\Big(1 + \int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta\gamma_i^{'}(\varpi).$$

Equivalently,

$$\frac{k_{4\varpi}(\varpi,\psi)}{\xi(a^{\frac{1}{m}}(\bar{\varpi},\psi)\exp(\frac{1}{m}k_4(\varpi,\psi)))} \leq b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\gamma_i(\varpi),\beta) \times\Big(1 + \int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta\gamma_i^{'}(\varpi). \quad (2.\ 46)$$

Keeping ψ fixed in (2.46) , setting $\varpi\to s$ and integrating with respect to s over $[\varpi_0,\varpi]$, making change of variable techniques and by using the definition of $J_4(u)$, we have

$$J_4(k_4(\varpi,\psi)) \leq J_4(k_4(\varpi_0,\psi)) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta) \times\Big(1 + \int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta d\alpha.$$

$$k_4(\varpi,\psi) \leq J_4^{-1}\Big\{J_4(k_4(\varpi_0,\psi)) + b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta d\alpha\Big\}. \quad (2.\ 47)$$

A combination of inequalities (2.42), (2.45) and (2.47), yields:

$$l(\varpi,\psi) \leq a^{\frac{1}{m}}(\bar{\varpi},\psi)\exp\Big[\frac{1}{m}J_4^{-1}\Big\{J_4(k_4(\varpi_0,\psi))+b(\bar{\varpi},\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} e_i(\alpha,\beta) \times\Big(1+\int_{\gamma_i(\varpi_0)}^{\alpha}\int_{\delta_i(\psi_0)}^{\beta} f_i(p,q)dqdp\Big)d\beta d\alpha\Big\}\Big]. \quad (2.\ 48)$$

Since $\bar{\varpi}$ is arbitrary therefore this completes the proof. □

Remark 2.5.
- *For* $f_i \equiv 0, 1 \leq i \leq n$, *Theorem 2.1 reduces to* [7, Theorem 2.3]
- *For* $f_i \equiv 0, 1 \leq i \leq n$, *Corollary* 1 *reduces to* [7, Corollary 2.4]

3. Applications

In this section, the boundedness of partial integro-differential equations would be achieved by the help of achieved inequalities, with various retarded arguments, of the forms

$$\frac{\partial}{\partial\psi}(z^{p-1}(\varpi,\psi)z_{\varpi}(\varpi,\psi)) = F[\varpi,\psi,z(\varpi-\alpha_1(\varpi),\psi-\beta_1(\psi)),...,z(\varpi-\alpha_n(\varpi),\psi-\beta_n(\psi)), \int_{\varpi_0}^{\varpi}\int_{\psi_0}^{\psi} Q(\varpi,\psi,\sigma,\tau,z(\varpi-\alpha_1(\varpi),\psi-\beta_1(\psi)),...,z(\varpi-\alpha_n(\varpi),\psi-\beta_n(\psi)))d\sigma d\tau] \quad (3.\ 49)$$

and

$$D_2(D_1\eta(z(\varpi,\psi))) = F[\varpi,\psi,z(\varpi-\alpha_1(\varpi),\psi-\beta_1(\psi)),...,z(\varpi-\alpha_n(\varpi),\psi-\beta_n(\psi)), \int_{\varpi_0}^{\varpi}\int_{\psi_0}^{\psi} Q(\varpi,\psi,\sigma,\tau,z(\varpi-\alpha_1(\varpi),\psi-\beta_1(\psi)),...,z(\varpi-\alpha_n(\varpi),\psi-\beta_n(\psi)))d\sigma d\tau], \quad (3.\ 50)$$

with the given initial boundary conditions

$$z(\varpi,\psi_0) = a_1(\varpi),\ z(\varpi_0,\psi) = a_2(\psi),\ a_1(\varpi_0) = a_2(\psi_0) = 0, \quad (3.\ 51)$$

where $F \in C(\Delta\times\mathbf{R}^{n+1},\mathbf{R})$, $Q \in C(\Delta\times\Delta\times\mathbf{R}^n,\mathbf{R})$, $a_1 \in C^1(M_1,\mathbf{R})$, $a_2 \in C^1(M_2,\mathbf{R})$ and $\alpha_i \in C^1(M_1,\mathbf{R})$, $\beta_i \in C^1(M_2,\mathbf{R})$ are nonincreasing such that $\varpi-\alpha_i(\varpi) \geq 0$, $\varpi-\alpha_i(\varpi) \in C^1(\mathbf{M}_1,\mathbf{M}_1)$, $\psi-\beta_i(\psi) \geq 0$, $\psi-\beta_i(\psi) \in C^1(\mathbf{M}_2,\mathbf{M}_2)$, $\alpha_i'(\varpi) < 1$, $\beta_i'(\psi) < 1$ and $\alpha_i(\varpi_0) = \beta_i(\psi_0) = 0$, $1 \leq i \leq n$ for $(\varpi,\psi) \in \Delta$; let $\eta \in C^1(\mathbf{R},\mathbf{R})$ be an increasing function suchlike $\eta(|l|) \leq |\eta(l)|$; let $\eta(\mathfrak{a}(\varpi,\psi)) = \eta(a_1(\varpi)) + \eta(a_2(\psi))$ and

$$M_i = \max_{\varpi\in\mathbf{M}_1}\frac{1}{1-\alpha_i'(\varpi)};\quad N_i = \max_{\psi\in\mathbf{M}_2}\frac{1}{1-\beta_i'(\psi)},\ 1 \leq i \leq n. \quad (3.\ 52)$$

The following theorem concerns with a boundedness of the solution of (3.50).

Theorem 3.1. *Suppose that $F : \Delta \times \mathbf{R}^{n+1} \to \mathbf{R}$ is a continuous function for which there exist nonnegative continuous functions $e_i(\varpi,\psi)$, $f_i(\varpi,\psi)$ and $g_i(\varpi,\psi)$ for $(\varpi,\psi) \in \Delta$ such that:*

$$\begin{cases} |F(\varpi,\psi,l_1,...,l_n,j)| \le b(\varpi,\psi)\sum_{i=1}^{n}\eta(|l_i|)[e_i(\varpi,\psi)\xi(|l_i|)+|j|+g_i(\varpi,\psi)]. \\ |Q(\varpi,\psi,v_1,v_2,l_1,l_2,...,l_n)| \le f_i(v_1,v_2)\xi(|l_i|). \end{cases} \tag{3. 53}$$

If $z(\varpi,\psi)$ with the conditions (3.51) is a solution of (3.50), then

$$|z(\varpi,\psi)| \le \eta^{-1}[|\eta(\mathfrak{a}(\varpi,\psi))|\exp[\overline{J}^{-1}[\overline{J}(\overline{k}(\varpi_0,\psi))+b(\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\overline{e}_i(p,q)\times (1+\int_{\gamma_i(\varpi_0)}^{p}\int_{\delta_i(\psi_0)}^{q}\overline{f}_i(m,n)dndm)dqdp]]], \tag{3. 54}$$

provided that:

$$\begin{cases} \overline{e}_i(u,v) = M_iN_ie_i(u+\alpha_i(\alpha), v+\beta_i(\beta)), \\ \overline{f}_i(u,v) = M_i^2N_i^2f_i(u+\alpha_i(\sigma), v+\beta_i(\tau)), \\ \overline{k}(\varpi_0,\psi) = b(\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\overline{g}_i(p,q)dqdp, \\ \overline{g}_i(u,v) = M_iN_ig_i(u+\alpha_i(\alpha), v+\beta_i(\beta)). \end{cases} ; \overline{J}(u) := \int_{u_0}^{u}\frac{ds}{\xi(\eta^{-1}(|\eta(\mathfrak{a})|\exp(s)))}, u_0 \ge 0,$$

Proof. The solution $z(\varpi,\psi)$ of the problem (3.50) with (3.51) satisfies the following integral equation

$$\begin{aligned} &\eta(z(\varpi,\psi)) \\ &= \eta(\mathfrak{a}(\varpi,\psi)) + \int_{\varpi_0}^{\varpi}\int_{\psi_0}^{\psi}F[l,m,z(l-\alpha_1(l),m-\beta_1(m)),...,z(l-\alpha_n(l),m-\beta_n(m)), \\ &\int_{\varpi_0}^{l}\int_{\psi_0}^{m}Q(l,m,\sigma,\tau,z(l-\alpha_1(l),m-\beta_1(m)),..., \\ &z(l-\alpha_n(l),m-\beta_n(m)))d\sigma d\tau]dmdl. \end{aligned} \tag{3. 55}$$

By properties of modulus and condition (3.53), equation (3.55) takes the form

$$\begin{aligned} &|\eta(z(\varpi,\psi))| \\ &\le |\eta(\mathfrak{a}(\varpi,\psi))| + \int_{\varpi_0}^{\varpi}\int_{\psi_0}^{\psi}|F[l,m,z(l-\alpha_1(l),m-\beta_1(m)),...,z(l-\alpha_n(l),m-\beta_n(m)), \\ &\int_{\varpi_0}^{l}\int_{\psi_0}^{m}Q(l,m,\sigma,\tau,z(l-\alpha_1(l),m-\beta_1(m)),..., \\ &z(l-\alpha_n(l),m-\beta_n(m)))d\sigma d\tau]|\, dmdl \\ &\le |\eta(\mathfrak{a}(\varpi,\psi))| + b(\varpi,\psi)\int_{\varpi_0}^{\varpi}\int_{\psi_0}^{\psi}\sum_{i=1}^{n}\eta(|z(\alpha-\alpha_i(\alpha),\beta-\beta_i(\beta))|).[e_i(\alpha,\beta) \\ &\xi(|z(\alpha-\alpha_i(\alpha),\beta-\beta_i(\beta))|) + \int_{\varpi_0}^{\alpha}\int_{\psi_0}^{\beta}f_i(\sigma,\tau)\xi(|z(\sigma-\alpha_i(\sigma),\tau-\beta_i(\tau))|)d\tau d\sigma \\ &+g_i(\alpha,\beta)]d\beta d\alpha \end{aligned}$$

$$=|\eta(\mathfrak{a}(\varpi,\psi))| + b(\varpi,\psi)\sum_{i=1}^{n}\int_{\varpi_0}^{\varpi}\int_{\psi_0}^{\psi}[\eta(|z(\alpha-\alpha_i(\alpha),\beta-\beta_i(\beta))|).(e_i(\alpha,\beta)\times$$
$$\xi(|z(\alpha-\alpha_i(\alpha),\beta-\beta_i(\beta))|)+\int_{\varpi_0}^{\alpha}\int_{\psi_0}^{\beta}f_i(\sigma,\tau)\xi(|z(\sigma-\alpha_i(\sigma),\tau-\beta_i(\tau))|)d\tau d\sigma)$$
$$+g_i(\alpha,\beta)\eta(|z(\alpha-\alpha_i(\alpha),\beta-\beta_i(\beta))|)]d\beta d\alpha$$
$$\leq|\eta(\mathfrak{a}(\varpi,\psi))| + b(\varpi,\psi)\sum_{i=1}^{n}M_iN_i\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}[\eta(|z(\gamma_i(\alpha),\delta_i(\beta))|)$$
$$\times(e_i(\gamma_i(\alpha)+\alpha_i(\alpha),\delta_i(\beta)+\beta_i(\beta))\xi(|z(\gamma_i(\alpha),\delta_i(\beta))|)$$
$$+\int_{\gamma_i(\varpi_0)}^{\gamma_i(\alpha)}\int_{\delta_i(\psi_0)}^{\delta_i(\beta)}M_iN_if_i(\gamma_i(\sigma)+\alpha_i(\sigma),\delta_i(\tau)+\beta_i(\tau))\xi(|z(\gamma_i(\sigma),\delta_i(\tau))|)d\delta_i(\tau)d\gamma i(\sigma))$$
$$+g_i(\gamma_i(\alpha)+\alpha_i(\alpha),\delta_i(\beta)+\beta_i(\beta))\eta(|z(\gamma_i(\alpha),\delta_i(\beta))|)]d\delta_i(\beta)d\gamma_i(\alpha)$$
$$\Rightarrow|\eta(z(\varpi,\psi))|$$
$$\leq|\eta(\mathfrak{a}(\varpi,\psi))| + b(\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}[\eta(|z(p,q)|).(\overline{e}_i(p,q)$$
$$\times\xi(|z(p,q)|)+\int_{\gamma_i(\varpi_0)}^{p}\int_{\delta(\psi_0)}^{q}\overline{f}_i(m,n)\xi(|z(m,n)|)dndm)$$
$$+\overline{g}_i(p,q)\eta(|z(p,q)|)]dqdp. \tag{3. 56}$$

Now an expeditious application of the inequality given in Theorem 2.1 to (3.56) provide the desired result (3.54). ☐

Theorem 3.2. *Suppose that $F:\Delta\times\mathbf{R}^{n+1}\to\mathbf{R}$ is a continuous function for which there exist non-negative continuous functions $e_i(\varpi,\psi)$, $f_i(\varpi,\psi)$ and $g_i(\varpi,\psi)$ for $(\varpi,\psi)\in\Delta$ such that:*

$$\begin{cases} |F(\varpi,\psi,l_1,...,l_n,j)|\leq\sum_{i=1}^{n}|l_i|^p\ [e_i(\varpi,\psi)\xi(|l_i|)+|j|+g_i(\varpi,\psi)]. \\ |Q(\varpi,\psi,v_1,v_2,l_1,l_2,...,l_n)|\leq f_i(v_1,v_2)\xi(|l_i|). \\ |a_1^p(\varpi)+a_2^p(\psi)|\leq\varpi. \end{cases} \tag{3. 57}$$

If $z(\varpi,\psi)$ with the condition (3.51) *is a solution of* (3.49), *then*

$$z(\varpi,\psi)\leq\varpi^{\frac{1}{p}}\exp\left[\frac{1}{p}\tilde{J}_4^{-1}\left[\tilde{J}_4(\tilde{k}_4(\varpi_0,\psi))+p\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\tilde{e}_i(s,q)\right.\right.$$
$$\left.\left.\times\left(1+\int_{\gamma_i(\varpi_0)}^{s}\int_{\delta_i(\psi_0)}^{q}\tilde{f}_i(m,n)dndm\right)dqds\right]\right], \tag{3. 58}$$

provided that:

$$\begin{cases}\tilde{e}_i(u,v)=M_iN_ie_i(u+\alpha_i(\alpha),v+\beta_i(\beta))\\ \tilde{f}_i(u,v)=M_i^2N_i^2f_i(u+\alpha_i(\sigma),v+\beta_i(\tau))\\ \tilde{k}(\varpi_0,\psi)=b(\varpi,\psi)\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}\tilde{g}_i(s,q)dqds\\ \tilde{g}_i(u,v)=M_iN_ig_i(u+\alpha_i(\alpha),v+\beta_i(\beta)).\end{cases};\ \tilde{J}_4(u):=\int_{u_0}^{u}\frac{ds}{\xi(\varpi^{\frac{1}{p}}\exp(\frac{1}{p}s))},u_0\geq 0.$$

Proof. It is easy to see that the solution $z(\varpi, \psi)$ of the problem (3.49) with (3.51) satisfies the following integral equation:

$$z^p(\varpi, \psi) = a_1^p(\varpi) + a_2^p(\psi) + p\int_{\varpi_0}^{\varpi}\int_{\psi_0}^{\psi} F[l, m, z(l-\alpha_1(l), m-\beta_1(m)), ...,$$
$$z(l-\alpha_n(l), m-\beta_n(m)), \int_{\varpi_0}^{l}\int_{\psi_0}^{m} Q(l, m, \sigma, \tau, z(l-\alpha_1(l), m-\beta_1(m)), ...,$$
$$z(l-\alpha_n(l), m-\beta_n(m)))d\sigma d\tau]dmdl. \quad (3.\ 59)$$

By modulus properties and condition (3.57), equation (3.59) has the form

$$\begin{aligned}
&|z^p(\varpi, \psi)| \\
&\leq \varpi + p\int_{\varpi_0}^{\varpi}\int_{\psi_0}^{\psi} |F[l, m, z(l-\alpha_1(l), m-\beta_1(m)), ..., z(l-\alpha_n(l), m-\beta_n(m)), \\
&\int_{\varpi_0}^{l}\int_{\psi_0}^{m} Q(l, m, \sigma, \tau, z(l-\alpha_1(l), m-\beta_1(m)), ..., z(l-\alpha_n(l), m-\beta_n(m)))d\sigma d\tau]|\ dmdl \\
&\leq \varpi + p\int_{\varpi_0}^{\varpi}\int_{\psi_0}^{\psi}\sum_{i=1}^{n}[|z(l-\alpha_i(l), m-\beta_i(m))|^p\ .[e_i(l, m)\xi(|z(l-\alpha_i(l), m-\beta_i(m))|) \\
&\quad + \int_{\varpi_0}^{l}\int_{\psi_0}^{m} f_i(\sigma, \tau)\xi(|z(\sigma-\alpha_i(\sigma), \tau-\beta_i(\tau))|)d\tau d\sigma] \\
&\qquad + g_i(l, m)\ |z(l-\alpha_i(l), m-\beta_i(m))|^p]dmdl \\
&\leq \varpi + p\sum_{i=1}^{n} M_i N_i \int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)}[|z(\gamma_i(\alpha), \delta_i(\beta))|^p\ .[e_i(\gamma_i(\alpha)+\alpha_i(\alpha), \beta_i(\beta)+\delta_i(\beta)) \\
&\quad \xi(|z(\gamma_i(\alpha), \delta_i(\beta))|) + \int_{\gamma_i(\varpi_0)}^{\gamma_i(\alpha)}\int_{\delta_i(\varpi_0)}^{\delta_i(\beta)} M_i N_i f_i(\alpha_i(\sigma)+\gamma_i(\sigma), \beta_i(\tau)+\delta_i(\tau)) \\
&\quad \xi(|z(\gamma_i(\sigma), \delta_i(\tau))|)d\delta_i(\tau)d\gamma_i(\sigma)] + g_i(\gamma_i(\alpha)+\alpha_i(\alpha), \beta_i(\beta)+\delta_i(\beta)) \\
&\quad |z(\gamma_i(\alpha), \delta_i(\beta))|^p]d\delta_i(\beta)d\gamma_i(\alpha) \\
&\leq \varpi + p\sum_{i=1}^{n}\int_{\gamma_i(\varpi_0)}^{\gamma_i(\varpi)}\int_{\delta_i(\psi_0)}^{\delta_i(\psi)} |z(s, q)|^p\ .[\widetilde{e}_i(s, q)(\xi(|z(s, q)|) \\
&\quad + \int_{\gamma_i(\varpi_0)}^{s}\int_{\delta_i(\psi_0)}^{q} \widetilde{f}_i(m, n)\xi(|z(m, n)|)dndm) + \widetilde{g}_i(s, q)]dqds. \qquad (3.\ 60)
\end{aligned}$$

Now an immediate application of the inequality given in Corollary 2.4 to (3.60) yields the desired result (3.58). □

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The authors declare no conflict of interest.

AUTHOR CONTRIBUTIONS

A significant contribution to technique, research, methodology, validation, and formal analysis has been made by Sabir Hussain. Shazia Shahadat made a substantial contribution to the writing of the manuscript as well as the production of the main results and applications. The main analysis was done by Sobia Rafeeq, who also proposed the concepts of preparing the initial draft, formal analysis, investigation, methodology, reviewing, and editing. The final manuscript has been read and approved by all writers.

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