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New Integral Inequalities For $r-$Convex Functions♦

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Abstract. We demonstrated Hermite-Hadamard type results via $r-$convex function class. The proofs of results are based on the identities which are given in [16] and [3], established by Set et al. and Chun and Qi respectively.

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1. Introduction and Preliminaries

Let us start this section by remembering how convex functions are defined, which have a very important place in the theory of inequalities and in different disciplines of mathematics.

Definition 1.1. *A real valued function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is defined convex if the following inequality holds for all $\mu_1, \mu_2 \in I$ and $\xi \in [0,1]$:*

$$f\left(\xi\mu_1 + (1-\xi)\mu_2\right) \leq \xi f(\mu_1) + (1-\xi)f(\mu_2).$$

Some results for convex functions and some generalizations of convex functions (e.g. $s-$convex functions, $h-$convex functions and $\varphi-$convex functions) can be found in [1], [3], [4], [6], [8] and [13]-[17].

Many researchers are studying inequalities to achieve optimal boundaries and approaches. Hermite-Hadamard inequality is valued as one of the most essential and remarkable inequalities. This inequality which is embodied below confers us upper and lower bounds for the mean value of convex functions.

Let $f : [\mu_1, \mu_2] \to \mathbb{R}$ be a convex function. Then

$$f\left(\frac{\mu_1+\mu_2}{2}\right) \le \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu \le \frac{f(\mu_1)+f(\mu_2)}{2} \tag{1. 1}$$

holds.

References [1]-[5], [7]-[14] and [18]-[20] can be examined for results that include this aesthetic inequality.

Now, let us define $r-$convex functions which are another generalized version of convex functions.

$M_r(\mu_1, \mu_2; \xi)$ is referred to power mean of order r of two positive numbers μ_1, μ_2 is defined by (See [5])

$$M_r(\mu_1,\mu_2;\xi) = \begin{cases} (\xi\mu_1^r + (1-\xi)\mu_2^r)^{\frac{1}{r}}, & \text{if } r \neq 0 \\ \mu_1^{\xi}\mu_2^{1-\xi}, & \text{if } r = 0. \end{cases}$$

Definition 1.2. [5] *A positive function f is $r-$convex on $[\mu, \eta]$ if*

$$f(\xi\mu_1 + (1-\xi)\mu_2) \le M_r(f(\mu_1), f(\mu_2); \xi)$$

holds for all $\mu_1, \mu_2 \in [\mu, \eta]$ and $\xi \in [0, 1]$.

In [5], Gill et al. obtained the inequality in (1. 2).

Theorem 1.3. *Suppose f is a positive $r-$convex function on $[\mu_1, \mu_2]$. Then*

$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu \le L_r(f(\mu_1), f(\mu_2)). \tag{1. 2}$$

Here L_r is the generalized logarithmic mean of order r of positive numbers μ_1, μ_2 is defined by

$$L_r(\mu_1,\mu_2) = \begin{cases} \frac{r}{r+1}\cdot\frac{\mu_1^{r+1}-\mu_2^{r+1}}{\mu_1^r-\mu_2^r}, & r \neq 0,-1,\ \mu_1 \neq \mu_2 \\ \frac{\mu_1-\mu_2}{\ln\mu_1-\ln\mu_2}, & r = 0,\ \mu_1 \neq \mu_2 \\ \mu_1\mu_2 \cdot \frac{\ln\mu_1-\ln\mu_2}{\mu_1-\mu_2}, & r = -1,\ \mu_1 \neq \mu_2 \\ \mu_1, & \mu_1 = \mu_2. \end{cases}$$

There are some results for $r-$convex functions in the references [1], [2], [4], [5], [7], [11], [12] and [18]-[20].

The principal objective of our study is to establish several new results via r-convexity. To derive our results we consult the identities which are embodied below.

Lemma 1.4. [16] *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, where $\mu_1, \mu_2 \in I$ with $\mu_1 < \mu_2$. If $f'' \in L[\mu_1, \mu_2]$,*

$$\begin{aligned} &\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu - f(\mu_3) + \left(\mu_3 - \frac{\mu_1+\mu_2}{2}\right)f'(\mu_3) \\ = \ &\frac{(\mu_3-\mu_1)^3}{2(\mu_2-\mu_1)}\int_0^1 \xi^2 f''(\xi\mu_3 + (1-\xi)\mu_1)d\xi \\ &+\frac{(\mu_2-\mu_3)^3}{2(\mu_2-\mu_1)}\int_0^1 \xi^2 f''(\xi\mu_3 + (1-\xi)\mu_2)d\xi \end{aligned}$$

holds for all $\mu_3 \in [\mu_1, \mu_2]$.

Lemma 1.5. [3] *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a third order differentiable function on* I°, *where* $\mu_1, \mu_2 \in I$ *with* $\mu_1 < \mu_2$. *If* $f''' \in L[\mu_1, \mu_2]$,

$$\begin{aligned} & \frac{f(\mu_1)+f(\mu_2)}{2} - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu - \frac{\mu_2-\mu_1}{12}\left[f'(\mu_2)-f'(\mu_1)\right] \\ = & \frac{(\mu_2-\mu_1)^3}{12}\int_0^1 \xi(1-\xi)(2\xi-1)f'''(\xi\mu_1+(1-\xi)\mu_2)d\xi \end{aligned}$$

holds.

2. Inequalities for twice differentiable $r-$convex functions

In this section, we give results for twice differentiable $r-$convex functions in the Theorems 2.1-2.4.

Theorem 2.1. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ *be a twice differentiable function on* I° *such that* $f'' \in L[\mu_1, \mu_2]$ *with* $\mu_1, \mu_2 \in I$ *and* $\mu_1 < \mu_2$. *If* $|f''|^q$ *is* $r-$*convex on* $[\mu_1, \mu_2]$, *we obtain the following inequality for all* $\mu_3 \in [\mu_1, \mu_2]$ *and* $q > 1$, $\frac{1}{p}+\frac{1}{q}=1$:

$$\begin{aligned} & \left|\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu - f(\mu_3) + \left(\mu_3 - \frac{\mu_1+\mu_2}{2}\right)f'(\mu_3)\right| \qquad (2.\ 3) \\ \leq & \frac{(\mu_3-\mu_1)^3}{2(\mu_2-\mu_1)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[L_r\left(|f''(\mu_1)|^q, |f''(\mu_3)|^q\right)\right]^{\frac{1}{q}} \\ & + \frac{(\mu_2-\mu_3)^3}{2(\mu_2-\mu_1)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[L_r\left(|f''(\mu_3)|^q, |f''(\mu_2)|^q\right)\right]^{\frac{1}{q}}. \end{aligned}$$

Proof. We can write

$$\begin{aligned} & \left|\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu - f(\mu_3) + \left(\mu_3 - \frac{\mu_1+\mu_2}{2}\right)f'(\mu_3)\right| \qquad (2.\ 4) \\ \leq & \frac{(\mu_3-\mu_1)^3}{2(\mu_2-\mu_1)}\int_0^1 \xi^2\left|f''(\xi\mu_3+(1-\xi)\mu_1)\right|d\xi \\ & + \frac{(\mu_2-\mu_3)^3}{2(\mu_2-\mu_1)}\int_0^1 \xi^2\left|f''(\xi\mu_3+(1-\xi)\mu_2)\right|d\xi \end{aligned}$$

via Lemma 1.4 and property of modulus. If we use Hölder inequality in (2. 4) we obtain

$$\begin{aligned} & \left|\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu - f(\mu_3) + \left(\mu_3 - \frac{\mu_1+\mu_2}{2}\right)f'(\mu_3)\right| \qquad (2.\ 5) \\ \leq & \frac{(\mu_3-\mu_1)^3}{2(\mu_2-\mu_1)}\left(\int_0^1 \xi^{2p}d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(\xi\mu_3+(1-\xi)\mu_1)\right|^q d\xi\right)^{\frac{1}{q}} \\ & + \frac{(\mu_2-\mu_3)^3}{2(\mu_2-\mu_1)}\left(\int_0^1 \xi^{2p}d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(\xi\mu_3+(1-\xi)\mu_2)\right|^q d\xi\right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f''|^q$ is $r-$convex on $[\mu_1, \mu_2]$, we have

$$\int_0^1 |f''(\xi\mu_3 + (1-\xi)\mu_1)|^q \, d\xi \leq L_r \left(|f''(\mu_1)|^q, |f''(\mu_3)|^q \right) \tag{2. 6}$$

and

$$\int_0^1 |f''(\xi\mu_3 + (1-\xi)\mu_2)|^q \, d\xi \leq L_r \left(|f''(\mu_3)|^q, |f''(\mu_2)|^q \right) \tag{2. 7}$$

via the inequality in (1. 2). If we use (2. 6) and (2. 7) in (2. 5) and if we calculate the integrals in (2. 5), we obtain the inequality in (2. 3). □

Theorem 2.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a twice differentiable function on I° such that $f'' \in L[\mu_1, \mu_2]$ with $\mu_1, \mu_2 \in I$ and $\mu_1 < \mu_2$. If $|f''|$ is $r-$convex on $[\mu_1, \mu_2]$, we obtain the following inequality for all $\mu_3 \in [\mu_1, \mu_2]$ and $r > 1$:*

$$\begin{aligned} & \left| \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} f(\mu) d\mu - f(\mu_3) + \left(\mu_3 - \frac{\mu_1 + \mu_2}{2} \right) f'(\mu_3) \right| \\ \leq \quad & \frac{r}{3r+1} \left(\frac{(\mu_3 - \mu_1)^3 + (\mu_2 - \mu_3)^3}{2(\mu_2 - \mu_1)} \right) |f''(\mu_3)| \\ & + \frac{r^3}{(1+r)(1+2r)(1+3r)} \left(\frac{(\mu_3 - \mu_1)^3 |f''(\mu_1)| + (\mu_2 - \mu_3)^3 |f''(\mu_2)|}{\mu_2 - \mu_1} \right). \end{aligned} \tag{2. 8}$$

Proof. Using Lemma 1.4, property of modulus and $r-$convexity of $|f''|$, we have

$$\begin{aligned} & \left| \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} f(\mu) d\mu - f(\mu_3) + \left(\mu_3 - \frac{\mu_1 + \mu_2}{2} \right) f'(\mu_3) \right| \\ \leq \quad & \frac{(\mu_3 - \mu_1)^3}{2(\mu_2 - \mu_1)} \int_0^1 \xi^2 \left[\xi |f''(\mu_3)|^r + (1-\xi) |f''(\mu_1)|^r \right]^{\frac{1}{r}} d\xi \\ & + \frac{(\mu_2 - \mu_3)^3}{2(\mu_2 - \mu_1)} \int_0^1 \xi^2 \left[\xi |f''(\mu_3)|^r + (1-\xi) |f''(\mu_2)|^r \right]^{\frac{1}{r}} d\xi. \end{aligned} \tag{2. 9}$$

If the following inequality in (2.10) is taken into account in (2.9),

$$\sum_{i=1}^m (\omega_i + \varpi_i)^\varepsilon \leq \sum_{i=1}^m \omega_i^\varepsilon + \sum_{i=1}^m \varpi_i^\varepsilon \tag{2. 10}$$

for $0 < \varepsilon < 1, \omega_1, \omega_2, ..., \omega_m \geq 0$ and $\varpi_1, \varpi_2, ..., \varpi_m \geq 0$, we obtain

$$\begin{aligned} & \left| \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} f(\mu) d\mu - f(\mu_3) + \left(\mu_3 - \frac{\mu_1 + \mu_2}{2} \right) f'(\mu_3) \right| \\ \leq \quad & \frac{(\mu_3 - \mu_1)^3}{2(\mu_2 - \mu_1)} \int_0^1 \left[\xi^{2+\frac{1}{r}} |f''(\mu_3)| + \xi^2 (1-\xi)^{\frac{1}{r}} |f''(\mu_1)| \right] d\xi \\ & + \frac{(\mu_2 - \mu_3)^3}{2(\mu_2 - \mu_1)} \int_0^1 \left[\xi^{2+\frac{1}{r}} |f''(\mu_3)| + \xi^2 (1-\xi)^{\frac{1}{r}} |f''(\mu_2)| \right] d\xi. \end{aligned} \tag{2. 11}$$

We have the required inequality in (2.8) with the calculations of the integrals in (2.11). □

Theorem 2.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a twice differentiable function on I° such that $f'' \in L[\mu_1, \mu_2]$ with $\mu_1, \mu_2 \in I$ and $\mu_1 < \mu_2$. If $|f''|^q$ is $r-$convex on $[\mu_1, \mu_2]$, we obtain the following inequality for all $\mu_3 \in [\mu_1, \mu_2]$ and $r > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$:*

$$\begin{aligned} & \left| \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} f(\mu) d\mu - f(\mu_3) + \left(\mu_3 - \frac{\mu_1 + \mu_2}{2} \right) f'(\mu_3) \right| \qquad (2.\ 12) \\ \leq & \quad \frac{1}{2(\mu_2 - \mu_1)\,(2p+1)^{\frac{1}{p}}} \left(\frac{r}{1+r} \right)^{\frac{1}{q}} \left[(\mu_3 - \mu_1)^3 \left[|f''(\mu_1)|^q + |f''(\mu_3)|^q \right]^{\frac{1}{q}} \right. \\ & \left. + (\mu_2 - \mu_3)^3 \left[|f''(\mu_3)|^q + |f''(\mu_2)|^q \right]^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. If we use Lemma 1.4, property of modulus and Hölder inequality we obtain the inequality in (2.5). If we use $r-$convexity of $|f''|^q$, we can write

$$\begin{aligned} & \left| \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} f(\mu) d\mu - f(\mu_3) + \left(\mu_3 - \frac{\mu_1 + \mu_2}{2} \right) f'(\mu_3) \right| \qquad (2.\ 13) \\ \leq & \quad \frac{(\mu_3 - \mu_1)^3}{2(\mu_2 - \mu_1)} \left(\int_0^1 \xi^{2p} d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \left[\xi \, |f''(\mu_3)|^{qr} + (1-\xi) \, |f''(\mu_1)|^{qr} \right]^{\frac{1}{r}} d\xi \right)^{\frac{1}{q}} \\ & + \frac{(\mu_2 - \mu_3)^3}{2(\mu_2 - \mu_1)} \left(\int_0^1 \xi^{2p} d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \left[\xi \, |f''(\mu_3)|^{qr} + (1-\xi) \, |f''(\mu_2)|^{qr} \right]^{\frac{1}{r}} d\xi \right)^{\frac{1}{q}}. \end{aligned}$$

In (2.13), if we use the inequality in (2.10) we obtain

$$\begin{aligned} & \left| \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} f(\mu) d\mu - f(\mu_3) + \left(\mu_3 - \frac{\mu_1 + \mu_2}{2} \right) f'(\mu_3) \right| \qquad (2.\ 14) \\ \leq & \quad \frac{(\mu_3 - \mu_1)^3}{2(\mu_2 - \mu_1)} \left(\int_0^1 \xi^{2p} d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \left[\xi^{\frac{1}{r}} \, |f''(\mu_3)|^{q} + (1-\xi)^{\frac{1}{r}} \, |f''(\mu_1)|^{q} \right] d\xi \right)^{\frac{1}{q}} \\ & + \frac{(\mu_2 - \mu_3)^3}{2(\mu_2 - \mu_1)} \left(\int_0^1 \xi^{2p} d\xi \right)^{\frac{1}{p}} \left(\int_0^1 \left[\xi^{\frac{1}{r}} \, |f''(\mu_3)|^{q} + (1-\xi)^{\frac{1}{r}} \, |f''(\mu_2)|^{q} \right] d\xi \right)^{\frac{1}{q}}. \end{aligned}$$

If we calculate the integrals in (2.14), we get the inequality in (2.12). □

Theorem 2.4. *Under the assumptions of Theorem 2.3, we have the following inequality:*

$$\begin{aligned} & \left| \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} f(\mu) d\mu - f(\mu_3) + \left(\mu_3 - \frac{\mu_1 + \mu_2}{2} \right) f'(\mu_3) \right| \qquad (2.\ 15) \\ \leq & \quad \frac{(\mu_3 - \mu_1)^3}{2(\mu_2 - \mu_1)\,(p+1)^{\frac{1}{p}}} \left[\frac{r}{qr + r + 1} \, |f''(\mu_3)|^q + \beta\left(1 + \frac{1}{r}, q+1\right) |f''(\mu_1)|^q \right]^{\frac{1}{q}} \\ & + \frac{(\mu_2 - \mu_3)^3}{2(\mu_2 - \mu_1)\,(p+1)^{\frac{1}{p}}} \left[\frac{r}{qr + r + 1} \, |f''(\mu_3)|^q + \beta\left(1 + \frac{1}{r}, q+1\right) |f''(\mu_2)|^q \right]^{\frac{1}{q}}. \end{aligned}$$

Here β is Euler Beta function.

Proof. If we use Lemma 1.4, property of modulus and Hölder inequality we obtain

$$\begin{aligned} & \left|\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu - f(\mu_3) + \left(\mu_3-\frac{\mu_1+\mu_2}{2}\right)f'(\mu_3)\right| \qquad (2.16) \\ \leq & \frac{(\mu_3-\mu_1)^3}{2(\mu_2-\mu_1)}\left(\int_0^1 \xi^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \xi^q \left|f''(\xi\mu_3+(1-\xi)\mu_1)\right|^q d\xi\right)^{\frac{1}{q}} \\ & +\frac{(\mu_2-\mu_3)^3}{2(\mu_2-\mu_1)}\left(\int_0^1 \xi^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \xi^q \left|f''(\xi\mu_3+(1-\xi)\mu_2)\right|^q d\xi\right)^{\frac{1}{q}}. \end{aligned}$$

We can write

$$\begin{aligned} & \left|\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu - f(\mu_3) + \left(\mu_3-\frac{\mu_1+\mu_2}{2}\right)f'(\mu_3)\right| \qquad (2.17) \\ \leq & \frac{(\mu_3-\mu_1)^3}{2(\mu_2-\mu_1)}\left(\int_0^1 \xi^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \xi^q \left[\xi^{\frac{1}{r}}\left|f''(\mu_3)\right|^q+(1-\xi)^{\frac{1}{r}}\left|f''(\mu_1)\right|^q\right]d\xi\right)^{\frac{1}{q}} \\ & +\frac{(\mu_2-\mu_3)^3}{2(\mu_2-\mu_1)}\left(\int_0^1 \xi^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \xi^q \left[\xi^{\frac{1}{r}}\left|f''(\mu_3)\right|^q+(1-\xi)^{\frac{1}{r}}\left|f''(\mu_2)\right|^q\right]d\xi\right)^{\frac{1}{q}} \end{aligned}$$

via $r-$convexity of $|f''|^q$ and the inequality in (2.10). If we calculate the integrals in (2.17), we obtain the inequality in (2.15). □

Remark 2.5. *In the Theorems 2.1-2.4 one can obtain some results for special values of μ_3. It is left to the interested reader.*

3. Inequalities for third order differentiable $r-$convex functions

In this section, two theorems are given for third-order differentiable $r-$convex functions.

Theorem 3.1. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a third order differentiable function on I° such that $f''' \in L[\mu_1,\mu_2]$ with $\mu_1,\mu_2 \in I$ and $\mu_1<\mu_2$. If $|f'''|$ is $r-$convex on $[\mu_1,\mu_2]$, we obtain the following inequality for $r>1$:*

$$\begin{aligned} & \left|\frac{f(\mu_1)+f(\mu_2)}{2}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu-\frac{\mu_2-\mu_1}{12}\left[f'(\mu_2)-f'(\mu_1)\right]\right| \qquad (3.18) \\ \leq & \frac{(\mu_2-\mu_1)^3 r^2}{12(1+2r)(1+3r)(1+4r)}\left(1+\frac{1+6r}{2^{2+\frac{1}{r}}}\right)\left[\left|f'''(\mu_1)\right|+\left|f'''(\mu_2)\right|\right]. \end{aligned}$$

Proof. If we use Lemma 1.5, property of modulus and $r-$convexity of $|f'''|$ we can write

$$\begin{aligned} & \left|\frac{f(\mu_1)+f(\mu_2)}{2}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu-\frac{\mu_2-\mu_1}{12}\left[f'(\mu_2)-f'(\mu_1)\right]\right| \qquad (3.19) \\ \leq & \frac{(\mu_2-\mu_1)^3}{12}\left[\int_0^{\frac{1}{2}} \xi(1-\xi)(1-2\xi)\left[\xi\left|f'''(\mu_1)\right|^r+(1-\xi)\left|f'''(\mu_2)\right|^r\right]^{\frac{1}{r}}d\xi\right. \\ & \left.+\int_{\frac{1}{2}}^1 \xi(1-\xi)(2\xi-1)\left[\xi\left|f'''(\mu_1)\right|^r+(1-\xi)\left|f'''(\mu_2)\right|^r\right]^{\frac{1}{r}}d\xi\right]. \end{aligned}$$

In (3.19), if we use the inequality in (2.10) we obtain

$$\begin{aligned}
&\left|\frac{f(\mu_1)+f(\mu_2)}{2}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu-\frac{\mu_2-\mu_1}{12}\left[f'(\mu_2)-f'(\mu_1)\right]\right| \quad (3.\ 20)\\
\leq\ & \frac{(\mu_2-\mu_1)^3}{12}\left[\int_0^{\frac{1}{2}}\xi(1-\xi)(1-2\xi)\left[\xi^{\frac{1}{r}}\left|f'''(\mu_1)\right|+(1-\xi)^{\frac{1}{r}}\left|f'''(\mu_2)\right|\right]d\xi\right.\\
&\left.+\int_{\frac{1}{2}}^{1}\xi(1-\xi)(2\xi-1)\left[\xi^{\frac{1}{r}}\left|f'''(\mu_1)\right|+(1-\xi)^{\frac{1}{r}}\left|f'''(\mu_2)\right|\right]d\xi\right].
\end{aligned}$$

Further we have

$$\begin{aligned}
\int_0^{\frac{1}{2}}\xi^{1+\frac{1}{r}}(1-\xi)(1-2\xi)d\xi &= \int_{\frac{1}{2}}^{1}\xi(1-\xi)^{1+\frac{1}{r}}(2\xi-1)d\xi \quad (3.\ 21)\\
&= \frac{2r^2+12r^3}{2^{4+\frac{1}{r}}(1+2r)(1+3r)(1+4r)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\xi(1-\xi)^{1+\frac{1}{r}}(1-2\xi)d\xi \quad (3.\ 22)\\
=\ &\int_{\frac{1}{2}}^{1}\xi^{1+\frac{1}{r}}(1-\xi)(2\xi-1)d\xi\\
=\ &\frac{r^2}{(1+2r)(1+3r)(1+4r)}+\frac{2r^2+12r^3}{2^{4+\frac{1}{r}}(1+2r)(1+3r)(1+4r)}.
\end{aligned}$$

If the equations in (3.21) and (3.22) are used in (3.20), we get the inequality in (3.18) . □

Theorem 3.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a third order differentiable function on I° such that $f''' \in L[\mu_1, \mu_2]$ with $\mu_1, \mu_2 \in I$ and $\mu_1 < \mu_2$. If $|f'''|^q$ is $r-$convex on $[\mu_1, \mu_2]$, we obtain the following inequality for $r > 1$ and $q > 1$, $\frac{1}{p}+\frac{1}{q}=1$:*

$$\begin{aligned}
&\left|\frac{f(\mu_1)+f(\mu_2)}{2}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu-\frac{\mu_2-\mu_1}{12}\left[f'(\mu_2)-f'(\mu_1)\right]\right| \quad (3.\ 23)\\
\leq\ & \frac{(\mu_2-\mu_1)^3}{12}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\beta\left(q+1,\frac{r+rq+1}{r}\right)\right)^{\frac{1}{q}}\left[\left|f'''(\mu_1)\right|^q+\left|f'''(\mu_2)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. If we use Lemma 1.5, property of modulus, $r-$convexity of $|f'''|^q$ and the inequality in (2.10) we can write

$$\begin{aligned}
&\left|\frac{f(\mu_1)+f(\mu_2)}{2}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(\mu)d\mu-\frac{\mu_2-\mu_1}{12}\left[f'(\mu_2)-f'(\mu_1)\right]\right| \quad (3.\ 24)\\
\leq\ & \frac{(\mu_2-\mu_1)^3}{12}\left(\int_0^1|2\xi-1|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\xi^q(1-\xi)^q\left[\xi\left|f'''(\mu_1)\right|^{qr}+(1-\xi)\left|f'''(\mu_2)\right|^{qr}\right]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}\\
\leq\ & \frac{(\mu_2-\mu_1)^3}{12}\left(\int_0^1|2\xi-1|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left(\xi^{q+\frac{1}{r}}(1-\xi)^q\left|f'''(\mu_1)\right|^q+\xi^q(1-\xi)^{q+\frac{1}{r}}\left|f'''(\mu_2)\right|^q\right)d\xi\right)^{\frac{1}{q}}.
\end{aligned}$$

With the help of calculations of the integrals in (3.24), we get the required inequality in (3.23). $\square$

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