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# New Inequalities Involving k-Fractional Integral for $h$-Convex Functions and Their Applications

Muhammad Amer Latif
Department of Basic Sciences,
Deanship of Preparatory Year, King Faisal
University, Hofuf 31982, Al-Hasa,Saudi Arabia
Email: m amer latif@hotmail.com; mlatif@kfu.edu.sa

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**Abstract.**: In this paper, we study new integral inequalities by using $\mathbf{k}$-fractional integral and the $h$-convex property of the the function $|\epsilon'|^q$, where $q \geq 1$ or $q > 1$ with $q^{-1} + p^{-1} = 1$. We also discuss some particular cases of the results for the classes contained in the class of $h$-convex functions.

**AMS (MOS) Subject Classification Codes: Primary: 26A51; Secondary: 05A30, 26A33, 26D07, 26D10, 26D15.**
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## 1. Introduction

Let $\hat{I} \neq \emptyset$ be interval in the set of reals. Convexity of a function $\epsilon : \hat{I} \to \mathbb{R}$ is defined as follows:

$$\epsilon\left(eu + (1-e)\,v\right) \leq e\epsilon\left(u\right) + (1-e)\,\epsilon\left(v\right)$$

for $u, v \in \hat{I}$ and $e \in [0,1]$.

The following well-known inequality, called Hermite-Hadamard inequality, is true for convex functions $\epsilon : \hat{I} \to \mathbb{R}$:

$$\epsilon\left(\frac{\rho+\zeta}{2}\right) \leq \frac{1}{\zeta-\rho}\int_{\rho}^{\zeta}\epsilon(\sigma)d\sigma \leq \frac{\epsilon(\rho)+\epsilon(\zeta)}{2}. \tag{1. 1}$$

For concave function $\epsilon$ inequality ( 1. 1 ) holds in reverse direction. Geometric significance and applications in different results of scientific fields, many scientists, mathematicians, and researchers are focusing on ( 1. 1 ). Additional research on this topic can be found in a number of articles. Varosanec [31] developed the notion of $h$-convex function in 2007 to generalize the concept of convex function. As particular cases for this class, there are $s$-convex functions [4], Godunova-Levin functions [7], and $P$-functions [6]. The Weir et al. [32] preinvex function which describes another profound generalisability of a convex function. Some generalized Hermite-Hadamard type inequalities had been previously proven and now Sundas et al. [8] had also discovered additional Hermite-Hadamard type inequalities related to Godunova-Levin types of preinvex functions, Brckner types of preinvex functions, Godunova-Levin types of $s$-preinvex functions, and $P$-preinvex functions. The work of the last decade has focused on finding creative and innovative ways to generalize existing inequalities. Researchers discovered that fractional integral operators are the most popular among them. It has been found that nontrivial and positive solutions of several classes of fractional differential equations exist when integral inequalities involving fractional integrals are involved.

The generalization of numerous integral inequalities, as well as their practical applications, are under investigation by researchers [19, 25]. Mubeen et al. [21, 22] acquired the improved generalized Grüss type and Ostrowski type integral inequalities for fractional $\varepsilon$-Riemann-Liouville integrals. According to Agarwal et al. [2], Hermite-Hadamard type inequalities for generalized $\varepsilon$-fractional integrals were investigated. The prescriptive Hermite-Hadamard inequalities from Set et al. [26] are best suited for the Riemann-Liouville fractional integral. In addition to using general Ostrowski type inequalities, In addition to using general Ostrowski type inequalities, Sarikaya et al. [27] reviewed Ostrowski type inequalities specific to local fractional integrals. As a way of further understanding the topic, we direct the reader to [10]-[18], [28, 29, 33] and the sources cited in them. Having proved new Hermite-Hadamard type inequalities for classes of $h$-preinvex functions and related classes by using techniques of mathematical analysis, we are motivated to pursue more progressive research in this direction. In this paper, the following organization was implemented: The presentation is then completed in Section 2, where introductory concepts are presented, in Section 3, where primary results about the topic are described, and in Section 4, where applied results are discussed.

## 2. PRELIMINARIES

**Definition 2.1.** [32] *Let $\lambda \neq \emptyset$ be in $\mathbb{R}^{\kappa}$ and $\hat{\varphi} : \lambda \times \lambda \to \mathbb{R}^{\kappa}$. Let $u \in \lambda$, then the set $\lambda$ is said to be invex at $u$ with respect to $\hat{\varphi}(\cdot, \cdot)$, if*

$$u + e\hat{\varphi}(v, u) \in \lambda$$

*for all $u, v \in \lambda, e \in [0, 1]$. Set $\lambda$ is called an invex set with regard to the function $\hat{\varphi}$ if at each $u \in \lambda$, $\lambda$ is invex. The invex set $\lambda$ is also called an $\hat{\varphi}$-connected set.*

**Remark 2.2.** *That is remarkable; there is a path that starts from a point known as $\sigma$ and which goes through $\lambda$. However, the final point does not have to be $v$. For $\hat{\varphi}(\nu, u) = v - u$, $v$ is the point at which the path terminates is in $\lambda$ implying invexity reduces to convexity.*

**Definition 2.3.** [32] *A function $\epsilon : \lambda \to \mathbb{R}$, where $\lambda \subset \mathbb{R}^{\kappa}$ is invex, is preinvex with respect to $\hat{\varphi}$, if*

$$\epsilon(u + e\hat{\varphi}(v, u)) \leq (1 - e)\,\epsilon(u) + e\epsilon(v)$$

*holds for all $u, v \in \lambda, e \in [0, 1]$. The function $\epsilon$ is preincave if and only if $-\epsilon$ is preinvex. A comment is in order to point out that every convex function is preinvex function on the map $\hat{\varphi}(v, u) = v - u$, the converse may not be true* [1].

**Definition 2.4.** [23] *Let $(0, 1) \subset \mathcal{J} \subset \mathbb{R}$ and let $h : \mathcal{J} \to \mathbb{R}$ be a non-negative function with $h \not\equiv 0$. A function $\epsilon : \lambda \to \mathbb{R}$ defined on $\lambda \subset \mathbb{R}^{\kappa}$, with the condition that $\lambda$ an invex* *set, is $h-$preinvex with respect to $\hat{\varphi}$, if $\forall\, u, v \in \lambda$ and $e \in [0,1]$, we have*

$$\epsilon(u + e\hat{\varphi}(v,u)) \leq h\,(1-e)\,\epsilon(u) + h\,(e)\,\epsilon(v). \tag{2. 2}$$

*For $h$-preincave function the inequality sign in (2.2) is reversed. Convex functions are also $h$-preinvex with respect to $\hat{\varphi}(v,u) = v - u$ and $h\,(e) = e$, $e \in [0,1]$.*

**Definition 2.5.** [22] *Let $\epsilon \in L_1\,([\rho,\zeta])$, the $\varepsilon$-fractional integrals ${}_{\varepsilon}\mathbb{J}^{\iota}_{\rho+}$ and ${}_{\varepsilon}\mathbb{J}^{\iota}_{\zeta-}$ of order $\iota > 0$ are defined by:*

$${}_{\varepsilon}\mathbb{J}^{\iota}_{\rho+}\epsilon\,(\sigma) := \frac{1}{\varepsilon\Gamma_{\varepsilon}\,(\iota)}\int_{\rho}^{\sigma}(\sigma-\xi)^{\frac{\iota}{\varepsilon}-1}\,\epsilon\,(\xi)\,d\xi,\ 0 \leq \rho < \sigma < \zeta$$

$${}_{\varepsilon}\mathbb{J}^{\iota}_{\zeta-}\epsilon\,(\sigma) := \frac{1}{\varepsilon\Gamma_{\varepsilon}\,(\iota)}\int_{\sigma}^{\zeta}(\xi-\sigma)^{\frac{\iota}{\varepsilon}-1}\,\epsilon\,(\xi)\,d\xi,\ 0 \leq \rho < \sigma < \zeta,$$

*respectively, where $\varepsilon > 0$ and $\Gamma_{\varepsilon}$ is the $\varepsilon$-gamma function defined as:*

$$\Gamma_{\varepsilon}\,(\sigma) := \int_{0}^{\infty}\xi^{\sigma-1}e^{\frac{\xi^{\varepsilon}}{\varepsilon}}\,d\xi,\ \operatorname{Re}(\sigma) > 0,$$

*where $\Gamma_{\varepsilon}$ satisifies the property $\Gamma_{\varepsilon}\,(\sigma+\varepsilon) = \sigma\Gamma_{\varepsilon}\,(\sigma) = 1$ and $\Gamma_{\varepsilon}\,(\varepsilon) = 1$.*

## 3. Main Results

Throughout the whole discussion, $\mathbb{N}$ is taken to be set of positive integers, $\iota$, $\varepsilon$, $\kappa$ are positive real numbers, $e \in [0,1]$, $\lambda \subset \mathbb{R}$ is an open invex set with respect to the mapping $\hat{\varphi} : \lambda \times \lambda \to \mathbb{R}\backslash\,\{0\}$; let $\epsilon : \lambda \to \mathbb{R}$ be a differentiable mapping on $[\rho, \rho + \hat{\varphi}\,(\zeta,\rho)]$ for $\hat{\varphi}\,(\zeta,\rho) > 0$ and

$$\begin{aligned}
\varphi_{\hat{\varphi}}\,(\varepsilon,\iota,e,\kappa;\sigma) &:= \left(\kappa^{\frac{\iota}{\varepsilon}} - e\right) \\
&\times \frac{(\hat{\varphi}\,(\sigma,\rho))^{\frac{\iota}{\varepsilon}}\,[\epsilon\,(\rho+\hat{\varphi}\,(\sigma,\rho)) + \epsilon\,(\rho)] + (\hat{\varphi}\,(\zeta,\sigma))^{\frac{\iota}{\varepsilon}}\,[\epsilon\,(\sigma+\hat{\varphi}\,(\zeta,\sigma)) + \epsilon\,(\sigma)]}{\hat{\varphi}\,(\zeta,\rho)} \\
&+ 2e\frac{(\hat{\varphi}\,(\sigma,\rho))^{\frac{\iota}{\varepsilon}}\,\epsilon\left(\rho+\frac{\hat{\varphi}(\sigma,\rho)}{2}\right) + (\hat{\varphi}\,(\zeta,\sigma))^{\frac{\iota}{\varepsilon}}\,\epsilon\left(\sigma+\frac{\hat{\varphi}(\zeta,\sigma)}{2}\right)}{\hat{\varphi}\,(\zeta,\rho)} \\
&- \frac{(2\kappa)^{\frac{\iota}{\varepsilon}}\,\Gamma_{\varepsilon}\,(\iota+\varepsilon)}{\hat{\varphi}\,(\zeta,\rho)}\left[{}_{\varepsilon}\mathbb{J}^{\iota}_{(\rho+\hat{\varphi}(\sigma,\rho))^-}\epsilon\left(\rho+\frac{\hat{\varphi}\,(\sigma,\rho)}{2}\right) + {}_{\varepsilon}\,\mathbb{J}^{\iota}_{\rho+}\epsilon\left(\rho+\frac{\hat{\varphi}\,(\sigma,\rho)}{2}\right)\right. \\
&\left. + {}_{\varepsilon}\mathbb{J}^{\iota}_{(\sigma+\hat{\varphi}(\zeta,\sigma))^-}\epsilon\left(\sigma+\frac{\hat{\varphi}\,(\zeta,\sigma)}{2}\right) + {}_{\varepsilon}\,\mathbb{J}^{\iota}_{\sigma+}\epsilon\left(\sigma+\frac{\hat{\varphi}\,(\zeta,\sigma)}{2}\right)\right].
\end{aligned} \tag{3. 3}$$

Particular cases of ( 3. 3 ) are as follows:

$$\begin{aligned}
\varphi_{\hat{\varphi}}\,(1,1,1,1;\sigma) &= \frac{2\left\{\hat{\varphi}\,(\sigma,\rho)\,\epsilon\left(\rho+\frac{\hat{\varphi}(\sigma,\rho)}{2}\right) + \hat{\varphi}\,(\zeta,\sigma)\,\epsilon\left(\sigma+\frac{\hat{\varphi}(\zeta,\sigma)}{2}\right)\right\}}{\hat{\varphi}\,(\zeta,\rho)} \\
&- \frac{2}{\hat{\varphi}\,(\zeta,\rho)}\left[\int_{\rho}^{\rho+\hat{\varphi}(\sigma,\rho)}\epsilon\,(\xi)\,d\xi + \int_{\sigma}^{\rho+\hat{\varphi}(\zeta,\sigma)}\epsilon\,(\xi)\,d\xi\right],
\end{aligned} \tag{3. 4}$$

$$\varphi_{\hat{\varphi}}(1,1,0,1;\sigma) = -\frac{2}{\hat{\varphi}(\zeta,\rho)}\left[\int_{\rho}^{\rho+\hat{\varphi}(\sigma,\rho)} \epsilon(\xi)\,d\xi + \int_{\sigma}^{\rho+\hat{\varphi}(\zeta,\sigma)} \epsilon(\xi)\,d\xi\right] + \frac{\hat{\varphi}(\sigma,\rho)\left[\epsilon(\rho+\hat{\varphi}(\sigma,\rho))+\epsilon(\rho)\right]+\hat{\varphi}(\zeta,\sigma)\left[\epsilon(\sigma+\hat{\varphi}(\zeta,\sigma))+\epsilon(\sigma)\right]}{\hat{\varphi}(\zeta,\rho)} \quad (3.\ 5)$$

and

$$\varphi(\varepsilon,\iota,e,\kappa;\sigma) := \frac{\left(\kappa^{\frac{\iota}{\varepsilon}}-e\right)\left\{(\sigma-\rho)^{\frac{\iota}{\varepsilon}}\left(\epsilon(\sigma)+\epsilon(\rho)\right)+(\zeta-\sigma)^{\frac{\iota}{\varepsilon}}\left(\epsilon(\zeta)+\epsilon(\sigma)\right)\right\}}{\zeta-\rho} + \frac{2e\left\{(\sigma-\rho)^{\frac{\iota}{\varepsilon}}\,\epsilon\left(\frac{\sigma-\rho}{2}\right)+(\zeta-\sigma)^{\frac{\iota}{\varepsilon}}\,\epsilon\left(\frac{\zeta+\sigma}{2}\right)\right\}}{\zeta-\rho}\left[{}_{\varepsilon}\mathbb{J}^{\iota}_{\sigma^-}\epsilon\left(\frac{\rho+\sigma}{2}\right)+{}_{\varepsilon}\mathbb{J}^{\iota}_{\rho^+}\epsilon\left(\frac{\rho+\sigma}{2}\right)+{}_{\varepsilon}\mathbb{J}^{\iota}_{\zeta^-}\epsilon\left(\frac{\zeta+\sigma}{2}\right)+{}_{\varepsilon}\mathbb{J}^{\iota}_{\sigma^+}\epsilon\left(\frac{\zeta+\sigma}{2}\right)\right]. \quad (3.\ 6)$$

Suppose also that

$$Q_1(\varepsilon,\iota,\kappa,e;h) = \int_0^{\kappa}\left|\xi^{\frac{\iota}{\varepsilon}}-e\right| \times\left[h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\rho)|^q + h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q\right]d\xi, \quad (3.\ 7)$$

$$Q_2(\varepsilon,\iota,\kappa,e;h) = \int_0^{\kappa}\left|\xi^{\frac{\iota}{\varepsilon}}-e\right| \times\left[h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\rho)|^q + h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q\right]d\xi, \quad (3.\ 8)$$

$$Q_3(\varepsilon,\iota,\kappa,e;h) = \int_0^{\kappa}\left|\xi^{\frac{\iota}{\varepsilon}}-e\right| \times\left[h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q + h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\zeta)|^q\right]d\xi, \quad (3.\ 9)$$

$$Q_4(\varepsilon,\iota,\kappa,e;h) = \int_0^{\kappa}\left|\xi^{\frac{\iota}{\varepsilon}}-e\right| \times\left[h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q + h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\zeta)|^q\right]d\xi, \quad (3.\ 10)$$

$$Q_5(\varepsilon,\iota,\kappa;e) = \frac{\iota e\left(2e^{\frac{\varepsilon}{\iota}}-\kappa\right)+\varepsilon\left(\kappa^{\frac{\varepsilon+\iota}{\varepsilon}}-\kappa e\right)}{\varepsilon+\iota}, \quad (3.\ 11)$$

$$\alpha_1(\varepsilon,\iota,\kappa;e)=\frac{\varepsilon^2\cdot\kappa^{\frac{\varepsilon+\iota}{\varepsilon}}}{2(\varepsilon+\iota)(2\varepsilon+\iota)}+\varepsilon e^{\frac{\varepsilon+\iota}{\iota}}\left(\frac{e^{\frac{\varepsilon}{\iota}}}{\kappa(2\varepsilon+\iota)}-\frac{1}{\varepsilon+\iota}\right)-e\frac{\kappa^2-4\kappa e^{\frac{\varepsilon}{\iota}}+2e^{\frac{2\varepsilon}{\iota}}}{4\kappa}, \quad (3.\ 12)$$

$$\alpha_2(\varepsilon,\iota,\kappa;e)=e\frac{2e^{\frac{2\varepsilon}{\iota}}+4\kappa e^{\frac{\varepsilon}{\iota}}-3\kappa^2}{4\kappa}-\varepsilon\iota\frac{\kappa e^{\frac{\varepsilon+\iota}{\iota}}+e^{\frac{2\varepsilon+\iota}{\iota}}-\kappa^{\frac{\iota+2\varepsilon}{\varepsilon}}}{\kappa(\varepsilon+\iota)(2\varepsilon+\iota)}-\varepsilon^2\frac{4\kappa e^{\frac{\varepsilon+\iota}{\iota}}+2e^{\frac{2\varepsilon+\iota}{\iota}}-3\kappa^{\frac{\iota+2\varepsilon}{\varepsilon}}}{2\kappa(\varepsilon+\iota)(2\varepsilon+\iota)}, \quad (3.\ 13)$$

$$\iota_1(\varepsilon,\iota,\kappa;e)=-\frac{e\left(\kappa-e^{\frac{\varepsilon}{\iota}}\right)^{s+1}-e\kappa^s\left(\kappa+\left(e^{\frac{\varepsilon}{\iota}}-\kappa\right)\left(1-\frac{e^{\frac{\varepsilon}{\iota}}}{\kappa}\right)^s\right)}{2^s\kappa^s(s+1)}-\frac{\varepsilon e^{\frac{\varepsilon+\iota}{\iota}}}{2^s(\varepsilon+\iota)}\,{}_2F_1\left(-s,\frac{\varepsilon+\iota}{\varepsilon},\frac{2\varepsilon+\iota}{\varepsilon},\frac{e^{\frac{\varepsilon}{\iota}}}{\kappa}\right), \quad (3.\ 14)$$

$$\iota_2(\varepsilon,\iota,\kappa;e)=\frac{2e\kappa\left(\frac{\kappa+e^{\frac{\varepsilon}{\iota}}}{\kappa}\right)^{s+1}-e\kappa-2^{s+1}\kappa e}{2^s(s+1)}+\frac{\varepsilon\kappa^{\frac{\varepsilon+\iota}{\varepsilon}}}{2^s(\varepsilon+\iota)}\,{}_2F_1\left(-s,\frac{\varepsilon+\iota}{\varepsilon},\frac{2\varepsilon+\iota}{\varepsilon},-1\right)-\frac{2\varepsilon e^{\frac{\varepsilon+\iota}{\iota}}}{2^s(\varepsilon+\iota)}\,{}_2F_1\left(-s,\frac{\varepsilon+\iota}{\varepsilon},\frac{2\varepsilon+\iota}{\varepsilon},-\frac{e^{\frac{\varepsilon}{\iota}}}{\kappa}\right) \quad (3.\ 15)$$

$$\gamma_1(\varepsilon,\iota,\kappa;e)=\frac{2^s\kappa^s e\left(\kappa-e^{\frac{\varepsilon}{\iota}}\right)^{1-s}+2^s e\left(\left(e^{\frac{\varepsilon}{\iota}}-\kappa\right)\left(1-\frac{e^{\frac{\varepsilon}{\iota}}}{\kappa}\right)^{-s}-\kappa\right)}{1-s}-\frac{2^s\varepsilon e^{\frac{\varepsilon+\iota}{\iota}}}{(\varepsilon+\iota)}\,{}_2F_1\left(s,\frac{\varepsilon+\iota}{\varepsilon},\frac{2\varepsilon+\iota}{\varepsilon},\frac{e^{\frac{\varepsilon}{\iota}}}{\kappa}\right), \quad (3.\ 16)$$

$$\gamma_2(\varepsilon,\iota,\kappa;e)=\frac{2^s\kappa e+2e\kappa-2^{s+1}\kappa e\left(\frac{\kappa+e^{\frac{\varepsilon}{\iota}}}{\kappa}\right)^{1-s}}{1-s}+\frac{2^s\varepsilon\kappa^{\frac{\varepsilon+\iota}{\varepsilon}}}{\varepsilon+\iota}\,{}_2F_1\left(s,\frac{\varepsilon+\iota}{\varepsilon},\frac{2\varepsilon+\iota}{\varepsilon},-1\right)-\frac{2^{s+1}\varepsilon e^{\frac{\varepsilon+\iota}{\varepsilon}}}{\varepsilon+\iota}\,{}_2F_1\left(s,\frac{\varepsilon+\iota}{\varepsilon},\frac{2\varepsilon+\iota}{\varepsilon},-\frac{e^{\frac{\varepsilon}{\iota}}}{\kappa}\right), \quad (3.\ 17)$$

$$\delta\left(\varepsilon,\iota,\kappa;e\right)=\frac{3\varepsilon^{2}\kappa^{3}\left(\kappa^{\frac{\iota}{\varepsilon}}-e\right)-\varepsilon\iota e\left(4\kappa^{3}-9\kappa^{2}e^{\frac{\varepsilon}{\iota}}+e^{\frac{3\varepsilon}{\iota}}\right)}{6\kappa^{2}\left(\varepsilon+\iota\right)\left(3\varepsilon+\iota\right)} -\iota^{2}e\frac{4\kappa^{3}-3\kappa^{2}e^{\frac{\varepsilon}{\iota}}+e^{\frac{3\varepsilon}{\iota}}}{6\kappa^{2}\left(\varepsilon+\iota\right)\left(3\varepsilon+\iota\right)}, \quad (3.\ 18)$$

$$Y_{1}\left(\kappa,e,q;h\right)=\int_{0}^{\kappa}\left[h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\rho)|^{q}+h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\sigma)|^{q}\right]d\xi, \quad (3.\ 19)$$

$$Y_{2}\left(\kappa,e,q;h\right)=\int_{0}^{\kappa}\left[h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\rho)|^{q}+h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\sigma)|^{q}\right]d\xi, \quad (3.\ 20)$$

$$Y_{3}\left(\kappa,e,q;h\right)=\int_{0}^{\kappa}\left[h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\sigma)|^{q}+h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\zeta)|^{q}\right]d\xi, \quad (3.\ 21)$$

$$Y_{4}\left(\kappa,e,q;h\right)=\int_{0}^{\kappa}\left[h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\sigma)|^{q}+h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\zeta)|^{q}\right]d\xi \quad (3.\ 22)$$

and

$$\mu\left(\varepsilon,\kappa,e,\iota,p\right)=\frac{\varepsilon e^{\frac{\varepsilon}{\iota}+p}B(p+1,\frac{\varepsilon}{\iota})}{\iota} -\frac{\kappa(\kappa^{\frac{\iota}{\varepsilon}}-e)\ {}_{2}F_{1}(1,1+p+\frac{\varepsilon}{\iota};\frac{\varepsilon+\iota}{\iota};\frac{\kappa^{\frac{\iota}{\varepsilon}}}{e})}{e}. \quad (3.\ 23)$$

**Lemma 3.1.** *Let $\epsilon:\lambda\to\mathbb{R}$ be a function with the above defined conditions, then*

$$\varphi_{\hat{\varphi}}\left(\varepsilon,\iota,e,\kappa;\sigma\right)=\frac{\left(\hat{\varphi}\left(\sigma,\rho\right)\right)^{\frac{\iota+\varepsilon}{\varepsilon}}}{2\kappa\hat{\varphi}\left(\zeta,\rho\right)}\left[\int_{0}^{\kappa}\left(\xi^{\frac{\iota}{\varepsilon}}-e\right)\left\{\epsilon'\left(\rho+\frac{\kappa+\xi}{2\kappa}\hat{\varphi}\left(\sigma,\rho\right)\right)\right.\right. \left.\left.-\epsilon'\left(\rho+\frac{\kappa-\xi}{2\kappa}\hat{\varphi}\left(\sigma,\rho\right)\right)\right\}d\xi\right]-\frac{\left(\hat{\varphi}\left(\zeta,\sigma\right)\right)^{\frac{\iota+\varepsilon}{\varepsilon}}}{2\kappa\hat{\varphi}\left(\zeta,\rho\right)}\left[\int_{0}^{\kappa}\left(\xi^{\frac{\iota}{\varepsilon}}-e\right)\right. \left.\times\left\{\epsilon'\left(\sigma+\frac{\kappa+\xi}{2\kappa}\hat{\varphi}\left(\zeta,\sigma\right)\right)-\epsilon'\left(\sigma+\frac{\kappa-\xi}{2\kappa}\hat{\varphi}\left(\zeta,\sigma\right)\right)\right\}d\xi\right] \quad (3.\ 24)$$

*holds for all $\sigma\in[\rho,\zeta]$.*

*Proof.* Consider,

$$\hat{I}_{1}:=\int_{0}^{\kappa}\left(\xi^{\frac{\iota}{\varepsilon}}-e\right)\epsilon'\left(\rho+\frac{\kappa+\xi}{2\kappa}\hat{\varphi}\left(\sigma,\rho\right)\right)d\xi,$$
$$\hat{I}_{2}:=\int_{0}^{\kappa}\left(\xi^{\frac{\iota}{\varepsilon}}-e\right)\epsilon'\left(\rho+\frac{\kappa-\xi}{2\kappa}\hat{\varphi}\left(\sigma,\rho\right)\right)d\xi,$$
$$\hat{I}_{3}:=\int_{0}^{\kappa}\left(\xi^{\frac{\iota}{\varepsilon}}-e\right)\epsilon'\left(\sigma+\frac{\kappa+\xi}{2\kappa}\hat{\varphi}\left(\zeta,\sigma\right)\right)d\xi$$

and

$$\hat{I}_{4}:=\int_{0}^{\kappa}\left(\xi^{\frac{\iota}{\varepsilon}}-e\right)\epsilon'\left(\sigma+\frac{\kappa-\xi}{2\kappa}\hat{\varphi}\left(\zeta,\sigma\right)\right)d\xi.$$

Repeatedly, integrating by parts

$$\begin{aligned}
\hat{I}_1 &= \int_0^{\kappa} \left(\xi^{\frac{\iota}{\varepsilon}} - e\right) \epsilon' \left(\rho + \frac{\kappa+\xi}{2\kappa}\hat{\varphi}(\sigma,\rho)\right) d\xi \\
&= \frac{2\kappa}{\hat{\varphi}(\sigma,\rho)} \left(\kappa^{\frac{\iota}{\varepsilon}} - e\right) \epsilon\left(\rho + \hat{\varphi}(\sigma,\rho)\right) + \frac{2\kappa e}{\hat{\varphi}(\sigma,\rho)} \epsilon\left(\rho + \frac{\hat{\varphi}(\sigma,\rho)}{2}\right) \\
&\quad + \frac{2\kappa\iota\Gamma_{\varepsilon}(\iota)}{\hat{\varphi}(\sigma,\rho)} \cdot \frac{1}{\varepsilon\Gamma_{\varepsilon}(\iota)} \int_0^{\kappa} \xi^{\frac{\iota-\varepsilon}{\varepsilon}} \epsilon\left(\rho + \frac{\kappa+\xi}{2\kappa}\hat{\varphi}(\sigma,\rho)\right) d\xi \\
&= \frac{2\kappa}{\hat{\varphi}(\sigma,\rho)} \left(\kappa^{\frac{\iota}{\varepsilon}} - e\right) \epsilon\left(\rho + \hat{\varphi}(\sigma,\rho)\right) + \frac{2\kappa e}{\hat{\varphi}(\sigma,\rho)} \epsilon\left(\rho + \frac{\hat{\varphi}(\sigma,\rho)}{2}\right) \\
&\quad - \left(\frac{2\kappa}{\hat{\varphi}(\sigma,\rho)}\right)^{\frac{\iota+\varepsilon}{\varepsilon}} \Gamma_{\varepsilon}(\iota+\varepsilon)\, {}_{\varepsilon}\mathbb{J}^{\iota}_{(\rho+\hat{\varphi}(\sigma,\rho))^-} \epsilon\left(\rho + \frac{\hat{\varphi}(\sigma,\rho)}{2}\right). \qquad (3.\ 25)
\end{aligned}$$

$$\begin{aligned}
\hat{I}_2 &= \int_0^{\kappa} \left(\xi^{\frac{\iota}{\varepsilon}} - e\right) \epsilon' \left(\rho + \frac{\kappa-\xi}{2\kappa}\hat{\varphi}(\sigma,\rho)\right) d\xi \\
&= -\frac{2\kappa}{\hat{\varphi}(\sigma,\rho)} \left(\kappa^{\frac{\iota}{\varepsilon}} - e\right) \epsilon(\rho) + \frac{2\kappa e}{\hat{\varphi}(\sigma,\rho)} \epsilon\left(\rho + \frac{\hat{\varphi}(\sigma,\rho)}{2}\right) \\
&\quad + \frac{2\kappa\iota\Gamma_{\varepsilon}(\iota)}{\hat{\varphi}(\sigma,\rho)} \cdot \frac{1}{\varepsilon\Gamma_{\varepsilon}(\iota)} \int_0^{\kappa} \xi^{\frac{\iota-\varepsilon}{\varepsilon}} \epsilon\left(\rho + \frac{\kappa-\xi}{2\kappa}\hat{\varphi}(\sigma,\rho)\right) d\xi \\
&= -\frac{2\kappa}{\hat{\varphi}(\sigma,\rho)} \left(\kappa^{\frac{\iota}{\varepsilon}} - e\right) \epsilon(\rho) - \frac{2\kappa e}{\hat{\varphi}(\sigma,\rho)} \epsilon\left(\rho + \frac{\hat{\varphi}(\sigma,\rho)}{2}\right) \\
&\quad + \left(\frac{2\kappa}{\hat{\varphi}(\sigma,\rho)}\right)^{\frac{\iota+\varepsilon}{\varepsilon}} \Gamma_{\varepsilon}(\iota+\varepsilon)\, {}_{\varepsilon}\mathbb{J}^{\iota}_{\rho^+} \epsilon\left(\rho + \frac{\hat{\varphi}(\sigma,\rho)}{2}\right). \qquad (3.\ 26)
\end{aligned}$$

$$\begin{aligned}
\hat{I}_3 &= \int_0^{\kappa} \left(\xi^{\frac{\iota}{\varepsilon}} - e\right) \epsilon' \left(\sigma + \frac{\kappa+\xi}{2\kappa}\hat{\varphi}(\zeta,\sigma)\right) d\xi \\
&= \frac{2\kappa}{\hat{\varphi}(\zeta,\sigma)} \left(\kappa^{\frac{\iota}{\varepsilon}} - e\right) \epsilon\left(\sigma + \hat{\varphi}(\zeta,\sigma)\right) + \frac{2\kappa e}{\hat{\varphi}(\zeta,\sigma)} \epsilon\left(\sigma + \frac{\hat{\varphi}(\zeta,\sigma)}{2}\right) \\
&\quad + \frac{2\kappa\iota\Gamma_{\varepsilon}(\iota)}{\hat{\varphi}(\zeta,\sigma)} \cdot \frac{1}{\varepsilon\Gamma_{\varepsilon}(\iota)} \int_0^{\kappa} \xi^{\frac{\iota-\varepsilon}{\varepsilon}} \epsilon\left(\sigma + \frac{\kappa+\xi}{2\kappa}\hat{\varphi}(\zeta,\sigma)\right) d\xi \\
&= \frac{2\kappa}{\hat{\varphi}(\zeta,\sigma)} \left(\kappa^{\frac{\iota}{\varepsilon}} - e\right) \epsilon\left(\sigma + \hat{\varphi}(\zeta,\sigma)\right) + \frac{2\kappa e}{\hat{\varphi}(\zeta,\sigma)} \epsilon\left(\sigma + \frac{\hat{\varphi}(\zeta,\sigma)}{2}\right) \\
&\quad + \left(\frac{2\kappa}{\hat{\varphi}(\zeta,\sigma)}\right)^{\frac{\iota+\varepsilon}{\varepsilon}} \Gamma_{\varepsilon}(\iota+\varepsilon)\ {}_{\varepsilon}\mathbb{J}^{\iota}_{(\sigma+\hat{\varphi}(\zeta,\sigma))^-} \epsilon\left(\sigma + \frac{\hat{\varphi}(\zeta,\sigma)}{2}\right) \qquad (3.\ 27)
\end{aligned}$$

$$\hat{I}_4 = \int_0^\kappa \left(\xi^{\frac{\iota}{\varepsilon}} - e\right) \epsilon' \left(\sigma + \frac{\kappa - \xi}{2\kappa} \hat{\varphi}(\zeta, \sigma)\right) d\xi$$
$$= -\frac{2\kappa}{\hat{\varphi}(\zeta,\sigma)} \left(\kappa^{\frac{\iota}{\varepsilon}} - e\right) \epsilon(\sigma) - \frac{2\kappa e}{\hat{\varphi}(\zeta,\sigma)} \epsilon\left(\sigma + \frac{\hat{\varphi}(\zeta,\sigma)}{2}\right)$$
$$+ \frac{2\kappa\iota\Gamma_\varepsilon(\iota)}{\hat{\varphi}(\zeta,\sigma)} \cdot \frac{1}{\varepsilon\Gamma_\varepsilon(\iota)} \int_0^\kappa \xi^{\frac{\iota-\varepsilon}{\varepsilon}} \epsilon\left(\sigma + \frac{\kappa-\xi}{2\kappa}\hat{\varphi}(\zeta,\sigma)\right) d\xi$$
$$= -\frac{2\kappa}{\hat{\varphi}(\zeta,\sigma)} \left(\kappa^{\frac{\iota}{\varepsilon}} - e\right) \epsilon(\sigma) - \frac{2\kappa e}{\hat{\varphi}(\zeta,\sigma)} \epsilon\left(\sigma + \frac{\hat{\varphi}(\zeta,\sigma)}{2}\right)$$
$$+ \left(\frac{2\kappa}{\hat{\varphi}(\zeta,\sigma)}\right)^{\frac{\iota+\varepsilon}{\varepsilon}} \Gamma_\varepsilon(\iota+\varepsilon)\ {}_\varepsilon\mathbb{J}^\iota_{\sigma^+} \epsilon\left(\sigma + \frac{\hat{\varphi}(\zeta,\sigma)}{2}\right). \quad (3.\ 28)$$

Subtracting $(3.26)$ and $(3.28)$ from $(3.25)$ and $(3.27)$ respectively

$$\left(\hat{I}_1 - \hat{I}_2\right) \frac{\left(\hat{\varphi}(\sigma,\rho)\right)^{\frac{\iota+\varepsilon}{\varepsilon}}}{2\kappa\hat{\varphi}(\zeta,\rho)} = \frac{\left(\hat{\varphi}(\sigma,\rho)\right)^{\frac{\iota}{\varepsilon}}}{\hat{\varphi}(\zeta,\rho)} \left(\kappa^{\frac{\iota}{\varepsilon}} - e\right) \left[\epsilon\left(\rho + \hat{\varphi}(\sigma,\rho)\right) + \epsilon(\rho)\right]$$
$$+ \frac{2e\left(\hat{\varphi}(\sigma,\rho)\right)^{\frac{\iota}{\varepsilon}}}{\hat{\varphi}(\zeta,\rho)} \epsilon\left(\rho + \frac{\hat{\varphi}(\sigma,\rho)}{2}\right) - \frac{(2\kappa)^{\frac{\iota}{\varepsilon}}}{\hat{\varphi}(\zeta,\rho)} \Gamma_\varepsilon(\iota+\varepsilon)$$
$$\times \left[{}_\varepsilon\mathbb{J}^\iota_{(\rho+\hat{\varphi}(\sigma,\rho))^-} \epsilon\left(\rho + \frac{\hat{\varphi}(\sigma,\rho)}{2}\right) +_\varepsilon \mathbb{J}^\iota_{\rho^+} \epsilon\left(\rho + \frac{\hat{\varphi}(\sigma,\rho)}{2}\right)\right]. \quad (3.\ 29)$$

$$\left(\hat{I}_3 - \hat{I}_4\right) \frac{\left(\hat{\varphi}(\zeta,\sigma)\right)^{\frac{\iota+\varepsilon}{\varepsilon}}}{2\kappa\left(\hat{\varphi}(\zeta,\rho)\right)} = \frac{\left(\hat{\varphi}(\zeta,\sigma)\right)^{\frac{\iota}{\varepsilon}}}{\hat{\varphi}(\zeta,\rho)} \left(\kappa^{\frac{\iota}{\varepsilon}} - e\right) \left[\epsilon\left(\sigma + \hat{\varphi}(\zeta,\sigma)\right) + \epsilon(\sigma)\right]$$
$$+ \frac{2e\left(\hat{\varphi}(\zeta,\sigma)\right)^{\frac{\iota}{\varepsilon}}}{\hat{\varphi}(\zeta,\rho)} \epsilon\left(\sigma + \frac{\hat{\varphi}(\zeta,\sigma)}{2}\right) - \frac{(2\kappa)^{\frac{\iota}{\varepsilon}}}{\hat{\varphi}(\zeta,\rho)} \Gamma_\varepsilon(\iota+\varepsilon)$$
$$\times \left[{}_\varepsilon\mathbb{J}^\iota_{(\sigma+\hat{\varphi}(\zeta,\sigma))^-} \epsilon\left(\sigma + \frac{\hat{\varphi}(\zeta,\sigma)}{2}\right) +_\varepsilon \mathbb{J}^\iota_{\sigma^+} \epsilon\left(\sigma + \frac{\hat{\varphi}(\zeta,\sigma)}{2}\right)\right]. \quad (3.\ 30)$$

Addition of $(3.29)$ and $(3.30)$ yields the desired result $(3.24)$. □

**Remark 3.2.** *It is remarkable to note that*

- *for* $\kappa = 1$, *Lemma 3.1 reduces to* [5, Lemma 2.1].
- *for* $\hat{\varphi}(\zeta,\rho) = \zeta - \rho$, *the following identity holds:*

$$\varphi(\varepsilon,\iota,e,\kappa;\sigma) = \frac{(\sigma-\rho)^{\frac{\iota+\varepsilon}{\varepsilon}}}{2\kappa(\zeta-\rho)} \left[\int_0^\kappa \left(\xi^{\frac{\iota}{\varepsilon}} - e\right) \left\{\epsilon'\left(\frac{\kappa-\xi}{2\kappa}\rho + \frac{\kappa+\xi}{2\kappa}\sigma\right)\right.\right.$$
$$\left.\left. -\epsilon'\left(\frac{\kappa-\xi}{2\kappa}\rho + \frac{\kappa+\xi}{2\kappa}\sigma\right)\right\} d\xi\right] - \frac{(\zeta-\sigma)^{\frac{\iota+\varepsilon}{\varepsilon}}}{2\kappa(\zeta-\rho)} \left[\int_0^\kappa \left(\xi^{\frac{\iota}{\varepsilon}} - e\right)\right.$$
$$\left. \times \left\{\epsilon'\left(\frac{\kappa-\xi}{2\kappa}\sigma + \frac{\kappa+\xi}{2\kappa}\zeta\right) - \epsilon'\left(\frac{\kappa+\xi}{2\kappa}\sigma + \frac{\kappa-\xi}{2\kappa}\zeta\right)\right\} d\xi\right], \quad (3.\ 31)$$

*for* $\sigma \in [\rho,\zeta]$, *provided that* $\varphi(\varepsilon,\iota,e,\kappa;\sigma)$ *is defined by* $(3.6)$.

- *for* $e=0$ *and* $\iota,\varepsilon,\kappa=1$*, the identity* $(3.24)$ *reduces to*

$$\varphi_{\hat{\varphi}}(1,1,0,1;\sigma)=\frac{(\hat{\varphi}(\sigma,\rho))^2}{2\hat{\varphi}(\zeta,\rho)}\left[\int_0^1\xi\left\{\epsilon'\left(\rho+\frac{1+\xi}{2}\hat{\varphi}(\sigma,\rho)\right)\right.\right.$$
$$\left.\left.-\epsilon'\left(\rho+\frac{1-\xi}{2}\hat{\varphi}(\sigma,\rho)\right)\right\}d\xi\right]-\frac{(\hat{\varphi}(\zeta,\sigma))^2}{2\hat{\varphi}(\zeta,\rho)}$$
$$\times\left[\int_0^1\xi\left\{\epsilon'\left(\sigma+\frac{1+\xi}{2}\hat{\varphi}(\zeta,\sigma)\right)-\epsilon'\left(\sigma+\frac{1-\xi}{2}\hat{\varphi}(\zeta,\sigma)\right)\right\}d\xi\right],\quad (3.32)$$

*for* $\sigma\in[\rho,\rho+\hat{\varphi}(\zeta,\rho)]$*, provided that* $\varphi_{\hat{\varphi}}(1,1,0,1;\sigma)$ *is defined by* $(3.5)$.

- *for* $e,\iota,\varepsilon,\kappa=1$*, the identity* $(3.24)$ *reduces to*

$$\varphi_{\hat{\varphi}}(1,1,1,1;\sigma)=\frac{(\hat{\varphi}(\sigma,\rho))^2}{2\hat{\varphi}(\zeta,\rho)}\left[\int_0^1\xi\left\{\epsilon'\left(\rho+\frac{1+\xi}{2}\hat{\varphi}(\sigma,\rho)\right)\right.\right.$$
$$\left.\left.-\epsilon'\left(\rho+\frac{1-\xi}{2}\hat{\varphi}(\sigma,\rho)\right)\right\}d\xi\right]-\frac{(\hat{\varphi}(\zeta,\sigma))^2}{2\hat{\varphi}(\zeta,\rho)}$$
$$\times\left[\int_0^1\xi\left\{\epsilon'\left(\sigma+\frac{1+\xi}{2}\hat{\varphi}(\zeta,\sigma)\right)-\epsilon'\left(\sigma+\frac{1-\xi}{2}\hat{\varphi}(\zeta,\sigma)\right)\right\}d\xi\right]\quad (3.33)$$

*for* $\sigma\in[\rho,\rho+\hat{\varphi}(\zeta,\rho)]$*, provide that* $\varphi_{\hat{\varphi}}(1,1,1,1;\sigma)$ *is defined by* $(3.4)$.

**Theorem 3.3.** *Let the function* $\epsilon:\lambda\to\mathbb{R}$ *satisfies the above defined conditions and* $|\epsilon'|^q$ *is* $h$*-preinvex function on* $[\rho,\rho+\hat{\varphi}(\zeta,\rho)]$ *for* $q\geq 1$*, then*

$$|\varphi_{\hat{\varphi}}(\varepsilon,\iota,e,\kappa;\sigma)|\leq\frac{[Q_5(\varepsilon,\iota,\kappa;e)]^{\frac{1-q}{q}}}{2\kappa\hat{\varphi}(\zeta,\rho)}$$
$$\times\left\{(\hat{\varphi}(\sigma,\rho))^{\frac{\iota+\varepsilon}{\varepsilon}}\left[\sqrt[q]{Q_1(\varepsilon,\iota,\kappa,e;h)}+\sqrt[q]{Q_2(\varepsilon,\iota,\kappa,e;h)}\right]\right.$$
$$\left.+(\hat{\varphi}(\zeta,\sigma))^{\frac{\iota+\varepsilon}{\varepsilon}}\left[\sqrt[q]{Q_3(\varepsilon,\iota,\kappa,e;h)}+\sqrt[q]{Q_4(\varepsilon,\iota,\kappa,e;h)}\right]\right\},\quad (3.34)$$

*provided that* $Q_1(\varepsilon,\iota,\kappa,e;h)$ *to* $Q_5(\varepsilon,\iota,\kappa,e;h)$ *are defined by* $(3.7)$ *to* $(3.11)$ *respectively.*

*Proof.* By properties of modulus and Lemma 3.1, we have

$$|\varphi_{\hat{\varphi}}(\varepsilon,\iota,e,\kappa;\sigma)|\leq\frac{[\hat{\varphi}(\sigma,\rho)]^{\frac{\iota+\varepsilon}{\varepsilon}}}{2\kappa\hat{\varphi}(\zeta,\rho)}\left[\int_0^\kappa\left|\xi^{\frac{\iota}{\varepsilon}}-e\right|\left\{\left|\epsilon'\left(\rho+\left(\frac{\kappa+\xi}{2\kappa}\right)\hat{\varphi}(\sigma,\rho)\right)\right|\right.\right.$$
$$\left.\left.+\left|\epsilon'\left(\rho+\left(\frac{\kappa-\xi}{2\kappa}\right)\hat{\varphi}(\sigma,\rho)\right)\right|\right\}d\xi\right]+\frac{[\hat{\varphi}(\zeta,\sigma)]^{\frac{\iota+\varepsilon}{\varepsilon}}}{2\kappa\hat{\varphi}(\zeta,\rho)}\left[\int_0^\kappa\left|\xi^{\frac{\iota}{\varepsilon}}-e\right|\right.$$
$$\left.\times\left\{\left|\epsilon'\left(\sigma+\left(\frac{\kappa+\xi}{2\kappa}\right)\hat{\varphi}(\zeta,\sigma)\right)\right|+\left|\epsilon'\left(\sigma+\left(\frac{\kappa-\xi}{2\kappa}\right)\hat{\varphi}(\zeta,\sigma)\right)\right|\right\}d\xi\right]\quad (3.35)$$

For simplicity, we write

$$\begin{aligned} X_1 &:= \int_0^{\kappa} \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| \left|\epsilon'\left(\rho + \left(\frac{\kappa+\xi}{2\kappa}\right)\hat{\varphi}(\sigma,\rho)\right)\right| d\xi, \\ X_2 &:= \int_0^{\kappa} \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| \left|\epsilon'\left(\rho + \left(\frac{\kappa-\xi}{2\kappa}\right)\hat{\varphi}(\sigma,\rho)\right)\right| d\xi, \\ X_3 &:= \int_0^{\kappa} \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| \left|\epsilon'\left(\sigma + \left(\frac{\kappa+\xi}{2\kappa}\right)\hat{\varphi}(\zeta,\sigma)\right)\right| d\xi \\ X_4 &=: \int_0^{\kappa} \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| \left|\epsilon'\left(\sigma + \left(\frac{\kappa-\xi}{2\kappa}\right)\hat{\varphi}(\zeta,\sigma)\right)\right| d\xi. \end{aligned}$$

Repeatedly by power mean inequality and $h$-preinvexity of $|\epsilon'|^q$, we have

$$\begin{aligned} X_1 &\le \left(\int_0^{\kappa} \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| d\xi\right)^{\frac{q-1}{q}} \\ &\quad \times \sqrt[q]{\int_0^{\kappa} \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| \left[h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\rho)|^q + h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q\right] d\xi} \\ &= \left[\frac{\iota e\left(2e^{\frac{\varepsilon}{\iota}} - \kappa\right) + \varepsilon\left(\kappa^{\frac{\varepsilon+\iota}{\varepsilon}} - \kappa e\right)}{\varepsilon+\iota}\right]^{\frac{q-1}{q}} \sqrt[q]{Q_1(\varepsilon,\iota,\kappa,e;h)}. \end{aligned} \tag{3. 36}$$

$$\begin{aligned} X_2 &\le \left(\int_0^{\kappa} \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| d\xi\right)^{\frac{q-1}{q}} \\ &\quad \times \sqrt[q]{\int_0^{\kappa} \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| \left[h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\rho)|^q + h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q\right] d\xi} \\ &= \left[\frac{\iota e\left(2e^{\frac{\varepsilon}{\iota}} - \kappa\right) + \varepsilon\left(\kappa^{\frac{\varepsilon+\iota}{\varepsilon}} - \kappa e\right)}{\varepsilon+\iota}\right]^{\frac{q-1}{q}} \sqrt[q]{Q_2(\varepsilon,\iota,\kappa,e;h)}. \end{aligned} \tag{3. 37}$$

$$\begin{aligned} X_3 &\le \left(\int_0^{\kappa} \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| d\xi\right)^{\frac{q-1}{q}} \\ &\quad \times \sqrt[q]{\int_0^{\kappa} \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| \left[h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q + h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\zeta)|^q\right] d\xi} \\ &= \left[\frac{\iota e\left(2e^{\frac{\varepsilon}{\iota}} - \kappa\right) + \varepsilon\left(\kappa^{\frac{\varepsilon+\iota}{\varepsilon}} - \kappa e\right)}{\varepsilon+\iota}\right]^{\frac{q-1}{q}} \sqrt[q]{Q_3(\varepsilon,\iota,\kappa,e;h)}. \end{aligned} \tag{3. 38}$$

$$X_4 \leq \left(\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| d\xi\right)^{\frac{q-1}{q}}$$
$$\times \sqrt[q]{\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}} - e\right| \left[h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q + h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\zeta)|^q\right] d\xi}$$
$$= \left[\frac{\iota e\left(2e^{\frac{\varepsilon}{\iota}} - \kappa\right) + \varepsilon\left(\kappa^{\frac{\varepsilon+\iota}{\varepsilon}} - \kappa e\right)}{\varepsilon+\iota}\right]^{\frac{q-1}{q}} \sqrt[q]{Q_4\left(\varepsilon,\iota,\kappa,e;h\right)}. \quad (3.\ 39)$$

A combination of (3.35)-(3.39) yields the desired result (3.34). □

**Corollary 3.4.** *Let the conditions of Theorem 3.3 be satisfied for* $h \equiv 1$, *then the following result for* $P-$*preinvex function on* $[\rho, \rho + \hat{\varphi}(\zeta,\rho)]$ *with* $\hat{\varphi}(\zeta,\rho) > 0$ *holds*

$$|\varphi_{\hat{\varphi}}(\varepsilon,\iota,e,\kappa;\sigma)| \leq \frac{\kappa^{\frac{1-q}{q}} Q_5(\varepsilon,\iota,\kappa;e)}{\hat{\varphi}(\zeta,\rho)} \left[(\hat{\varphi}(\sigma,\rho))^{\frac{\iota+\varepsilon}{\varepsilon}} \sqrt[q]{|\epsilon'(\rho)|^q + |\epsilon'(\sigma)|^q}\right.$$
$$\left. + (\hat{\varphi}(\zeta,\sigma))^{\frac{\iota+\varepsilon}{\varepsilon}} \sqrt[q]{|\epsilon'(\sigma)|^q + |\epsilon'(\zeta)|^q}\right]. \quad (3.\ 40)$$

**Corollary 3.5.** *Let the conditions of Theorem 3.3 be satisfied for* $h = \hat{I}$, *identity map, then the following result for preinvex function on* $[\rho, \rho + \hat{\varphi}(\zeta,\rho)]$ *with* $\hat{\varphi}(\zeta,\rho) > 0$ *holds*

$$|\varphi_{\hat{\varphi}}(\varepsilon,\iota,e,\kappa;\sigma)| \leq \frac{(Q_5(\varepsilon,\iota,\kappa;e))^{\frac{q-1}{q}}}{2\kappa\hat{\varphi}(\zeta,\rho)}$$
$$\times \left[(\hat{\varphi}(\sigma,\rho))^{\frac{\iota+\varepsilon}{\varepsilon}} \left\{\sqrt[q]{\alpha_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\rho)|^q + \alpha_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q}\right.\right.$$
$$\left. + \sqrt[q]{\alpha_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\rho)|^q + \alpha_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q}\right\}$$
$$+ (\hat{\varphi}(\zeta,\sigma))^{\frac{\iota+\varepsilon}{\varepsilon}} \left\{\sqrt[q]{\alpha_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q + \alpha_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\zeta)|^q}\right.$$
$$\left.\left. + \sqrt[q]{\alpha_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q + \alpha_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\zeta)|^q}\right\}\right], \quad (3.\ 41)$$

*provided that* $\alpha_1(\varepsilon,\iota,\kappa;e)$, $\alpha_2(\varepsilon,\iota,\kappa;e)$ *and* $Q_5(\varepsilon,\iota,\kappa;e)$ *are defined by* (3.12), (3.13) *and* (3.11) *respectively.*

**Corollary 3.6.** *Let the conditions of Theorem 3.3 be satisfied for* $h(\xi) = \xi^s$, *then the following result for s-preinvex function of Breckner type on* $[\rho, \rho + \hat{\varphi}(\zeta,\rho)]$ *with* $\hat{\varphi}(\zeta,\rho) >$

0 *holds*

$$
\begin{aligned}
|\varphi_{\hat{\varphi}}(\varepsilon,\iota,e,\kappa;\sigma)| \leq & \frac{(Q_5(\varepsilon,\iota,\kappa;e))^{\frac{q-1}{q}}}{2\kappa\hat{\varphi}(\zeta,\rho)} \\
& \times \Big[(\hat{\varphi}(\sigma,\rho))^{\frac{\iota+\varepsilon}{\varepsilon}} \Big\{ \sqrt[q]{\iota_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\rho)|^q + \iota_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q} \\
& + \sqrt[q]{\iota_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\rho)|^q + \iota_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q}\Big\} \\
& + (\hat{\varphi}(\zeta,\sigma))^{\frac{\iota+\varepsilon}{\varepsilon}} \Big\{ \sqrt[q]{\iota_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q + \iota_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\zeta)|^q} \\
& + \sqrt[q]{\iota_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q + \iota_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\zeta)|^q}\Big\}\Big], \quad (3.\ 42)
\end{aligned}
$$

*provided that* $\iota_1(\varepsilon,\iota,\kappa;e)$, $\iota_2(\varepsilon,\iota,\kappa;e)$ *and* $Q_5(\varepsilon,\iota,\kappa;e)$ *are defined by* (3.14), (3.15) *and* (3.11) *respectively.*

**Corollary 3.7.** *Let the conditions of Theorem 3.3 be satisfied for* $h(\xi) = \xi^{-s}$, *then the following result for* $s$*-preinvex function of Godunova-Levin type on* $[\rho, \rho + \hat{\varphi}(\zeta,\rho)]$ *with* $\hat{\varphi}(\zeta,\rho) > 0$ *holds*

$$
\begin{aligned}
|\varphi_{\hat{\varphi}}(\varepsilon,\iota,e,\kappa;\sigma)| \leq & \frac{(Q_5(\varepsilon,\iota,\kappa;e))^{\frac{q-1}{q}}}{2\kappa\hat{\varphi}(\zeta,\rho)} \\
& \times \Big[(\hat{\varphi}(\sigma,\rho))^{\frac{\iota+\varepsilon}{\varepsilon}} \Big\{ \sqrt[q]{\gamma_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\rho)|^q + \gamma_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q} \\
& + \sqrt[q]{\gamma_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\rho)|^q + \gamma_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q}\Big\} \\
& + (\hat{\varphi}(\zeta,\sigma))^{\frac{\iota+\varepsilon}{\varepsilon}} \Big\{ \sqrt[q]{\gamma_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q + \gamma_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\zeta)|^q} \\
& + \sqrt[q]{\gamma_2(\varepsilon,\iota,\kappa;e)|\epsilon'(\sigma)|^q + \gamma_1(\varepsilon,\iota,\kappa;e)|\epsilon'(\zeta)|^q}\Big\}\Big], \quad (3.\ 43)
\end{aligned}
$$

*provided that* $\gamma_1(\varepsilon,\iota,\kappa;e)$, $\gamma_2(\varepsilon,\iota,\kappa;e)$ *and* $Q_5(\varepsilon,\iota,\kappa;e)$ *are defined by* (3.16), (3.17) *and* (3.11) *respectively.*

**Corollary 3.8.** *Let the conditions of Theorem 3.3 be satisfied for* $h(\xi) = \xi(1-\xi)$, *then the following result for* $\xi gs$*-preinvex function on* $[\rho, \rho + \hat{\varphi}(\zeta,\rho)]$ *with* $\hat{\varphi}(\zeta,\rho) > 0$ *holds*

$$
\begin{aligned}
|\varphi_{\hat{\varphi}}(\varepsilon,\iota,e,\kappa;\sigma)| \leq & \frac{(Q_5(\varepsilon,\iota,\kappa;e))^{\frac{q-1}{q}} \sqrt[q]{\delta(\varepsilon,\iota,\kappa;e)}}{\kappa\hat{\varphi}(\zeta,\rho)} \Big\{(\hat{\varphi}(\sigma,\rho))^{\frac{\iota+\varepsilon}{\varepsilon}} \\
& \times \sqrt[q]{|\epsilon'(\rho)|^q + |\epsilon'(\sigma)|^q} + (\hat{\varphi}(\zeta,\sigma))^{\frac{\iota+\varepsilon}{\varepsilon}} \sqrt[q]{|\epsilon'(\sigma)|^q + |\epsilon'(\zeta)|^q}\Big\}, \quad (3.\ 44)
\end{aligned}
$$

*provided that* $\delta(\varepsilon,\iota,\kappa;e)$ *and* $Q_5(\varepsilon,\iota,\kappa;e)$ *are defined by* (3.18) *and* (3.11) *respectively.*

**Theorem 3.9.** *Let the function* $\epsilon : \lambda \to \mathbb{R}$ *satisfies the above defined conditions and* $|\epsilon'|^q$ *is* $h$*-preinvex function on* $[\rho, \rho + \hat{\varphi}(\zeta,\rho)]$ *for* $q > 1$ *and* $p = \frac{q}{q-1}$, *then*

$$
\begin{aligned}
|\varphi_{\hat{\varphi}}(\varepsilon,\iota,e,\kappa;\sigma)| \leq & \frac{\sqrt[p]{\mu(\varepsilon,\kappa,e,\iota,p)}}{2\kappa\hat{\varphi}(\zeta,\rho)} \Big\{(\hat{\varphi}(\sigma,\rho))^{\frac{\iota+\varepsilon}{\varepsilon}} [\sqrt[q]{Y_1(\kappa,e,q;h)} \\
& + \sqrt[q]{Y_2(\kappa,e,q;h)}] + (\hat{\varphi}(\zeta,\sigma))^{\frac{\iota+\varepsilon}{\varepsilon}} [\sqrt[q]{Y_3(\kappa,e,q;h)} + \sqrt[q]{Y_4(\kappa,e,q;h)}]\Big\}, \quad (3.\ 45)
\end{aligned}
$$

*provided that* $Y_1(\kappa,e,q;h)$, $Y_2(\kappa,e,q;h)$, $Y_3(\kappa,e,q;h)$, $Y_4(\kappa,e,q;h)$ *and* $\mu(\varepsilon,\kappa,e,\iota,p)$ *are defined by* (3.19), (3.20), (3.21), (3.22) *and* (3.23) *respectively.*

*Proof.* By properties of modulus and Lemma 3.1, we have

$$\begin{aligned}\left|\varphi_{\hat{\varphi}}(\varepsilon,\iota,e,\kappa;\sigma)\right| &\leq \frac{(\hat{\varphi}(\sigma,\rho))^{\frac{\iota+\varepsilon}{\varepsilon}}}{2\kappa\hat{\varphi}(\zeta,\rho)}\left[\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}}-e\right|\left\{\left|\epsilon'\left(\rho+\frac{\kappa+\xi}{2\kappa}\hat{\varphi}(\sigma,\rho)\right)\right|\right.\right.\\ &\left.\left.+\left|\epsilon'\left(\rho+\frac{\kappa-\xi}{2\kappa}\hat{\varphi}(\sigma,\rho)\right)\right|\right\}d\xi\right]+\frac{(\hat{\varphi}(\zeta,\sigma))^{\frac{\iota+\varepsilon}{\varepsilon}}}{2\kappa\hat{\varphi}(\zeta,\rho)}\left[\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}}-e\right|\right.\\ &\left.\times\left\{\left|\epsilon'\left(\sigma+\frac{\kappa+\xi}{2\kappa}\hat{\varphi}(\zeta,\sigma)\right)\right|+\left|\epsilon'\left(\sigma+\frac{\kappa-\xi}{2\kappa}\hat{\varphi}(\zeta,\sigma)\right)\right|\right\}d\xi\right]. \quad (3.\,46)\end{aligned}$$

Repeatedly by Hölder inequality and $h$-preinvexity of $|\epsilon'|^q$, we have

$$\begin{aligned}&\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}}-e\right|\left|\epsilon'\left(\rho+\frac{\kappa+\xi}{2\kappa}\hat{\varphi}(\sigma,\rho)\right)\right|d\xi\\ &\leq \sqrt[p]{\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}}-e\right|^p d\xi}\sqrt[q]{\int_0^\kappa\left[h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\rho)|^q+h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q\right]d\xi}\\ &=\sqrt[p]{\mu(\varepsilon,\kappa,e,\iota,p)}\sqrt[q]{Y_1(\kappa,e,q;h)} \quad (3.\,47)\end{aligned}$$

$$\begin{aligned}&\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}}-e\right|\left|\epsilon'\left(\rho+\frac{\kappa-\xi}{2\kappa}\hat{\varphi}(\sigma,\rho)\right)\right|d\xi\\ &\leq \sqrt[p]{\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}}-e\right|^p d\xi}\sqrt[q]{\int_0^\kappa\left[h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\rho)|^q+h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q\right]d\xi}\\ &=\sqrt[p]{\mu(\varepsilon,\kappa,e,\iota,p)}\sqrt[q]{Y_2(\kappa,e,q;h)} \quad (3.\,48)\end{aligned}$$

$$\begin{aligned}&\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}}-e\right|\left|\epsilon'\left(\rho+\frac{\kappa+\xi}{2\kappa}\hat{\varphi}(\sigma,\rho)\right)\right|d\xi\\ &\leq \sqrt[p]{\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}}-e\right|^p d\xi}\sqrt[q]{\int_0^\kappa\left[h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q+h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\zeta)|^q\right]d\xi}\\ &=\sqrt[p]{\mu(\varepsilon,\kappa,e,\iota,p)}\sqrt[q]{Y_3(\kappa,e,q;h)} \quad (3.\,49)\end{aligned}$$

$$\begin{aligned}&\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}}-e\right|\left|\epsilon'\left(\rho+\frac{\kappa-\xi}{2\kappa}\hat{\varphi}(\sigma,\rho)\right)\right|d\xi\\ &\leq \sqrt[p]{\int_0^\kappa \left|\xi^{\frac{\iota}{\varepsilon}}-e\right|^p d\xi}\sqrt[q]{\int_0^\kappa\left[h\left(\frac{\kappa+\xi}{2\kappa}\right)|\epsilon'(\sigma)|^q+h\left(\frac{\kappa-\xi}{2\kappa}\right)|\epsilon'(\zeta)|^q\right]d\xi}\\ &=\sqrt[p]{\mu(\varepsilon,\kappa,e,\iota,p)}\sqrt[q]{Y_4(\kappa,e,q;h)} \quad (3.\,50)\end{aligned}$$

A combination of (3.46)-(3.50) yields the desired result (3.45). □

**Corollary 3.10.** *Let the conditions of Theorem 3.9 be satisfied for $h \equiv 1$, then the following result for $P-$preinvex function on $[\rho, \rho + \hat{\varphi}(\zeta, \rho)]$ with $\hat{\varphi}(\zeta, \rho) > 0$ holds*

$$|\varphi_{\hat{\varphi}}(\varepsilon, \iota, e, \kappa; \sigma)| \leq \frac{\kappa^{\frac{1-q}{q}} \sqrt[p]{\mu(\varepsilon, \kappa, e, \iota, p)}}{\hat{\varphi}(\zeta, \rho)} \times \Big\{ (\hat{\varphi}(\sigma, \rho))^{\frac{\iota+\varepsilon}{\varepsilon}} \sqrt[q]{|\epsilon'(\rho)|^q + |\epsilon'(\sigma)|^q} + (\hat{\varphi}(\zeta, \sigma))^{\frac{\iota+\varepsilon}{\varepsilon}} \sqrt[q]{|\epsilon'(\zeta)|^q + |\epsilon'(\sigma)|^q} \Big\}, \quad (3.51)$$

*provided that $\mu(\varepsilon, \kappa, e, \iota, p)$ is defined by* (3.23).

**Corollary 3.11.** *Let the conditions of Theorem 3.9 be satisfied for $h = \hat{I}$, identity map, then the following result for preinvex function on $[\rho, \rho + \hat{\varphi}(\zeta, \rho)]$ with $\hat{\varphi}(\zeta, \rho) > 0$ holds:*

$$|\varphi_{\hat{\varphi}}(\varepsilon, \iota, e, \kappa; \sigma)| \leq \frac{\kappa^{\frac{1-q}{q}} \sqrt[p]{\mu(\varepsilon, \kappa, e, \iota, p)}}{2^{\frac{q+2}{q}} \hat{\varphi}(\zeta, \rho)} \Big\{ [\hat{\varphi}(\sigma, \rho)]^{\frac{\iota+\varepsilon}{\varepsilon}} \Big[ \sqrt[q]{|\epsilon'(\rho)|^q + 3|\epsilon'(\sigma)|^q} + \sqrt[q]{3|\epsilon'(\rho)|^q + |\epsilon'(\sigma)|^q} \Big] + [\hat{\varphi}(\zeta, \sigma)]^{\frac{\iota+\varepsilon}{\varepsilon}} \Big[ \sqrt[q]{|\epsilon'(\sigma)|^q + 3|\epsilon'(\zeta)|^q} + \sqrt[q]{3|\epsilon'(\sigma)|^q + |\epsilon'(\zeta)|^q} \Big] \Big\}, \quad (3.52)$$

*provided that $\mu(\varepsilon, \kappa, e, \iota, p)$ is defined by* (3.23).

**Corollary 3.12.** *Let the conditions of Theorem 3.9 be satisfied for $h(\xi) = \xi^s$, then the following result for s-preinvex function of Breckner type on $[\rho, \rho + \hat{\varphi}(\zeta, \rho)]$ with $\hat{\varphi}(\zeta, \rho) > 0$ holds:*

$$|\varphi_{\hat{\varphi}}(\varepsilon, \iota, e, \kappa; \sigma)| \leq \frac{\kappa^{\frac{1-q}{q}} \sqrt[p]{\mu(\varepsilon, \kappa, e, \iota, p)}}{2\hat{\varphi}(\zeta, \rho) \sqrt[q]{s+1}} \times \{ (\hat{\varphi}(\sigma, \rho))^{\frac{\iota+\varepsilon}{\varepsilon}} \Big[ \sqrt[q]{2^{-s}|\epsilon'(\rho)|^q + (2 - 2^{-s})|\epsilon'(\sigma)|^q} + \sqrt[q]{(2 - 2^{-s})|\epsilon'(\rho)|^q + 2^{-s}|\epsilon'(\sigma)|^q} \Big] + (\hat{\varphi}(\zeta, \sigma))^{\frac{\iota+\varepsilon}{\varepsilon}} \times \Big[ \sqrt[q]{2^{-s}|\epsilon'(\sigma)|^q + (2 - 2^{-s})|\epsilon'(\zeta)|^q} + \sqrt[q]{(2 - 2^{-s})|\epsilon'(\sigma)|^q + 2^{-s}|\epsilon'(\zeta)|^q} \Big] \Big\}, \quad (3.53)$$

*provided that $\mu(\varepsilon, \kappa, e, \iota, p)$ is defined by* (3.23).

**Corollary 3.13.** *Let the conditions of Theorem 3.9 be satisfied for $h(\xi) = \xi^{-s}$, then the following result for s-preinvex function of Godunova-Levin type on $[\rho, \rho + \hat{\varphi}(\zeta, \rho)]$ with $\hat{\varphi}(\zeta, \rho) > 0$ holds:*

$$|\varphi_{\hat{\varphi}}(\varepsilon, \iota, e, \kappa; \sigma)| \leq \frac{\kappa^{\frac{1-q}{q}} \sqrt[p]{\mu(\varepsilon, \kappa, e, \iota, p)}}{2\hat{\varphi}(\zeta, \rho) \sqrt[q]{1-s}} \times \{ (\hat{\varphi}(\sigma, \rho))^{\frac{\iota+\varepsilon}{\varepsilon}} [ \sqrt[q]{2^s|\epsilon'(\rho)|^q + (2^{-s} - 2)|\epsilon'(\sigma)|^q} + \sqrt[q]{(2^{-s} - 2)|\epsilon'(\rho)|^q + 2^s|\epsilon'(\sigma)|^q} ] + (\hat{\varphi}(\zeta, \sigma))^{\frac{\iota+\varepsilon}{\varepsilon}} [ \sqrt[q]{2^s|\epsilon'(\sigma)|^q + (2^{-s} - 2)|\epsilon'(\zeta)|^q} + \sqrt[q]{(2^{-s} - 2)|\epsilon'(\sigma)|^q + 2^s|\epsilon'(\zeta)|^q} ] \}, \quad (3.54)$$

*provided that $\mu(\varepsilon, \kappa, e, \iota, p)$ is defined by* (3.23).

**Corollary 3.14.** *Let the conditions of Theorem 3.9 be satisfied for $h(\xi) = \xi(1-\xi)$, then the following result for $\xi gs$-preinvex function on $[\rho, \rho + \hat{\varphi}(\zeta, \rho)]$ with $\hat{\varphi}(\zeta, \rho) > 0$ holds:*

$$\left|\varphi_{\hat{\varphi}}(\varepsilon, \iota, e, \kappa; \sigma)\right| \leq \frac{\kappa^{\frac{1-q}{q}} \sqrt[p]{\mu(\varepsilon, \kappa, e, \iota, p)}}{\sqrt[q]{6}\hat{\varphi}(\zeta, \rho)} \times$$
$$\left\{ (\hat{\varphi}(\sigma, \rho))^{\frac{\iota+\varepsilon}{\varepsilon}} \sqrt[q]{|\epsilon'(\sigma)|^q + |\epsilon'(\rho)|^q} + (\hat{\varphi}(\zeta, \sigma))^{\frac{\iota+\varepsilon}{\varepsilon}} \sqrt[q]{|\epsilon'(\sigma)|^q + |\epsilon'(\zeta)|^q} \right\}, \quad (3.\ 55)$$

*provided that $\mu(\varepsilon, \kappa, e, \iota, p)$ is defined by* (3.23).

**Remark 3.15.** *It is remarkable to all the interested readers, for $\kappa \to 1$ in all the above mentioned results, some very interesting inequalities can be obtained. The details are omitted.*

## 4. Applications

4.1. **Applications to special means.** This section is devoted to some applications from Section 3. Here, We shall consider the following special means:

- The arithmetic mean: $A(\rho, \zeta) := \frac{\rho+\zeta}{2}$, $\rho, \zeta \geq 0$.
- The harmonic mean: $H(\rho, \zeta) := \frac{2\rho\zeta}{\rho+\zeta}$, $\rho, \zeta > 0$.
- The logarithmic mean for $\rho, \zeta > 0$:

$$L(\rho, \zeta) := \begin{cases} \rho, & a = b; \\ \frac{\zeta - \rho}{\ln \zeta - \ln \rho}, & a \neq b. \end{cases}$$

**Proposition 4.2.** *Let $\epsilon : \hat{I} \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on $\hat{I}^\circ$, interior of $\hat{I}$, $\rho, \zeta \in \hat{I}^\circ$ with $\rho < \zeta$ such that $|\epsilon'|^q$, $q \geq 1$ is convex, then*

$$\left|H^{-1}(\rho, \zeta) + A^{-1}(\rho, \zeta) - 2L^{-1}(\rho, \zeta)\right| \leq \frac{(\zeta - \rho)H^{-\frac{1}{q}}(\rho^{2q}, \zeta^{2q})}{2^{\frac{2q-1}{q}}}.$$

*Proof.* From Corollary 3.4, the assertion follows with $\epsilon(\sigma) = \sigma^{-1}$, $\sigma \in [\rho, \zeta]$ for $\varepsilon, \kappa, \rho, \iota = 1$; $e = \frac{1}{2}$; $\sigma = \zeta$ and $\hat{\varphi}(\zeta, \rho) = \zeta - \rho$. □

4.3. $\epsilon-$**divergence measures.** Here, we provide several applications that demonstrate how to use results from Section 3 in regards to $\epsilon$-divergence measure and probability density function. Given the set $\phi$ and the $\sigma$-finite measure $\mu$, define the set of all probability densities on $\mu$ by $\Omega := \{\chi | \chi : \phi \to \mathbb{R}, \chi(\varpi) > 0, \int_\phi \chi(\varpi) d\mu(\varpi) = 1\}$. Let $\epsilon : (0, \infty) \to \mathbb{R}$ be given mapping and consider $D_\epsilon(\chi, \psi)$ defined by:

$$D_\epsilon(\chi, \psi) := \int_\phi \chi(\varpi) \epsilon \left[\frac{\psi(\varpi)}{\chi(\varpi)}\right] d\mu(\varpi), \quad \chi, \psi \in \Omega. \quad (4.\ 56)$$

When $\epsilon$ is convex, ((4.56)) is referred to as the Csisźar-divergence. Consider the Hermite-Hadamard (HH) divergence as follows:

$$D^\epsilon_{HH}(\chi, \psi) := \int_\phi \chi(\varpi) \frac{\int_1^{\frac{\psi(\varpi)}{\chi(\varpi)}} \epsilon(\xi) d\xi}{\frac{\psi(\varpi)}{\chi(\varpi)} - 1} d\mu(\varpi), \quad \chi, \psi \in \Omega, \quad (4.\ 57)$$

where $\epsilon$ is convex on $(0, \infty)$ with $\epsilon(1) = 0$. Note that $D^\epsilon_{HH}(\chi, \psi) \geq 0$ with the equality holds if and only if $\chi = \psi$.

**Proposition 4.4.** *Let $\epsilon : \hat{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\hat{I}^\circ$, $\rho, \zeta \in \hat{I}^\circ$ such that $|\epsilon'^q$ is convex for $q \geq 1$ and $\epsilon(1) = 0$, then*

$$\left| \frac{D_\epsilon(\chi, \psi)}{2} + \int_\phi \chi(\varpi)\epsilon\left(\frac{\psi(\varpi)+\chi(\varpi)}{2\chi(\varpi)}\right) d\mu(\varpi) - 2D^\epsilon_{HH}(\chi,\psi)\right| \leq \int_\phi \frac{|\psi(\varpi)-\chi(\varpi)|}{4} \sqrt[q]{|\epsilon'(1)|^q + \left|\epsilon'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|^q} d\mu(\varpi). \quad (4.\ 58)$$

*Proof.* Let $\Phi_1 := \{\varpi \in \phi : \psi(\varpi) > \chi(\varpi)\}$; $\Phi_2 := \{\varpi \in \phi : \psi(\varpi) < \chi(\varpi)\}$ and $\Phi_3 := \{\varpi \in \phi : \psi(\varpi) = \chi(\varpi)\}$. Obviously, if $\varpi \in \Phi_3$, then equality holds in $(4.58)$. Now, if $\varpi \in \Phi_1$, then for $\varepsilon, \kappa, \rho, \iota = 1$; $e = \frac{1}{2}$; $\sigma = \zeta = \frac{\psi(\varpi)}{\chi(\varpi)}$ and $\hat{\varphi}(\zeta, \rho) = \zeta - \rho$ in Corollary 3.4, multiplication with $\chi(\varpi)$ to the acquired inequality and integrating over $\Phi_1$, we have

$$\left| \int_{\Phi_1} \chi(\varpi)\epsilon\left(\frac{\psi(\varpi)+\chi(\varpi)}{2\chi(\varpi)}\right) d\mu(\varpi) + \frac{1}{2}\int_{\Phi_1} \chi(\varpi)\epsilon\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right) d\mu(\varpi) - 2\int_{\Phi_1} \chi(\varpi)\frac{\int_1^{\frac{\psi(\varpi)}{\chi(\varpi)}} \epsilon(\xi)d\xi}{\frac{\psi(\varpi)}{\chi(\varpi)} - 1} d\mu(\varpi)\right| \leq \int_{\Phi_1} \frac{\psi(\varpi)-\chi(\varpi))}{4} \times \sqrt[q]{|\epsilon'(1)|^q + \left|\epsilon'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|^q} d\mu(\varpi) \quad (4.\ 59)$$

Similarly, if $\varpi \in \Phi_2$, then for $\varepsilon, \kappa, \iota = 1$; $e = \frac{1}{2}$; $\sigma = \zeta = 1$, $\rho = \frac{\psi(\varpi)}{\chi(\varpi)}$ and $\hat{\varphi}(\zeta, \rho) = \zeta - \rho$ in Corollary 3.4, taking product with $\chi(\varpi)$ to the acquired inequality and integrating over $\Phi_2$, we have

$$\left| \int_{\Phi_2} \chi(\varpi)\epsilon\left(\frac{\psi(\varpi)+\chi(\varpi)}{2\chi(\varpi)}\right) d\mu(\varpi) + \frac{1}{2}\int_{\Phi_2} \chi(\varpi)\epsilon\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right) d\mu(\varpi) - 2\int_{\Phi_2} \chi(\varpi)\frac{\int_1^{\frac{\psi(\varpi)}{\chi(\varpi)}} \epsilon(\xi)d\xi}{\frac{\psi(\varpi)}{\chi(\varpi)} - 1} d\mu(\varpi)\right| \leq \int_{\Phi_2} \frac{\chi(\varpi)-\psi(\varpi))}{4} \times \sqrt[q]{|\epsilon'(1)|^q + \left|\epsilon'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|^q} d\mu(\varpi) \quad (4.\ 60)$$

Adding inequalities $(4.59)$ and $(4.60)$, and utilizing triangular inequality, we get the desired result $(4.58)$. □

4.5. **Probability density functions.** Let $g : [\rho, \zeta] \to [0, 1]$ be the probability density function of a continuous random variable $X$ with the cumulative distribution function, $F$, given by:

$$F(\varrho) = Pr(X \leq \varrho) = \int_\rho^\varrho g(\xi)d\xi \text{ and } E(X) = \int_\rho^\zeta \xi dF(\xi) = \zeta - \int_\rho^\zeta F(\xi)d\xi. \quad (4.\ 61)$$

Then, from Corollary 3.4 for $\varepsilon, \kappa, \iota = 1$; $e = \frac{1}{2}$; $\sigma = \zeta$ and $\hat{\varphi}(\zeta, \rho) = \zeta - \rho$ we have the following result:

$$\left| \frac{1}{2} + Pr\left(X \leq \frac{\rho + \zeta}{2}\right) - \frac{2}{\zeta - \rho}(\zeta - E(X)) \right| \leq \frac{(\zeta - \rho)\sqrt[q]{|g(\rho)|^q + |g(\zeta)|^q}}{4}.$$

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