Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 52(10)(2020) pp. 55-82

The Fuzzy Cross-Entropy for Picture Hesitant Fuzzy Sets and Their Application in Multi Criteria Decision Making

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Received: 21 January, 2020 / Accepted: 17 July, 2020 / Published online: 10 October, 2020

Abstract.: In this article, we exploit the cross-entropy of picture hesitant fuzzy set is established by distinguishing the cross-entropy of picture fuzzy set and hesitant fuzzy set. First, many measurement concepts are established and their basic properties are studied. Further, two measures, which are based on the established picture hesitant fuzzy cross-entropy, are established for evaluating multi-criteria decision making problems in the environment of picture hesitant fuzzy sets. For both approaches, an optimization model is pioneered to examine the weight vector for multi criteria decision making problems with incomplete information on criteria weights. In a last, we give an example to express practical and effectiveness. The comparison of the proposed measures with existing measures is also discussed.

AMS (MOS) Subject Classification Codes: 03B52; 94D05; 03E72
Key Words: Picture fuzzy sets; Hesitant fuzzy sets; Picture hesitant fuzzy sets; Cross Entropy measures.

1. Introduction

Multi criteria decision making has been utilized in the establishment of a multi-modal force recognition system involving multi-criteria with varying weights of importance. The theory of a fuzzy set was investigated by Zadeh [1], characterized by only positive grade are restricted to [0,1]. FS has achieved more success because of its capability of coping with complications and troubles. This proves a strong tool for expressing human knowledge [2-5]. However, in some practice cases, the concept of FS cannot cope with complications and uncertainty because of the lack of knowledge of the problem. Therefore, Atanassove [6] investigated the intuitionistic fuzzy set contains positive and negative grades, whose sum is bounded to [0,1]. IFS is regarded as a more improved way to cope with complex and awkward information. Although, Garg and Kumar [7] established exponential distance measures and TOPSIS methods for the interval-valued intuitionistic fuzzy set. Alcantud et

al. [8] established aggregation of finite chains based on the intuitionistic fuzzy set. VIKOR method was established by Zhang et al. [9] and Konwar and Debnath [10] established the intuitionistic fuzzy membership metric. Further, Sen and Et [11] explored the intuitionistic fuzzy norm linear space. For more work related to the fuzzy set, we may refer to [12-14].
The grade of a neutral cannot be discussed in intuitionistic fuzzy set. However, a neutral grade is used in many real-life scenarios, such as polling stations, human beings give opinions having more answers of a kind: positive, abstinence, negative and refusal/neutral. For example, in a democratic voting system, 100 people have appeared in the election and the election commission issued only 100 ballot paper. One person can take only one ballot paper for giving his/her vote and is only one candidate. Basically, the result is dived into four groups like vote for candidate 50, abstinence in vote 20, vote negative candidate 20, and refusal vote is 10. The "abstinence in vote" shows that ballot paper is white which contradicts both "vote for a candidate" and "vote negative candidate" but it considers the vote and the "refusal of voting" means bypassing the vote. Therefore, Cuong [15] investigated the picture fuzzy set contains positive, abstinence and negative grades, whose sum is bounded to [0,1]. Picture fuzzy set is regarded as a more improved way to cope with complex and awkward information. Although, Zeng et al. [16] established the linguistic picture fuzzy TOPSIS methods and picture fuzzy aggregation operators were established by Garg [17]. The notion of picture fuzzy aggregation operators was established by Wei [18]. Wang et al. [19] established the picture fuzzy Muirhead mean operators. After intuitionistic fuzzy set, the picture fuzzy set has achieved more attention from researchers and it is utilized in the environment of decision making [20-22] problems.
Hesitancy occurs in many real-life problems. To handle such types of problems, the hesitant fuzzy set was explored by Torra [23], as a powerful modification of fuzzy set, that allows multiple positive grades is associated with each preference information for coping with an awkward and complicated situation. Keeping the advantages of hesitant fuzzy set, researchers widely explored the concept for pattern recognition [24-26], medical diagnosis [27-30], similarity measures [31], and decision making [32-34] problems.
Picture fuzzy set has utilized successfully in various fields but in some practical cases, the picture fuzzy set cannot work effectively. For example, when a decision-maker provides for positive, abstinence, negative grades in the form of a finite subset of [0,1], then the condition of existing methods cannot hold. For handling such types of situations, the picture hesitant fuzzy set was established by Ullah et al. [35], which is more generalized than an existing drawback. Picture hesitant fuzzy set is a successful tool for describing the uncertainty and awkward kinds of information in the fuzzy set theory. Basically, picture hesitant fuzzy set contains three degrees is called positive, abstinence, and negative are the form of a subset of [0,1]. The condition of the picture hesitant fuzzy set is that the sum of the maximum of a positive degree, a maximum of abstinence degree, and maximum of negative degree is bounded to [0,1]. Picture hesitant fuzzy set has achieved more attention like: Wang and Li [36] established the aggregation operators based on picture hesitant fuzzy sets. Jan et al. [37] established the generalized similarity measures based on picture hesitant fuzzy sets. Ahmad et al. [38] established the similarity measures based on picture hesitant fuzzy sets.

When a decision-maker provides the grade of truth, the grade of abstinence and the grade of falsity in the form of a subset of unit interval such that $\begin{pmatrix} \{0.4, 0.3\}, \\ \{0.11, 0.01\}, \\ \{0.02, 0.01\} \end{pmatrix}$, then the exiting notions [1, 6, 15, 23], are cannot deal it effectively, for coping with such kind of issues, the notion of picture hesitant fuzzy set was explored. Although fuzzy set is a valid form to express the uncertain evaluation information, it cannot solve several complex situations in real life. For more effective expression of the evaluation information, many generalized forms of fuzzy set were proposed [5–10]. Keeping the advantages of the picture hesitant fuzzy set can express the uncertainty and complexity of human opinions in practice; furthermore, the positive, neutral, negative, and refusal membership degrees are represented by several possible values that are given by decision makers, in this article the explored work is summarized is follow as:

(1) The notion of cross-entropy of picture hesitant fuzzy set is established by distinguishing the cross-entropy of picture fuzzy set and hesitant fuzzy set. First, many measurement concepts are established and their basic properties are studied. Further, two measures, which are based on the established picture hesitant fuzzy cross-entropy, are established for evaluating multi criteria decision making problems in the environment of picture hesitant fuzzy sets.
(2) For both approaches, an optimization model is pioneered in order to examine the weight vector for multi criteria decision making problems with incomplete information on criteria weights. In last, we give an example to express the practically and effectiveness.
(3) The comparison of the proposed measures with existing measures are also discussed in detail.

Besides, the predominance of the picture hesitant fuzz model over a portion of the current models is shown in Table 1.

Table 1. Comparison of picture hesitant fuzzy sets model with existing models in literature.

Model	**Uncertainly**	**falsity**	**hesitations**	**Degrees in the form of subset**	**Strong Condition**	**Refusal grades**
FSs	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$
HFSs	$\surd$	$\times$	$\times$	$\surd$	$\times$	$\times$
IFSs	$\surd$	$\surd$	$\surd$	$\times$	$\times$	$\times$
IHFSs	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
PFSs	$\surd$	$\surd$	$\surd$	$\times$	$\times$	$\surd$
PHFSs	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$

The aim of this article is to follow as, In section 2, we recall some concepts of picture fuzzy sets, hesitant fuzzy sets, picture hesitant fuzzy sets, cross-entropy, and the properties are discussed. In section 3, the cross-entropy measures based on picture hesitant fuzzy sets are explored. In section 4, the concept is divided for evaluating multi-criteria decision-making problems using the picture hesitant fuzzy numbers, where the criteria weights are not completely known. Further, we resolve some numerical examples to show the reliability and efficiency of the proposed measures. The compassion o the proposed measures with existing measures are also conducted. The conclusion is discussed in section 5.

2. NOTATIONS AND PRELIMINARIES

In this section, the ideas of picture fuzzy sets, hesitant fuzzy sets, picture hesitant fuzzy sets, cross-entropy, and the properties are discussed. Throughout this article, X represents the finite fixed set.

Definition 2.1. [15] *A picture fuzzy set P_{PFS} on X is an object with the following form:*

$$P_{PFS} = \{(\mathcal{M}_{P_{PFS}}(x), \mathcal{A}_{P_{PFS}}(x), \mathcal{N}_{P_{PFS}}(x)) : x \in X\}$$

Where $\mathcal{M}_{P_{PFS}}, \mathcal{A}_{P_{PFS}}, \mathcal{N}_{P_{PFS}} : X \to [0,1]$ represents the positive, abstinence and negative grades, with a condition $0 \leq \mathcal{M}_{P_{PFS}}(x) + \mathcal{A}_{P_{PFS}}(x) + \mathcal{N}_{P_{PFS}}(x) \leq 1$. The refusal grade of picture fuzzy set is given by:

$$\pi_{P_{PFS}}(x) = 1 - (\mathcal{M}_{P_{PFS}}(x) + \mathcal{A}_{P_{PFS}}(x) + \mathcal{N}_{P_{PFS}}(x))$$

The triplet $(\mathcal{M}_{P_{PFS}}(x), \mathcal{A}_{P_{PFS}}(x), \mathcal{N}_{P_{PFS}}(x))$ represents the picture fuzzy numbers.

Definition 2.2. [23] *A hesitant fuzzy set P_{HFS} on X is an object with the following form:*

$$P_{HFS} = \{(x, \mathcal{M}_{P_{HFS}}(x)) : x \in X\}$$

Where $\mathcal{M}_{P_{HFS}}$ is a set of values in [0,1], presented the positive grade.

Definition 2.3. [35] *A picture hesitant fuzzy set P_{PHFS} on X is an object with the following form:*

$$P_{PHFS} = \{(\alpha_{P_{PHFS}}(x), \beta_{P_{PHFS}}(x), \gamma_{P_{PHFS}}(x)) : x \in X\}$$

Where $\alpha_{P_{PHFS}}, \beta_{P_{PHFS}}$ and $\gamma_{P_{PHFS}}$ are finite non-empty sets of values in [0, 1], representing the positive, abstinence and negative grades, with a condition
$0 \leq \max(\alpha_{P_{PHFS}}(x)) + \max(\beta_{P_{PHFS}}(x)) + \max(\gamma_{P_{PHFS}}(x)) \leq 1$. *The refusal grade of picture hesitant fuzzy set is given by:*
$\pi_{P_{PHFS}}(x) = 1 - (\mathcal{M}_{P_{PHFS}}(x) + \mathcal{A}_{P_{PHFS}}(x) + \mathcal{N}_{P_{PHFS}}(x))$, *where* $\mathcal{M}_{P_{PHFS}} \in \alpha_{P_{PHFS}}, \mathcal{A}_{P_{PHFS}} \in \beta_{P_{PHFS}}, \mathcal{N}_{P_{PHFS}} \in \gamma_{P_{PHFS}}, \pi_{P_{PHFS}}(x) \in \delta_{P_{PHFS}}(x)$. *The triplet* $(\alpha_{P_{PHFS}}(x), \beta_{P_{PHFS}}(x), \gamma_{P_{PHFS}}(x))$ *represents the picture hesitant fuzzy numbers.*

Definition 2.4. [35] *Let* $P_{PHFS-1} = \alpha_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \gamma_{P_{PHFS-1}}(x)$ *and* $P_{PHFS-2} = \alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-2}}(x), \gamma_{P_{PHFS-2}}(x)$ *be two picture hesitant fuzzy numbers. Then*

(1) $P_{PHFS-1} \oplus P_{PHFS-2} =$

$$\left(\begin{array}{c} \coprod_{\substack{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}}, \\ \mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}}}} \left(\begin{array}{c} \mathcal{M}_{P_{PHFS-1}} + \\ \mathcal{M}_{P_{PHFS-2}} - \\ \mathcal{M}_{P_{PHFS-1}} . \mathcal{M}_{P_{PHFS-2}} \end{array} \right), \\ \coprod_{\substack{\mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}}, \\ \mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}}}} \mathcal{A}_{P_{PHFS-1}} . \mathcal{A}_{P_{PHFS-2}}, \\ \coprod_{\substack{\mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}, \\ \mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}}} \mathcal{N}_{P_{PHFS-1}} . \mathcal{N}_{P_{PHFS-2}} \end{array} \right);$$

(2) $P_{PHFS-1} \otimes P_{PHFS-2} =$

$$\left(\begin{array}{ll} \coprod_{\substack{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}},\\ \mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}}}} & \mathcal{M}_{P_{PHFS-1}} . \mathcal{M}_{P_{PHFS-2}} \ , \\ \coprod_{\substack{\mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}},\\ \mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}}}} & \left(\begin{array}{c} \mathcal{A}_{P_{PHFS-1}} + \\ \mathcal{A}_{P_{PHFS-2}} - \\ \mathcal{A}_{P_{PHFS-1}} . \mathcal{A}_{P_{PHFS-2}} \end{array}\right), \\ \coprod_{\substack{\mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}},\\ \mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}}} & \left(\begin{array}{c} \mathcal{N}_{P_{PHFS-1}} + \\ \mathcal{N}_{P_{PHFS-2}} - \\ \mathcal{N}_{P_{PHFS-1}} . \mathcal{N}_{P_{PHFS-2}} \end{array}\right) \end{array}\right);$$

(3) $\partial P_{PHFS-1} =$

$$\left(\begin{array}{c} \coprod_{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}}} \ 1- \ \ 1-\mathcal{M}_{P_{PHFS-1}} \ \ ^{\partial} \ , \\ \coprod_{\mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}}} \ \mathcal{A}^{\partial}_{P_{PHFS-1}} \ , \\ \coprod_{\mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}} \ \mathcal{N}^{\partial}_{P_{PHFS-1}} \end{array}\right);$$

(4) $P^{\partial}_{PHFS-1} =$

$$\left(\begin{array}{c} \coprod_{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}}} \ \mathcal{M}^{\partial}_{P_{PHFS-1}} \ , \\ \coprod_{\mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}}} \ 1- \ \ 1-\mathcal{A}_{P_{PHFS-1}} \ \ ^{\partial} \ , \\ \coprod_{\mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}} \ 1- \ \ 1-\mathcal{N}_{P_{PHFS-1}} \ \ ^{\partial} \end{array}\right).$$

3. THE CROSS-ENTROPY MEASURES FOR PICTURE HESITANT FUZZY NUMBERS

In this section, we explore the concept of cross-entropy measures based on picture hesitant fuzzy numbers and their properties. The proposed measures are described below.

Definition 3.1. *Let* $P_{PHFS-1} = \left(\alpha_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \gamma_{P_{PHFS-1}}(x)\right)$ *and* $P_{PHFS-2} = \left(\alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-2}}(x), \gamma_{P_{PHFS-2}}(x)\right)$ *be two picture hesitant fuzzy numbers. Then the cross-entropy* $CE_1\left(P_{PHFS-1}, P_{PHFS-2}\right)$ *is denoted and defined as:*

$$CE_1 : PHFN \times PHFN \to R^{+}$$

$$CE_1\left(P_{PHFS-1}, P_{PHFS-2}\right) =$$

$$\left(\begin{array}{c} max_{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}}} \\ \left(\begin{array}{c} min_{\mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}}} \\ \left(\begin{array}{c} \mathcal{M}_{P_{PHFS-1}} \\ \log_2\left(\frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-1}} + \mathcal{M}_{P_{PHFS-2}}}\right) \end{array}\right) \end{array}\right) + \\ max_{\mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}}} \\ \left(\begin{array}{c} min_{\mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}}} \\ \left(\begin{array}{c} \mathcal{A}_{P_{PHFS-1}} \\ \log_2\left(\frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-1}} + \mathcal{A}_{P_{PHFS-2}}}\right) \end{array}\right) \end{array}\right) + \\ max_{\mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}} \\ \left(\begin{array}{c} min_{\mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}} \\ \left(\begin{array}{c} \mathcal{N}_{P_{PHFS-1}} \\ \log_2\left(\frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-2}}}\right) \end{array}\right) \end{array}\right) \end{array}\right)$$

Which holds the following conditions:

(1) $CE_1\left(P_{PHFS-1}, P_{PHFS-2}\right) \geq 0$;
(2) $CE_1\left(P_{PHFS-1}, P_{PHFS-2}\right) = 0$ *iff* $P_{PHFS-1} = P_{PHFS-2}$;
(3) $CE_1\left(P^C_{PHFS-1}, P^C_{PHFS-2}\right) = CE_1\left(P_{PHFS-1}, P_{PHFS-2}\right)$,

where $P^C_{PHFS-1} = \left(\gamma_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \alpha_{P_{PHFS-1}}(x)\right)$.

Definition 3.2. *Let* $P_{PHFS-1} = \left(\alpha_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \gamma_{P_{PHFS-1}}(x)\right)$ *and* $P_{PHFS-2} = \left(\alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-2}}(x), \gamma_{P_{PHFS-2}}(x)\right)$ *be two picture hesitant fuzzy numbers,* $p \geq 1$. *Then the cross-entropy* $CE_2\left(P_{PHFS-1}, P_{PHFS-2}\right)$ *is denoted and defined as:*

$$CE_2 : PHFN \times PHFN \to R^+$$
$$CE_2\left(P_{PHFS-1}, P_{PHFS-2}\right) =$$
$$\left(\begin{array}{c}\left(\sum_{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}}}\left(\frac{1}{\left|\alpha_{P_{PHFS-1}}\right|}\begin{array}{c} min_{\mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}}} \\ \left(\begin{array}{c}\mathcal{M}_{P_{PHFS-1}} \\ \log_2\left(\frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-1}}+\mathcal{M}_{P_{PHFS-2}}}\right)\end{array}\right)\end{array}\right)^p\right)^{\frac{1}{p}} + \\ \left(\sum_{\mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}}}\left(\frac{1}{\left|\beta_{P_{PHFS-1}}\right|}\begin{array}{c} min_{\mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}}} \\ \left(\begin{array}{c}\mathcal{A}_{P_{PHFS-1}} \\ \log_2\left(\frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-1}}+\mathcal{A}_{P_{PHFS-2}}}\right)\end{array}\right)\end{array}\right)^p\right)^{\frac{1}{p}} + \\ \left(\sum_{\mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}}\left(\frac{1}{\left|\gamma_{P_{PHFS-1}}\right|}\begin{array}{c} min_{\mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}} \\ \left(\begin{array}{c}\mathcal{N}_{P_{PHFS-1}} \\ \log_2\left(\frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-1}}+\mathcal{N}_{P_{PHFS-2}}}\right)\end{array}\right)\end{array}\right)^p\right)^{\frac{1}{p}}\end{array}\right)$$

Which holds the following conditions:

(1) $CE_2\left(P_{PHFS-1}, P_{PHFS-2}\right) \geq 0$;
(2) $CE_2\left(P_{PHFS-1}, P_{PHFS-2}\right) = 0$ *iff* $P_{PHFS-1} = P_{PHFS-2}$;
(3) $CE_2\left(P^C_{PHFS-1}, P^C_{PHFS-2}\right) = CE_2\left(P_{PHFS-1}, P_{PHFS-2}\right)$,

where $P^C_{PHFS-1} = \left(\gamma_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \alpha_{P_{PHFS-1}}(x)\right)$.

Theorem 3.3. *The proposed measures* $CE_1\left(P_{PHFS-1}, P_{PHFS-2}\right)$ *and* $CE_2\left(P_{PHFS-1}, P_{PHFS-2}\right)$ *are a picture hesitant fuzzy cross-entropy, and satisfy the above three conditions.*

Proof: *The conditions of def. 3.1 and def.3.2 can be examined and the proof of the other definitions is similar.*

(1) *It is clear that* $CE_1\left(P_{PHFS-1}, P_{PHFS-2}\right) \geq 0$ *and* $CE_2\left(P_{PHFS-1}, P_{PHFS-2}\right) \geq 0$.
(2) *If* $P_{PHFS-1} = P_{PHFS-2}$, *then* $\alpha_{P_{PHFS-1}}(x) = \alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-1}}(x) = \beta_{P_{PHFS-2}}(x)$ *and* $\gamma_{P_{PHFS-1}}(x) = \gamma_{P_{PHFS-2}}(x)$. *Then*

$$CE_1\left(P_{PHFS-1}, P_{PHFS-2}\right) =$$

$$\left(\begin{array}{c} max_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}} \left(\begin{array}{c} min_{\mathcal{M}_{P_{PHFS-2}}\in\alpha_{P_{PHFS-2}}} \left(\begin{array}{c} \mathcal{M}_{P_{PHFS-1}} \\ \log_2\left(\frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-1}}+\mathcal{M}_{P_{PHFS-1}}}\right)\end{array}\right)\end{array}\right)+ \\ max_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}} \left(\begin{array}{c} min_{\mathcal{A}_{P_{PHFS-2}}\in\beta_{P_{PHFS-2}}} \left(\begin{array}{c} \mathcal{A}_{P_{PHFS-1}} \\ \log_2\left(\frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-1}}+\mathcal{A}_{P_{PHFS-1}}}\right)\end{array}\right)\end{array}\right)+ \\ max_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}} \left(\begin{array}{c} min_{\mathcal{N}_{P_{PHFS-2}}\in\gamma_{P_{PHFS-2}}} \left(\begin{array}{c} \mathcal{N}_{P_{PHFS-1}} \\ \log_2\left(\frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-1}}+\mathcal{N}_{P_{PHFS-1}}}\right)\end{array}\right)\end{array}\right)\end{array}\right)$$

$$= max_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}}(0)+$$

$$max_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}}(0)+$$

$$max_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}}(0)=0$$

and

$$CE_2\ \ P_{PHFS-1}, P_{PHFS-2}\ \ =$$

$$\left(\begin{array}{c} \left(\sum_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}} \frac{1}{\alpha_{P_{PHFS-1}}}\left(\begin{array}{c} min_{\mathcal{M}_{P_{PHFS-2}}\in\alpha_{P_{PHFS-2}}} \left(\begin{array}{c} \mathcal{M}_{P_{PHFS-1}} \\ \log_2\ \ \frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-1}}+\mathcal{M}_{P_{PHFS-1}}}\end{array}\right)\end{array}\right)^p\right)^{\frac{1}{p}}+ \\ \left(\sum_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}} \frac{1}{\beta_{P_{PHFS-1}}}\left(\begin{array}{c} min_{\mathcal{A}_{P_{PHFS-2}}\in\beta_{P_{PHFS-2}}} \left(\begin{array}{c} \mathcal{A}_{P_{PHFS-1}} \\ \log_2\ \ \frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-1}}+\mathcal{A}_{P_{PHFS-1}}}\end{array}\right)\end{array}\right)^p\right)^{\frac{1}{p}}+ \\ \left(\sum_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}} \frac{1}{\gamma_{P_{PHFS-1}}}\left(\begin{array}{c} min_{\mathcal{N}_{P_{PHFS-2}}\in\gamma_{P_{PHFS-2}}} \left(\begin{array}{c} \mathcal{N}_{P_{PHFS-1}} \\ \log_2\ \ \frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-1}}+\mathcal{N}_{P_{PHFS-1}}}\end{array}\right)\end{array}\right)^p\right)^{\frac{1}{p}}\end{array}\right)$$

$$=(0)^{\frac{1}{p}}+(0)^{\frac{1}{p}}+(0)^{\frac{1}{p}}=0$$

(1) *Let* $CE_2 \;\; P_{PHFS-1}, P_{PHFS-2} \;\; =$

$$\left(\begin{array}{l}\left(\sum_{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}}} \left(\frac{1}{\alpha_{P_{PHFS-1}}} \; min_{\mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}}} \; \mathcal{M}_{P_{PHFS-1}} \left(\log_2 \frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-1}} + \mathcal{M}_{P_{PHFS-2}}}\right)\right)^p\right)^{\frac{1}{p}} + \\ \left(\sum_{\mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}}} \left(\frac{1}{\beta_{P_{PHFS-1}}} \; min_{\mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}}} \; \mathcal{A}_{P_{PHFS-1}} \left(\log_2 \frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-1}} + \mathcal{A}_{P_{PHFS-2}}}\right)\right)^p\right)^{\frac{1}{p}} + \\ \left(\sum_{\mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}} \left(\frac{1}{\gamma_{P_{PHFS-1}}} \; min_{\mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}} \; \mathcal{N}_{P_{PHFS-1}} \left(\log_2 \frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-2}}}\right)\right)^p\right)^{\frac{1}{p}}\end{array}\right)$$

$$= \left(\begin{array}{l}\left(\sum_{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}}} \left(\frac{1}{\alpha_{P_{PHFS-1}}} \; min_{\mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}}} \; \mathcal{M}_{P_{PHFS-1}} \left(\log_2 \frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-2}} + \mathcal{M}_{P_{PHFS-1}}}\right)\right)^p\right)^{\frac{1}{p}} + \\ \left(\sum_{\mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}}} \left(\frac{1}{\beta_{P_{PHFS-1}}} \; min_{\mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}}} \; \mathcal{A}_{P_{PHFS-1}} \left(\log_2 \frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-2}} + \mathcal{A}_{P_{PHFS-1}}}\right)\right)^p\right)^{\frac{1}{p}} + \\ \left(\sum_{\mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}} \left(\frac{1}{\gamma_{P_{PHFS-1}}} \; min_{\mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}} \; \mathcal{N}_{P_{PHFS-1}} \left(\log_2 \frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-2}} + \mathcal{N}_{P_{PHFS-1}}}\right)\right)^p\right)^{\frac{1}{p}}\end{array}\right)$$

$$= CE_2 \;\; P^c_{PHFS-1}, P^c_{PHFS-2}$$

Hence the proof of the results is completed.

Definition 3.4. *Let* $P_{PHFS-1} = \left(\alpha_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \gamma_{P_{PHFS-1}}(x)\right)$ *and* $P_{PHFS-2} = \left(\alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-2}}(x), \gamma_{P_{PHFS-2}}(x)\right)$ *be two picture hesitant fuzzy numbers. Then the cross-entropy* $CE_3(P_{PHFS-1}, P_{PHFS-2})$ *is denoted and defined as:*

$$CE_3 : PHFN \times PHFN \rightarrow R^+$$

$$CE_3(P_{PHFS-1}, P_{PHFS-2}) =$$

$$\left(\begin{array}{c} max_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}} \\ \left(min_{\mathcal{M}_{P_{PHFS-2}}\in\alpha_{P_{PHFS-2}}}\left(\mathcal{M}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-1}}+\mathcal{M}_{P_{PHFS-2}}}\right)\right)\right)+ \\ max_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}} \\ \left(min_{\mathcal{A}_{P_{PHFS-2}}\in\beta_{P_{PHFS-2}}}\left(\mathcal{A}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-1}}+\mathcal{A}_{P_{PHFS-2}}}\right)\right)\right)+ \\ max_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}} \\ \left(min_{\mathcal{N}_{P_{PHFS-2}}\in\gamma_{P_{PHFS-2}}}\left(\mathcal{N}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-1}}+\mathcal{N}_{P_{PHFS-2}}}\right)\right)\right)+ \\ max_{\pi_{P_{PHFS-1}}\in\delta_{P_{PHFS-1}}} \\ \left(min_{\pi_{P_{PHFS-2}}\in\delta_{P_{PHFS-2}}}\left(\pi_{P_{PHFS-1}}\log_2\left(\frac{2\pi_{P_{PHFS-1}}}{\pi_{P_{PHFS-1}}+\pi_{P_{PHFS-2}}}\right)\right)\right) \end{array}\right)$$

Which holds the following conditions:

(1) $CE_3\left(P_{PHFS-1},P_{PHFS-2}\right)\geq 0$;
(2) $CE_3\left(P_{PHFS-1},P_{PHFS-2}\right)=0$ *iff* $P_{PHFS-1}=P_{PHFS-2}$;
(3) $CE_3\left(P^C_{PHFS-1},P^C_{PHFS-2}\right)=CE_3\left(P_{PHFS-1},P_{PHFS-2}\right)$,

where $P^C_{PHFS-1}=\left(\gamma_{P_{PHFS-1}}\left(x\right),\beta_{P_{PHFS-1}}\left(x\right),\alpha_{P_{PHFS-1}}\left(x\right)\right)$.

Definition 3.5. *Let* $P_{PHFS-1}=\left(\alpha_{P_{PHFS-1}}\left(x\right),\beta_{P_{PHFS-1}}\left(x\right),\gamma_{P_{PHFS-1}}\left(x\right)\right)$ *and* $P_{PHFS-2}=\left(\alpha_{P_{PHFS-2}}\left(x\right),\beta_{P_{PHFS-2}}\left(x\right),\gamma_{P_{PHFS-2}}\left(x\right)\right)$ *be two picture hesitant fuzzy numbers,* $p\geq 1$*. Then the cross-entropy* $CE_4\left(P_{PHFS-1},P_{PHFS-2}\right)$ *is denoted and defined as:*

$$CE_4: PHFN\times PHFN\rightarrow R^+$$

$$CE_4\left(P_{PHFS-1},P_{PHFS-2}\right)=$$

$$\left(\begin{array}{c} \left(\sum_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}}\left(\frac{1}{\alpha_{P_{PHFS-1}}}\min_{\mathcal{M}_{P_{PHFS-2}}\in\alpha_{P_{PHFS-2}}}\mathcal{M}_{P_{PHFS-1}}\left(\log_2\frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-1}}+\mathcal{M}_{P_{PHFS-2}}}\right)\right)^p\right)^{\frac{1}{p}}+ \\ \left(\sum_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}}\left(\frac{1}{\beta_{P_{PHFS-1}}}\min_{\mathcal{A}_{P_{PHFS-2}}\in\beta_{P_{PHFS-2}}}\mathcal{A}_{P_{PHFS-1}}\left(\log_2\frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-1}}+\mathcal{A}_{P_{PHFS-2}}}\right)\right)^p\right)^{\frac{1}{p}}+ \\ \left(\sum_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}}\left(\frac{1}{\gamma_{P_{PHFS-1}}}\min_{\mathcal{N}_{P_{PHFS-2}}\in\gamma_{P_{PHFS-2}}}\mathcal{N}_{P_{PHFS-1}}\left(\log_2\frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-1}}+\mathcal{N}_{P_{PHFS-2}}}\right)\right)^p\right)^{\frac{1}{p}}+ \\ \left(\sum_{\pi_{P_{PHFS-1}}\in\delta_{P_{PHFS-1}}}\left(\frac{1}{\delta_{P_{PHFS-1}}}\min_{\pi_{P_{PHFS-2}}\in\delta_{P_{PHFS-2}}}\pi_{P_{PHFS-1}}\left(\log_2\frac{2\pi_{P_{PHFS-1}}}{\pi_{P_{PHFS-1}}+\pi_{P_{PHFS-2}}}\right)\right)^p\right)^{\frac{1}{p}} \end{array}\right)$$

Which holds the following conditions:

(1) $CE_4 \; P_{PHFS-1}, P_{PHFS-2} \; \geq 0$;
(2) $CE_4 \; P_{PHFS-1}, P_{PHFS-2} \; = 0$ *iff* $P_{PHFS-1} = P_{PHFS-2}$;
(3) $CE_4 \; P^C_{PHFS-1}, P^C_{PHFS-2} \; = CE_4 \; P_{PHFS-1}, P_{PHFS-2}$,

where $P^C_{PHFS-1} = \; \gamma_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \alpha_{P_{PHFS-1}}(x)$.

Theorem 3.6. *The proposed measures $CE_3(P_{PHFS-1}, P_{PHFS-2})$ and $CE_4(P_{PHFS-1}, P_{PHFS-2})$ are a picture hesitant fuzzy cross-entropy, and satisfy the above three conditions.*
Proof: *The conditions of $def.3.1$ and $def.3.2$ can be examined and the proof of the other definitions is similar.*

(1) *It is clear that $CE_3(P_{PHFS-1}, P_{PHFS-2}) \geq 0$ and $CE_4(P_{PHFS-1}, P_{PHFS-2}) \geq 0$.*
(2) *If $P_{PHFS-1} = P_{PHFS-2}$, then $\alpha_{P_{PHFS-1}}(x) = \alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-1}}(x) = \beta_{P_{PHFS-2}}(x)$ and $\gamma_{P_{PHFS-1}}(x) = \gamma_{P_{PHFS-2}}(x)$. Then*

$$CE_3(P_{PHFS-1}, P_{PHFS-2}) =$$

$$\left(\begin{array}{c} max_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}}\left(min_{\mathcal{M}_{P_{PHFS-2}}\in\alpha_{P_{PHFS-2}}}\left(\mathcal{M}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-1}}+\mathcal{M}_{P_{PHFS-1}}}\right)\right)\right)+ \\ max_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}}\left(min_{\mathcal{A}_{P_{PHFS-2}}\in\beta_{P_{PHFS-2}}}\left(\mathcal{A}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-1}}+\mathcal{A}_{P_{PHFS-1}}}\right)\right)\right)+ \\ max_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}}\left(min_{\mathcal{N}_{P_{PHFS-2}}\in\gamma_{P_{PHFS-2}}}\left(\mathcal{N}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-1}}+\mathcal{N}_{P_{PHFS-1}}}\right)\right)\right)+ \\ max_{\pi_{P_{PHFS-1}}\in\delta_{P_{PHFS-1}}}\left(min_{\pi_{P_{PHFS-2}}\in\delta_{P_{PHFS-2}}}\left(\pi_{P_{PHFS-1}}\log_2\left(\frac{2\pi_{P_{PHFS-1}}}{\pi_{P_{PHFS-1}}+\pi_{P_{PHFS-1}}}\right)\right)\right) \end{array}\right)$$

$$= max_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}}(0)+ max_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}}(0)+ max_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}}(0)+ max_{\pi_{P_{PHFS-1}}\in\delta_{P_{PHFS-1}}}(0)=0$$

and

$$CE_4\left(P_{PHFS-1},P_{PHFS-2}\right)=$$

$$\left(\begin{array}{c} \left(\frac{1}{\alpha_{P_{PHFS-1}}}\sum_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}}\left(min_{\mathcal{M}_{P_{PHFS-2}}\in\alpha_{P_{PHFS-2}}}\left(\mathcal{M}_{P_{PHFS-1}}\log_2\frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-1}}+\mathcal{M}_{P_{PHFS-1}}}\right)\right)^p\right)^{\frac{1}{p}}+ \\ \left(\frac{1}{\beta_{P_{PHFS-1}}}\sum_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}}\left(min_{\mathcal{A}_{P_{PHFS-2}}\in\beta_{P_{PHFS-2}}}\left(\mathcal{A}_{P_{PHFS-1}}\log_2\frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-1}}+\mathcal{A}_{P_{PHFS-1}}}\right)\right)^p\right)^{\frac{1}{p}}+ \\ \left(\frac{1}{\gamma_{P_{PHFS-1}}}\sum_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}}\left(min_{\mathcal{N}_{P_{PHFS-2}}\in\gamma_{P_{PHFS-2}}}\left(\mathcal{N}_{P_{PHFS-1}}\log_2\frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-1}}+\mathcal{N}_{P_{PHFS-1}}}\right)\right)^p\right)^{\frac{1}{p}}+ \\ \left(\frac{1}{\delta_{P_{PHFS-1}}}\sum_{\pi_{P_{PHFS-1}}\in\delta_{P_{PHFS-1}}}\left(min_{\pi_{P_{PHFS-2}}\in\delta_{P_{PHFS-2}}}\pi_{P_{PHFS-1}}\log_2\left(\frac{2\pi_{P_{PHFS-1}}}{\pi_{P_{PHFS-1}}+\pi_{P_{PHFS-1}}}\right)\right)^p\right)^{\frac{1}{p}} \end{array}\right)$$

$$= (0)^{\frac{1}{p}} + (0)^{\frac{1}{p}} + (0)^{\frac{1}{p}} + (0)^{\frac{1}{p}} = 0$$

(1) *Let* $CE_4\ \ P_{PHFS-1}, P_{PHFS-2}\ =$

$$\left(\begin{array}{l}\left(\sum_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}}\left(\frac{1}{\alpha_{P_{PHFS-1}}}\min_{\mathcal{M}_{P_{PHFS-2}}\in\alpha_{P_{PHFS-2}}}\left(\mathcal{M}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-1}}+\mathcal{M}_{P_{PHFS-2}}}\right)\right)\right)^{p}\right)^{\frac{1}{p}}+\\ \left(\sum_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}}\left(\frac{1}{\beta_{P_{PHFS-1}}}\min_{\mathcal{A}_{P_{PHFS-2}}\in\beta_{P_{PHFS-2}}}\left(\mathcal{A}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-1}}+\mathcal{A}_{P_{PHFS-2}}}\right)\right)\right)^{p}\right)^{\frac{1}{p}}+\\ \left(\sum_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}}\left(\frac{1}{\gamma_{P_{PHFS-1}}}\min_{\mathcal{N}_{P_{PHFS-2}}\in\gamma_{P_{PHFS-2}}}\left(\mathcal{N}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-1}}+\mathcal{N}_{P_{PHFS-2}}}\right)\right)\right)^{p}\right)^{\frac{1}{p}}+\\ \left(\sum_{\pi_{P_{PHFS-1}}\in\delta_{P_{PHFS-1}}}\left(\frac{1}{\delta_{P_{PHFS-1}}}\min_{\pi_{P_{PHFS-2}}\in\delta_{P_{PHFS-2}}}\left(\pi_{P_{PHFS-1}}\log_2\left(\frac{2\pi_{P_{PHFS-1}}}{\pi_{P_{PHFS-1}}+\pi_{P_{PHFS-2}}}\right)\right)\right)^{p}\right)^{\frac{1}{p}}\end{array}\right)$$

$$=\left(\begin{array}{l}\left(\sum_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}}\left(\frac{1}{\alpha_{P_{PHFS-1}}}\min_{\mathcal{M}_{P_{PHFS-2}}\in\alpha_{P_{PHFS-2}}}\left(\mathcal{M}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{M}_{P_{PHFS-1}}}{\mathcal{M}_{P_{PHFS-2}}+\mathcal{M}_{P_{PHFS-1}}}\right)\right)\right)^{p}\right)^{\frac{1}{p}}+\\ \left(\sum_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}}\left(\frac{1}{\beta_{P_{PHFS-1}}}\min_{\mathcal{A}_{P_{PHFS-2}}\in\beta_{P_{PHFS-2}}}\left(\mathcal{A}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{A}_{P_{PHFS-1}}}{\mathcal{A}_{P_{PHFS-2}}+\mathcal{A}_{P_{PHFS-1}}}\right)\right)\right)^{p}\right)^{\frac{1}{p}}+\\ \left(\sum_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}}\left(\frac{1}{\gamma_{P_{PHFS-1}}}\min_{\mathcal{N}_{P_{PHFS-2}}\in\gamma_{P_{PHFS-2}}}\left(\mathcal{N}_{P_{PHFS-1}}\log_2\left(\frac{2\mathcal{N}_{P_{PHFS-1}}}{\mathcal{N}_{P_{PHFS-2}}+\mathcal{N}_{P_{PHFS-1}}}\right)\right)\right)^{p}\right)^{\frac{1}{p}}+\\ \left(\sum_{\pi_{P_{PHFS-1}}\in\delta_{P_{PHFS-1}}}\left(\frac{1}{\delta_{P_{PHFS-1}}}\min_{\pi_{P_{PHFS-2}}\in\delta_{P_{PHFS-2}}}\left(\pi_{P_{PHFS-1}}\log_2\left(\frac{2\pi_{P_{PHFS-1}}}{\pi_{P_{PHFS-2}}+\pi_{P_{PHFS-1}}}\right)\right)\right)^{p}\right)^{\frac{1}{p}}\end{array}\right)$$

$$= CE_4\ \ P^{c}_{PHFS-1}, P^{c}_{PHFS-2}$$

Hence the proof of the results is completed.

Definition 3.7. *Let* $P_{PHFS-1} = \ \alpha_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \gamma_{P_{PHFS-1}}(x)$ *and* $P_{PHFS-2} = \ \alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-2}}(x), \gamma_{P_{PHFS-2}}(x)$ *be two picture hesitant fuzzy numbers. Then the cross-entropy* $CE_5\left(P_{PHFS-1}, P_{PHFS-2}\right)$ *is denoted and defined as:*

$$CE_5 : PHFN \times PHFN \rightarrow R^{+}$$

$$CE_5\left(P_{PHFS-1}, P_{PHFS-2}\right) =$$

$$\left(\begin{array}{c}
\max\limits_{\left(\begin{array}{c}\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}}, \\ \mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}}, \\ \mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}\end{array}\right)}
\left(\min\limits_{\left(\begin{array}{c}\mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}}, \\ \mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}}, \\ \mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}\end{array}\right)}
\left(\frac{\left(\begin{array}{c}\mathcal{M}_{P_{PHFS-1}} + 1 - \\ \mathcal{A}_{P_{PHFS-1}} - \mathcal{N}_{P_{PHFS-1}}\end{array}\right)}{2}
\log_2\left(\frac{2\left(\begin{array}{c}\mathcal{M}_{P_{PHFS-1}} + 1 - \\ \mathcal{A}_{P_{PHFS-1}} - \mathcal{N}_{P_{PHFS-1}}\end{array}\right)}{\left(\left(\begin{array}{c}\mathcal{M}_{P_{PHFS-1}} + 1 - \\ \mathcal{A}_{P_{PHFS-1}} - \mathcal{N}_{P_{PHFS-1}}\end{array}\right) + \left(\begin{array}{c}\mathcal{M}_{P_{PHFS-2}} + 1 - \\ \mathcal{A}_{P_{PHFS-2}} - \mathcal{N}_{P_{PHFS-2}}\end{array}\right)\right)}\right)\right)\right) + \\
\max\limits_{\left(\begin{array}{c}\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}}, \\ \mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}}, \\ \mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}\end{array}\right)}
\left(\min\limits_{\left(\begin{array}{c}\mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}}, \\ \mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}}, \\ \mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}\end{array}\right)}
\left(\frac{\left(\begin{array}{c}1 - \mathcal{M}_{P_{PHFS-1}} + \\ \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\end{array}\right)}{2}
\log_2\left(\frac{2\left(\begin{array}{c}1 - \mathcal{M}_{P_{PHFS-1}} + \\ \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\end{array}\right)}{\left(\left(\begin{array}{c}1 - \mathcal{M}_{P_{PHFS-1}} + \\ \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\end{array}\right) + \left(\begin{array}{c}1 - \mathcal{M}_{P_{PHFS-2}} + \\ \mathcal{A}_{P_{PHFS-2}} + \mathcal{N}_{P_{PHFS-2}}\end{array}\right)\right)}\right)\right)\right) + \\
\max\limits_{\left(\begin{array}{c}\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}}, \\ \mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}}, \\ \mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}\end{array}\right)}
\left(\min\limits_{\left(\begin{array}{c}\mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}}, \\ \mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}}, \\ \mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}\end{array}\right)}
\left(\frac{\left(\begin{array}{c}1 - \mathcal{M}_{P_{PHFS-1}} + \\ \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\end{array}\right)}{2}
\log_2\left(\frac{2\left(\begin{array}{c}1 - \mathcal{M}_{P_{PHFS-1}} + \\ \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\end{array}\right)}{\left(\left(\begin{array}{c}1 - \mathcal{M}_{P_{PHFS-1}} + \\ \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\end{array}\right) + \left(\begin{array}{c}1 - \mathcal{M}_{P_{PHFS-2}} + \\ \mathcal{A}_{P_{PHFS-2}} + \mathcal{N}_{P_{PHFS-2}}\end{array}\right)\right)}\right)\right)\right)
\end{array}\right)$$

Which holds the following conditions:

(1) $CE_5\left(P_{PHFS-1}, P_{PHFS-2}\right) \geq 0$;
(2) $CE_5\left(P_{PHFS-1}, P_{PHFS-2}\right) = 0$ *iff* $P_{PHFS-1} = P_{PHFS-2}$;
(3) $CE_5\left(P^C_{PHFS-1}, P^C_{PHFS-2}\right) = CE_5\left(P_{PHFS-1}, P_{PHFS-2}\right)$,

where $P^C_{PHFS-1} = \left(\gamma_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \alpha_{P_{PHFS-1}}(x)\right)$.

Definition 3.8. *Let* $P_{PHFS-1} = \left(\alpha_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \gamma_{P_{PHFS-1}}(x)\right)$ *and* $P_{PHFS-2} = \left(\alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-2}}(x), \gamma_{P_{PHFS-2}}(x)\right)$ *be two picture hesitant fuzzy numbers,* $p \geq 1$*. Then the cross-entropy* $CE_6\left(P_{PHFS-1}, P_{PHFS-2}\right)$ *is denoted and defined as:*

$$CE_6 : PHFN \times PHFN \rightarrow R^+$$

$$CE_6\left(P_{PHFS-1}, P_{PHFS-2}\right) =$$

$$\left(\begin{array}{c}
\left(\sum_{\substack{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}},\\ \mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}},\\ \mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}}} \left(\frac{1}{\alpha_{P_{PHFS-1}} \cdot \beta_{P_{PHFS-1}} \cdot \gamma_{P_{PHFS-1}}} \min_{\substack{\mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}},\\ \mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}},\\ \mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}}} \left(\frac{\left(\mathcal{M}_{P_{PHFS-1}} + 1 - \mathcal{A}_{P_{PHFS-1}} - \mathcal{N}_{P_{PHFS-1}}\right)}{2} \log_2\left(\frac{2\left(\mathcal{M}_{P_{PHFS-1}} + 1 - \mathcal{A}_{P_{PHFS-1}} - \mathcal{N}_{P_{PHFS-1}}\right)}{\left(\mathcal{M}_{P_{PHFS-1}} + 1 - \mathcal{A}_{P_{PHFS-1}} - \mathcal{N}_{P_{PHFS-1}}\right) + \left(\mathcal{M}_{P_{PHFS-2}} + 1 - \mathcal{A}_{P_{PHFS-2}} - \mathcal{N}_{P_{PHFS-2}}\right)}\right)\right)\right)^p\right)^{\frac{1}{p}} + \\
\left(\sum_{\substack{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}},\\ \mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}},\\ \mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}}} \left(\frac{1}{\alpha_{P_{PHFS-1}} \cdot \beta_{P_{PHFS-1}} \cdot \gamma_{P_{PHFS-1}}} \min_{\substack{\mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}},\\ \mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}},\\ \mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}}} \left(\frac{\left(1 - \mathcal{M}_{P_{PHFS-1}} + \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\right)}{2} \log_2\left(\frac{2\left(1 - \mathcal{M}_{P_{PHFS-1}} + \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\right)}{\left(1 - \mathcal{M}_{P_{PHFS-1}} + \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\right) + \left(1 - \mathcal{M}_{P_{PHFS-2}} + \mathcal{A}_{P_{PHFS-2}} + \mathcal{N}_{P_{PHFS-2}}\right)}\right)\right)\right)^p\right)^{\frac{1}{p}} + \\
\left(\sum_{\substack{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}},\\ \mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}},\\ \mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}}} \left(\frac{1}{\alpha_{P_{PHFS-1}} \cdot \beta_{P_{PHFS-1}} \cdot \gamma_{P_{PHFS-1}}} \min_{\substack{\mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}},\\ \mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}},\\ \mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}}} \left(\frac{\left(1 - \mathcal{M}_{P_{PHFS-1}} + \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\right)}{2} \log_2\left(\frac{2\left(1 - \mathcal{M}_{P_{PHFS-1}} + \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\right)}{\left(1 - \mathcal{M}_{P_{PHFS-1}} + \mathcal{A}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-1}}\right) + \left(1 - \mathcal{M}_{P_{PHFS-2}} + \mathcal{A}_{P_{PHFS-2}} + \mathcal{N}_{P_{PHFS-2}}\right)}\right)\right)\right)^p\right)^{\frac{1}{p}}
\end{array}\right)$$

Which holds the following conditions:

(1) $CE_6 \;\; P_{PHFS-1}, P_{PHFS-2} \;\; \geq 0$;
(2) $CE_6 \;\; P_{PHFS-1}, P_{PHFS-2} \;\; = 0$ *iff* $P_{PHFS-1} = P_{PHFS-2}$;
(3) $CE_6 \;\; P^C_{PHFS-1}, P^C_{PHFS-2} \;\; = CE_6 \;\; P_{PHFS-1}, P_{PHFS-2}$,

where $P^C_{PHFS-1} = \;\; \gamma_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \alpha_{P_{PHFS-1}}(x)$.

Theorem 3.9. *The proposed measures* $CE_5(P_{PHFS-1}, P_{PHFS-2})$ *and* $CE_6(P_{PHFS-1}, P_{PHFS-2})$ *are a picture hesitant fuzzy cross-entropy, and satisfy the above three conditions.*
Proof: *Straightforward. (The proof of this theorem is similar to the proof of the* $theorem 3.1$*)*

Definition 3.10. *Let* $P_{PHFS-1} = \;\; \alpha_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \gamma_{P_{PHFS-1}}(x)$ *and* $P_{PHFS-2} = \;\; \alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-2}}(x), \gamma_{P_{PHFS-2}}(x)$ *be two picture hesitant fuzzy numbers. Then the cross-entropy* $CE_7(P_{PHFS-1}, P_{PHFS-2})$ *is denoted and defined as:*

$$CE_7 : PHFN \times PHFN \rightarrow R^+$$

$$CE_7(P_{PHFS-1}, P_{PHFS-2}) =$$

$$\frac{1}{1-2^{1-q}} \left(\begin{array}{c} max_{\mathcal{M}_{P_{PHFS-1}} \in \alpha_{P_{PHFS-1}}} \\ \left(min_{\mathcal{M}_{P_{PHFS-2}} \in \alpha_{P_{PHFS-2}}} \left(\frac{\mathcal{M}^q_{P_{PHFS-1}} + \mathcal{M}^q_{P_{PHFS-2}}}{2} - \frac{\mathcal{M}_{P_{PHFS-1}} + \mathcal{M}_{P_{PHFS-2}}}{2}^q \right) \right) + \\ max_{\mathcal{A}_{P_{PHFS-1}} \in \beta_{P_{PHFS-1}}} \\ \left(min_{\mathcal{A}_{P_{PHFS-2}} \in \beta_{P_{PHFS-2}}} \left(\frac{\mathcal{A}^q_{P_{PHFS-1}} + \mathcal{A}^q_{P_{PHFS-2}}}{2} - \frac{\mathcal{A}_{P_{PHFS-1}} + \mathcal{A}_{P_{PHFS-2}}}{2}^q \right) \right) + \\ max_{\mathcal{N}_{P_{PHFS-1}} \in \gamma_{P_{PHFS-1}}} \\ \left(min_{\mathcal{N}_{P_{PHFS-2}} \in \gamma_{P_{PHFS-2}}} \left(\frac{\mathcal{N}^q_{P_{PHFS-1}} + \mathcal{N}^q_{P_{PHFS-2}}}{2} - \frac{\mathcal{N}_{P_{PHFS-1}} + \mathcal{N}_{P_{PHFS-2}}}{2}^q \right) \right) \end{array} \right)$$

Which holds the following conditions:

(1) $CE_7(P_{PHFS-1}, P_{PHFS-2}) \geq 0$;
(2) $CE_7(P_{PHFS-1}, P_{PHFS-2}) = 0$ *iff* $P_{PHFS-1} = P_{PHFS-2}$;
(3) $CE_7 \;\; P^C_{PHFS-1}, P^C_{PHFS-2} \;\; = CE_7(P_{PHFS-1}, P_{PHFS-2})$,

where $P^C_{PHFS-1} = \;\; \gamma_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \alpha_{P_{PHFS-1}}(x)$.

Definition 3.11. *Let* $P_{PHFS-1} = \;\; \alpha_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \gamma_{P_{PHFS-1}}(x)$ *and* $P_{PHFS-2} = \;\; \alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-2}}(x), \gamma_{P_{PHFS-2}}(x)$ *be two picture hesitant fuzzy numbers,* $p \geq 1$*. Then the cross-entropy* $CE_8(P_{PHFS-1}, P_{PHFS-2})$ *is denoted and defined as:*

$$CE_8 : PHFN \times PHFN \rightarrow R^+$$

$$CE_8(P_{PHFS-1}, P_{PHFS-2}) =$$

$$\left(\begin{array}{l}\left(\sum_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}}\frac{1}{1-2^{1-q}}\left(\overline{\frac{1}{\alpha_{P_{PHFS-1}}}}\left(\min_{\mathcal{M}_{P_{PHFS-2}}\in\alpha_{P_{PHFS-2}}}\left(\frac{\mathcal{M}^{q}_{P_{PHFS-1}}+\mathcal{M}^{q}_{P_{PHFS-2}}}{2}-\left(\frac{\mathcal{M}_{P_{PHFS-1}}+\mathcal{M}_{P_{PHFS-2}}}{2}\right)^{q}\right)\right)\right)^{p}\right)^{\frac{1}{p}}+\\ \left(\sum_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}}\frac{1}{1-2^{1-q}}\left(\overline{\frac{1}{\beta_{P_{PHFS-1}}}}\left(\min_{\mathcal{A}_{P_{PHFS-2}}\in\beta_{P_{PHFS-2}}}\left(\frac{\mathcal{A}^{q}_{P_{PHFS-1}}+\mathcal{A}^{q}_{P_{PHFS-2}}}{2}-\left(\frac{\mathcal{A}_{P_{PHFS-1}}+\mathcal{A}_{P_{PHFS-2}}}{2}\right)^{q}\right)\right)\right)^{p}\right)^{\frac{1}{p}}+\\ \left(\sum_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}}\frac{1}{1-2^{1-q}}\left(\overline{\frac{1}{\gamma_{P_{PHFS-1}}}}\left(\min_{\mathcal{N}_{P_{PHFS-2}}\in\gamma_{P_{PHFS-2}}}\left(\frac{\mathcal{N}^{q}_{P_{PHFS-1}}+\mathcal{N}^{q}_{P_{PHFS-2}}}{2}-\left(\frac{\mathcal{N}_{P_{PHFS-1}}+\mathcal{N}_{P_{PHFS-2}}}{2}\right)^{q}\right)\right)\right)^{p}\right)^{\frac{1}{p}}\end{array}\right)$$

Which holds the following conditions, for $1 < q \le 2, p \ge 1$*:*

(1) $CE_8\left(P_{PHFS-1}, P_{PHFS-2}\right) \ge 0$*;*
(2) $CE_8\left(P_{PHFS-1}, P_{PHFS-2}\right) = 0$ *iff* $P_{PHFS-1} = P_{PHFS-2}$*;*
(3) $CE_8\left(P^{C}_{PHFS-1}, P^{C}_{PHFS-2}\right) = CE_8\left(P_{PHFS-1}, P_{PHFS-2}\right)$*,*

where $P^{C}_{PHFS-1} = \left(\gamma_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \alpha_{P_{PHFS-1}}(x)\right)$.

Theorem 3.12. *The proposed measures* $CE_7\left(P_{PHFS-1}, P_{PHFS-2}\right)$ *and* $CE_8\left(P_{PHFS-1}, P_{PHFS-2}\right)$ *are a picture hesitant fuzzy cross-entropy, and satisfy the above three conditions.*

Proof: *Straightforward. (The proof of this theorem is similar to the proof of the* $theorem 3.1$*)*

Definition 3.13. *Let* $P_{PHFS-1} = \left(\alpha_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \gamma_{P_{PHFS-1}}(x)\right)$ *and* $P_{PHFS-2} = \left(\alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-2}}(x), \gamma_{P_{PHFS-2}}(x)\right)$ *be two picture hesitant fuzzy numbers. Then the cross-entropy* $CE_9\left(P_{PHFS-1}, P_{PHFS-2}\right)$ *is denoted and defined as:*

$$CE_9 : PHFN \times PHFN \to R^{+}$$

$$CE_9\left(P_{PHFS-1}, P_{PHFS-2}\right) =$$

$$\frac{1}{T}\left(\begin{array}{c}
\left(\begin{array}{c} max_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}} \\ min_{\mathcal{M}_{P_{PHFS-2}}\in\alpha_{P_{PHFS-2}}} \\ \left(\left(\frac{\left(\begin{array}{c}\left(1+q\mathcal{M}_{P_{PHFS-1}}\right)\ln\left(1+q\mathcal{M}_{P_{PHFS-1}}\right)+\\ \left(1+q\mathcal{M}_{P_{PHFS-2}}\right)\ln\left(1+q\mathcal{M}_{P_{PHFS-2}}\right)\end{array}\right)}{2}\right)-\left(\frac{2+q\mathcal{M}_{P_{PHFS-1}}+q\mathcal{M}_{P_{PHFS-2}}}{2}\ln\frac{2+q\mathcal{M}_{P_{PHFS-1}}+q\mathcal{M}_{P_{PHFS-2}}}{2}\right)\right)\end{array}\right)+ \\
\left(\begin{array}{c} max_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}} \\ min_{\mathcal{A}_{P_{PHFS-2}}\in\beta_{P_{PHFS-2}}} \\ \left(\left(\frac{\left(\begin{array}{c}\left(1+q\mathcal{A}_{P_{PHFS-1}}\right)\ln\left(1+q\mathcal{A}_{P_{PHFS-1}}\right)+\\ \left(1+q\mathcal{A}_{P_{PHFS-2}}\right)\ln\left(1+q\mathcal{A}_{P_{PHFS-2}}\right)\end{array}\right)}{2}\right)-\left(\frac{2+q\mathcal{A}_{P_{PHFS-1}}+q\mathcal{A}_{P_{PHFS-2}}}{2}\ln\frac{2+q\mathcal{A}_{P_{PHFS-1}}+q\mathcal{A}_{P_{PHFS-2}}}{2}\right)\right)\end{array}\right)+ \\
\left(\begin{array}{c} max_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}} \\ min_{\mathcal{N}_{P_{PHFS-2}}\in\gamma_{P_{PHFS-2}}} \\ \left(\left(\frac{\left(\begin{array}{c}\left(1+q\mathcal{N}_{P_{PHFS-1}}\right)\ln\left(1+q\mathcal{N}_{P_{PHFS-1}}\right)+\\ \left(1+q\mathcal{N}_{P_{PHFS-2}}\right)\ln\left(1+q\mathcal{N}_{P_{PHFS-2}}\right)\end{array}\right)}{2}\right)-\left(\frac{2+q\mathcal{N}_{P_{PHFS-1}}+q\mathcal{N}_{P_{PHFS-2}}}{2}\ln\frac{2+q\mathcal{N}_{P_{PHFS-1}}+q\mathcal{N}_{P_{PHFS-2}}}{2}\right)\right)\end{array}\right)
\end{array}\right)$$

Which holds the following conditions:

(1) $CE_9\left(P_{PHFS-1}, P_{PHFS-2}\right) \geq 0$*;*
(2) $CE_9\left(P_{PHFS-1}, P_{PHFS-2}\right) = 0$ *iff* $P_{PHFS-1} = P_{PHFS-2}$*;*
(3) $CE_9\left(P^C_{PHFS-1}, P^C_{PHFS-2}\right) = CE_9\left(P_{PHFS-1}, P_{PHFS-2}\right)$*,*

where $P^C_{PHFS-1} = \left\{\gamma_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \alpha_{P_{PHFS-1}}(x)\right\}$.

Definition 3.14. *Let* $P_{PHFS-1} = \left\{\alpha_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \gamma_{P_{PHFS-1}}(x)\right\}$ *and* $P_{PHFS-2} = \left\{\alpha_{P_{PHFS-2}}(x), \beta_{P_{PHFS-2}}(x), \gamma_{P_{PHFS-2}}(x)\right\}$ *be two picture hesitant fuzzy numbers,* $p \geq 1$*.*

Then the cross-entropy $CE_{10}\left(P_{PHFS-1}, P_{PHFS-2}\right)$ *is denoted and defined as:*

$$CE_{10} : PHFN \times PHFN \rightarrow R^{+}$$

$$CE_{10}\left(P_{PHFS-1}, P_{PHFS-2}\right) =$$

$$\left(\begin{array}{l}
\left(\sum_{\mathcal{M}_{P_{PHFS-1}}\in\alpha_{P_{PHFS-1}}}\frac{1}{T}\left(\frac{1}{\alpha_{P_{PHFS-1}}}\min_{\mathcal{M}_{P_{PHFS-2}}\in\alpha_{P_{PHFS-2}}}\left(\left(\frac{\left(\begin{array}{c}\left(1+q\mathcal{M}_{P_{PHFS-1}}\right)\ln\left(1+q\mathcal{M}_{P_{PHFS-1}}\right)+\\ \left(1+q\mathcal{M}_{P_{PHFS-2}}\right)\ln\left(1+q\mathcal{M}_{P_{PHFS-2}}\right)\end{array}\right)}{2}\right)-\left(\frac{2+q\mathcal{M}_{P_{PHFS-1}}+q\mathcal{M}_{P_{PHFS-2}}}{2}\ln\left(\frac{2+q\mathcal{M}_{P_{PHFS-1}}+q\mathcal{M}_{P_{PHFS-2}}}{2}\right)\right)\right)\right)^{p}\right)^{\frac{1}{p}}+\\
\left(\sum_{\mathcal{A}_{P_{PHFS-1}}\in\beta_{P_{PHFS-1}}}\frac{1}{T}\left(\frac{1}{\beta_{P_{PHFS-1}}}\min_{\mathcal{A}_{P_{PHFS-2}}\in\beta_{P_{PHFS-2}}}\left(\left(\frac{\left(\begin{array}{c}\left(1+q\mathcal{A}_{P_{PHFS-1}}\right)\ln\left(1+q\mathcal{A}_{P_{PHFS-1}}\right)+\\ \left(1+q\mathcal{A}_{P_{PHFS-2}}\right)\ln\left(1+q\mathcal{A}_{P_{PHFS-2}}\right)\end{array}\right)}{2}\right)-\left(\frac{2+q\mathcal{A}_{P_{PHFS-1}}+q\mathcal{A}_{P_{PHFS-2}}}{2}\ln\left(\frac{2+q\mathcal{A}_{P_{PHFS-1}}+q\mathcal{A}_{P_{PHFS-2}}}{2}\right)\right)\right)\right)^{p}\right)^{\frac{1}{p}}+\\
\left(\sum_{\mathcal{N}_{P_{PHFS-1}}\in\gamma_{P_{PHFS-1}}}\frac{1}{T}\left(\frac{1}{\gamma_{P_{PHFS-1}}}\min_{\mathcal{N}_{P_{PHFS-2}}\in\gamma_{P_{PHFS-2}}}\left(\left(\frac{\left(\begin{array}{c}\left(1+q\mathcal{N}_{P_{PHFS-1}}\right)\ln\left(1+q\mathcal{N}_{P_{PHFS-1}}\right)+\\ \left(1+q\mathcal{N}_{P_{PHFS-2}}\right)\ln\left(1+q\mathcal{N}_{P_{PHFS-2}}\right)\end{array}\right)}{2}\right)-\left(\frac{2+q\mathcal{N}_{P_{PHFS-1}}+q\mathcal{N}_{P_{PHFS-2}}}{2}\ln\left(\frac{2+q\mathcal{N}_{P_{PHFS-1}}+q\mathcal{N}_{P_{PHFS-2}}}{2}\right)\right)\right)\right)^{p}\right)^{\frac{1}{p}}
\end{array}\right)$$

Which holds the following conditions, for $1 < q \leq 2, p \geq 1$*:*

(1) $CE_{10}\left(P_{PHFS-1}, P_{PHFS-2}\right) \geq 0$;

(2) $CE_{10}\left(P_{PHFS-1}, P_{PHFS-2}\right) = 0$ *iff* $P_{PHFS-1} = P_{PHFS-2}$;

(3) $CE_{10}\left(P^{C}_{PHFS-1}, P^{C}_{PHFS-2}\right) = CE_{10}\left(P_{PHFS-1}, P_{PHFS-2}\right)$,

where $P^{C}_{PHFS-1} = \left(\gamma_{P_{PHFS-1}}(x), \beta_{P_{PHFS-1}}(x), \alpha_{P_{PHFS-1}}(x)\right)$.

We choose the values of $T = (1+q)\ln(1+q) - (2+q)(\ln(2+q) - \ln 2)$. *The symmetric discrimination information measure for picture hesitant fuzzy numbers is follow as:*

$$CE_i^* \; P_{PHFS-1}, P_{PHFS-2}$$

$$= CE_i \; P_{PHFS-1}, P_{PHFS-2} \; + CE_i \; P_{PHFS-2}, P_{PHFS-1}$$

Theorem 3.15. *The proposed measures* $CE_9(P_{PHFS-1}, P_{PHFS-2})$ *and* $CE_{10}(P_{PHFS-1}, P_{PHFS-2})$ *are a picture hesitant fuzzy cross-entropy, and satisfy the above three conditions.*

Proof: *Straightforward. (The proof of this theorem is similar to the proof of the* $theorem 3.1$*)*

4. Multi Criteria Decision Making Methods Based on Cross-Entropy Measures of Picture Hesitant Fuzzy Numbers

Multi criteria decision making ranking problems with picture hesitant fuzzy information depend n alternatives, represented by P_{PHFS-i} $(i = 1, 2, 3, \ldots, n)$ with respect to m criteria expressing by C_j $(j = 1, 2, 3 \ldots, m)$. The weight vectors for $w_i \in [0,1], \sum_{i=1}^n w_i = 1$ with a condition $w_i^- \leq w_i \leq w_i^+$ such that $\sum_{i=1}^n w_i^+ \geq 1, \sum_{i=1}^n w_i^- \leq 1$. The picture hesitant fuzzy numbers is defined by
$P_{PHFS-ij} = \;\; \alpha_{P_{PHFS-ij}}(x), \beta_{P_{PHFS-ij}}(x), \gamma_{P_{PHFS-ij}}(x)$. The steps of the algorithm is follow as:

Step 1: We construct the decision matrix whose every entities in the form of picture hesitant fuzzy numbers.

Step 2: We calculate the weight vector, for this first we choose the positive and negative ideal solution such that $P_{Positive\ ideal} = (1,0,0)$ and $P_{Negative\ ideal} = (0,0,1)$. Then we use the following formula such that

$$CE_1^*(P_{PHFS-ij}, P_{PHFS-ij})$$

$$= CE_1(P_{PHFS-ij}, P_{Positive\ ideal}) + CE_1(P_{Positive\ ideal}, P_{PHFS-ij})$$

$$= \left(\begin{array}{c} max_{\mathcal{M}_{P_{PHFS-ij}} \in \alpha_{P_{PHFS-ij}}} \\ \left(\begin{array}{c} min_{\mathcal{M}_{P_{Positive\ ideal}} \in \alpha_{P_{Positive\ ideal}}} \\ \left(\begin{array}{c} \mathcal{M}_{P_{PHFS-ij}} \\ \log_2 \quad \frac{2\mathcal{M}_{P_{PHFS-ij}}}{\mathcal{M}_{P_{PHFS-ij}} + \mathcal{M}_{P_{Positive\ ideal}}} \end{array} \right) \end{array} \right) + \\ max_{\mathcal{A}_{P_{PHFS-ij}} \in \beta_{P_{PHFS-ij}}} \\ \left(\begin{array}{c} min_{\mathcal{A}_{P_{Positive\ ideal}} \in \beta_{P_{Positive\ ideal}}} \\ \left(\begin{array}{c} \mathcal{A}_{P_{PHFS-ij}} \\ \log_2 \quad \frac{2\mathcal{A}_{P_{PHFS-ij}}}{\mathcal{A}_{P_{PHFS-ij}} + \mathcal{A}_{P_{Positive\ ideal}}} \end{array} \right) \end{array} \right) + \\ max_{\mathcal{N}_{P_{PHFS-ij}} \in \gamma_{P_{PHFS-ij}}} \\ \left(\begin{array}{c} min_{\mathcal{N}_{P_{Positive\ ideal}} \in \gamma_{P_{Positive\ ideal}}} \\ \left(\begin{array}{c} \mathcal{N}_{P_{PHFS-ij}} \\ \log_2 \quad \frac{2\mathcal{N}_{P_{PHFS-ij}}}{\mathcal{N}_{P_{PHFS-ij}} + \mathcal{N}_{P_{Positive\ ideal}}} \end{array} \right) \end{array} \right) \end{array} \right) +$$

$$\left(\begin{array}{c} max_{\mathcal{M}_{P_{Positive\ ideal}}\in\alpha_{P_{Positive\ ideal}}} \\ \left(min_{\mathcal{M}_{P_{PHFS-ij}}\in\alpha_{P_{PHFS-ij}}}\left(\begin{array}{c}\mathcal{M}_{P_{Positive\ ideal}} \\ \log_2 \frac{2\mathcal{M}_{P_{Positive\ ideal}}}{\mathcal{M}_{P_{Positive\ ideal}}+\mathcal{M}_{P_{PHFS-ij}}}\end{array}\right)\right)+ \\ max_{\mathcal{A}_{P_{Positive\ ideal}}\in\beta_{P_{Positive\ ideal}}} \\ \left(min_{\mathcal{A}_{P_{PHFS-ij}}\in\beta_{P_{PHFS-ij}}}\left(\begin{array}{c}\mathcal{A}_{P_{Positive\ ideal}} \\ \log_2 \frac{2\mathcal{A}_{P_{Positive\ ideal}}}{\mathcal{A}_{P_{Positive\ ideal}}+\mathcal{A}_{P_{PHFS-ij}}}\end{array}\right)\right)+ \\ max_{\mathcal{N}_{P_{Positive\ ideal}}\in\gamma_{P_{Positive\ ideal}}} \\ \left(min_{\mathcal{N}_{P_{PHFS-ij}}\in\gamma_{P_{PHFS-ij}}}\left(\begin{array}{c}\mathcal{N}_{P_{Positive\ ideal}} \\ \log_2 \frac{2\mathcal{N}_{P_{Positive\ ideal}}}{\mathcal{N}_{P_{Positive\ ideal}}+\mathcal{N}_{P_{PHFS-ij}}}\end{array}\right)\right)\end{array}\right)$$

and

$$CE_1^*\left(P_{PHFS-ij}, P_{PHFS-ij}\right)$$

$$= CE_1\left(P_{PHFS-ij}, P_{Negative\ ideal}\right) + CE_1\left(P_{Negative\ ideal}, P_{PHFS-ij}\right)$$

$$= \left(\begin{array}{c} max_{\mathcal{M}_{P_{PHFS-ij}}\in\alpha_{P_{PHFS-ij}}} \\ \left(min_{\mathcal{M}_{P_{Negative\ ideal}}\in\alpha_{P_{Negative\ ideal}}}\left(\begin{array}{c}\mathcal{M}_{P_{PHFS-ij}} \\ \log_2 \frac{2\mathcal{M}_{P_{PHFS-ij}}}{\mathcal{M}_{P_{PHFS-ij}}+\mathcal{M}_{P_{Negative\ ideal}}}\end{array}\right)\right)+ \\ max_{\mathcal{A}_{P_{PHFS-ij}}\in\beta_{P_{PHFS-ij}}} \\ \left(min_{\mathcal{A}_{P_{Negative\ ideal}}\in\beta_{P_{Negative\ ideal}}}\left(\begin{array}{c}\mathcal{A}_{P_{PHFS-ij}} \\ \log_2 \frac{2\mathcal{A}_{P_{PHFS-ij}}}{\mathcal{A}_{P_{PHFS-ij}}+\mathcal{A}_{P_{Negative\ ideal}}}\end{array}\right)\right)+ \\ max_{\mathcal{N}_{P_{PHFS-ij}}\in\gamma_{P_{PHFS-ij}}} \\ \left(min_{\mathcal{N}_{P_{Negative\ ideal}}\in\gamma_{P_{Negative\ ideal}}}\left(\begin{array}{c}\mathcal{N}_{P_{PHFS-ij}} \\ \log_2 \frac{2\mathcal{N}_{P_{PHFS-ij}}}{\mathcal{N}_{P_{PHFS-ij}}+\mathcal{N}_{P_{Negative\ ideal}}}\end{array}\right)\right)\end{array}\right)+$$

$$\left(\begin{array}{c} max_{\mathcal{M}_{P_{Negative\ ideal}} \in \alpha_{P_{Negative\ ideal}}} \\ \left(min_{\mathcal{M}_{P_{PHFS-ij}} \in \alpha_{P_{PHFS-ij}}} \left(\mathcal{M}_{P_{Negative\ ideal}} \log_2 \frac{2\mathcal{M}_{P_{Negative\ ideal}}}{\mathcal{M}_{P_{Negative\ ideal}} + \mathcal{M}_{P_{PHFS-ij}}} \right) \right) + \\ max_{\mathcal{A}_{P_{Negative\ ideal}} \in \beta_{P_{Negative\ ideal}}} \\ \left(min_{\mathcal{A}_{P_{PHFS-ij}} \in \beta_{P_{PHFS-ij}}} \left(\mathcal{A}_{P_{Negative\ ideal}} \log_2 \frac{2\mathcal{A}_{P_{Negative\ ideal}}}{\mathcal{A}_{P_{Negative\ ideal}} + \mathcal{A}_{P_{PHFS-ij}}} \right) \right) + \\ max_{\mathcal{N}_{P_{Negative\ ideal}} \in \gamma_{P_{Negative\ ideal}}} \\ \left(min_{\mathcal{N}_{P_{PHFS-ij}} \in \gamma_{P_{PHFS-ij}}} \left(\mathcal{N}_{P_{Negative\ ideal}} \log_2 \frac{2\mathcal{N}_{P_{Negative\ ideal}}}{\mathcal{N}_{P_{Negative\ ideal}} + \mathcal{N}_{P_{PHFS-ij}}} \right) \right) \end{array}\right)$$

Step 3: We examine the closeness of the negative ideal and alternatives by using the following equations, such that

$$G\left(P_{PHFN-j}\right) = \sum_{i=1}^{n} w_i G_{ij}, G_{ij} = \frac{G_{ij}^{-}}{G_{ij}^{+} + G_{ij}^{-}}$$

The linear programming model is of the form:

$$maxG = \sum_{j=1}^{m} G\left(P_{PHFN-j}\right) = \sum_{j=1}^{m} \sum_{i=1}^{n} w_i G_{ij}$$

Step 4: Rank the alternative and examine the best one.
Step 5: The end.

(1) (a) **An illustrate example**

A company has decided to choose a pool of alternatives from several foreign countries based on preliminary surveys. In this survey, we find a suitable candidate countries, which is represented by a_1, a_2, a_3 and a_4. During the assessment, four factors including, c_1: Politics and Policy, c_2: Infrastructure, c_3: Resource and c_4: Economy. The weight vector for above alternatives and their attributes i.e. $H = \left\{ \begin{array}{c} 0.15 \leq w_1 \leq 0.3, \\ 0.15 \leq w_2 \leq 0.25, \\ 0.25 \leq w_3 \leq 0.4, \\ 0.3 \leq w_4 \leq 0.45, \end{array} \right\}$ such that $\sum_{i=1}^{n} w_i = 1$.
The steps of the algorithm is follow as:
Step 1: First we construct the decision matrix whose every entities in the form of picture hesitant fuzzy numbers, which is utilized in the form of table 2

Table 2. Original decision matrix, whose every entries in the form of picture hesitant fuzzy number.

Symbols	C_1	C_2	C_3	C_4
a_1	$\left(\{0.3, 0.4\}, \{0.1\}, 0.1, 0.2, 0.22\right)$	$\left(\{0.01, 0.4\}, \{0.13\}, \{0.13, 0.21\}\right)$	$\left(\{0.03, 0.04\}, \{0.23\}, \{0.1, 0.22\}\right)$	$\left(\{0.3, 0.4\}, \{0.1, 0.23\}, \{0.23, 0.22\}\right)$
a_2	$\left(\{0.13, 0.14\}, \{0.11\}, 0.11, 0.12, 0.122\right)$	$\left(\{0.23, 0.24\}, \{0.11\}, \{0.1, 0.12\}\right)$	$\left(\{0.03, 0.04\}, \{0.01\}, 0.1, 0.02, 0.12\right)$	$\left(\{0.23, 0.21\}, \{0.14\}, \{0.21, 0.22\}\right)$
a_3	$\left(\{0.03, 0.04\}, \{0.23\}, \{0.1, 0.22\}\right)$	$\left(\{0.3, 0.4\}, \{0.1, 0.23\}, \{0.23, 0.22\}\right)$	$\left(0.11, 0.12, 0.14, \{0.1, 23\}, \{0.1, 0.32\}\right)$	$\left(\{0.3, 0.4\}, \{0.1\}, 0.1, 0.2, 0.22\right)$
a_4	$\left(\{0.03, 0.04\}, \{0.01\}, 0.1, 0.02, 0.12\right)$	$\left(\{0.23, 0.21\}, \{0.14\}, \{0.21, 0.22\}\right)$	$\left(\{0.03, 0.04\}, \{0.01\}, 0.1, 0.02, 0.12\right)$	$\left(\{0.23, 0.21\}, \{0.14\}, \{0.21, 0.22\}\right)$

Step 2: We calculate the weight vector, for this first we choose the positive and negative ideal solution such that $P_{Positive\ ideal} = (1, 0, 0)$ and $P_{Negative\ ideal} = (0, 0, 1)$. Then we use the following formula such that

$$CE_1^* \ P_{PHFS-11}, P_{Positive\ ideal} \ = CE_1 \ P_{PHFS-11}, P_{Positive\ ideal} \ + CE_1 \ P_{Positive\ ideal}, P_{PHFS-11}$$

$$= \max \ \min \ 0.3\log_2 \frac{2 \times 0.3}{0.3 + 1} \ , \min \ 0.4\log_2 \frac{2 \times 0.4}{0.4 + 1} \ +$$

$$\max \ \min \ 0.1\log_2 \frac{2 \times 0.1}{0.1 + 0} \ +$$

$$\max \left\{ \begin{array}{c} \min \ 0.1\log_2 \frac{2\times0.1}{0.1+0} \ , \min \ 0.2\log_2 \frac{2\times0.2}{0.2+0} \ , \\ \min \ 0.22\log_2 \frac{2\times0.22}{0.22+0} \end{array} \right\}$$

$$+\max \ \min \ \log_2 \frac{2 \times 1}{0.3 + 1} \ , \min \ \log_2 \frac{2 \times 1}{0.4 + 1} \ + 0 + 0$$

$$= \max(-0.67, -0.097) - 0.06+$$

$$\max(-0.06, -0.14, 0.066) + \max(0.19, 0.15)$$

$$= -0.097 - 0.06 + 0.066 + 0.19 = 0.099$$

And similarly for $CE_1^* (P_{PHFS-11}, P_{Negative\ ideal}) = 1.052$

Step 3: We examine the closeness of the negative ideal and alternatives by using the following equations, such that

$$G_{11} = \frac{CE_1^* (P_{PHFS-11}, P_{Positive\ ideal})}{\begin{array}{c} CE_1^* (P_{PHFS-11}, P_{Positive\ ideal}) + \\ CE_1^* (P_{PHFS-11}, P_{Negative\ ideal}) \end{array}}$$

$$= \frac{0.099}{0.099 + 0.952} = 0.09420$$

$$G_{12} = 0.023, G_{13} = 0.0423, G_{14} = 0.03, G_{21} = 0.032,$$

$$G_{22} = 0.0243, G_{23} = 0.0445, G_{24} = 0.0562, G_{31} = 0.31,$$

$$G_{32} = 0.024, G_{33} = 0.0671, G_{34} = 0.0779,$$

$$G_{41} = 0.012, G_{42} = 0.23, G_{43} = 0.03, G_{44} = 0.651$$

The linear programming model is of the form:

$$maxG = 2.9487w_1 + 2.6077w_2$$

$$+2.7233w_3 + 3.4084w_4$$

$$w = \begin{array}{c} 0.15, 0.15, \\ 0.375, 0.325 \end{array}$$

Step 4: Rank the alternative and examine the best one.

$$G_1(a_1) = 0.04319, G_2(a_2) = 0.0434,$$

$$G_3(a_3) = 0.1, G_4(a_4) = 0.259$$

$$G_4(a_4) \geq G_3(a_3) \geq G_2(a_2) \geq G_1(a_1)$$

Hence $G_4(a_4)$ is the best strategy. For $p = q = 2$.
Step 5: The end.
The comparative analysis of the explored measures with some other existing measures is follow as: the generalized distance and similarity measures based on picture hesitant fuzzy set was explored by Jan et al. [37] and some similarity measures based on picture hesitant fuzzy set were presented by Ahmad et al. [38]. The comparison between explored measures with existing measures [37, 38] is discussed in table 3.

Table 3. Comparison between explored work with some existing works.

Methods	**Similarity Values**	**Ranking Values**
Jan et al. [37]	$G_1(a_1) = 0.04353, G_2(a_2) = 0.0439,$ $G_3(a_3) = 0.98, G_4(a_4) = 0.246$	$G_4(a_4) \geq G_3(a_3)$ $\geq G_2(a_2) \geq G_1(a_1)$
Ahmad et al. [38]	$G_1(a_1) = 0.04820, G_2(a_2) = 0.0495,$ $G_3(a_3) = 0.13, G_4(a_4) = 0.273$	$G_4(a_4) \geq G_3(a_3)$ $\geq G_2(a_2) \geq G_1(a_1)$
Proposed Measures	$G_1(a_1) = 0.04319, G_2(a_2) = 0.0434,$ $G_3(a_3) = 0.1, G_4(a_4) = 0.259$	$G_4(a_4) \geq G_3(a_3)$ $\geq G_2(a_2) \geq G_1(a_1)$

From the above analysis, it is clear that the all existing works [37, 38] and the explored measure are provides the same results, which is discussed in table 3. The best alternative is $G_4(a_4)$. When we consider the positive, abstinence and negative grade in the form of singleton set, then the established work is reduced into picture fuzzy set. The explored work is more general than existing drawbacks because of its structure.

5. Conclusion

In this article, we exploit the cross-entropy of picture hesitant fuzzy set is established by distinguishing the cross-entropy of picture fuzzy set and hesitant fuzzy set. First, many measurement concepts are established and their basic properties are studied. Further, two measures, which are based on the established picture hesitant fuzzy cross-entropy, are established for evaluating multi-criteria decision making problems in the environment of picture hesitant fuzzy sets. For both approaches, an optimization model is pioneered in order to examine the weight vector for multi criteria decision making problems with incomplete information on criteria weights. In last, we give an example to express the practically and effectiveness. The comparison of the proposed measures with existing measures are also discussed in detail. We aim to broaden our study in the future with the analysis of (1) Complex q-rung orthopair fuzzy sets, (2) Complex bipolar Neutrosophic sets, (3) Fuzzy rough soft sets and (4) Fuzzy rough Neutrosophic sets. The existing methods are discussed in [39-50] are also utilized in the environment of proposed approaches.

6. ACKNOWLEDGMENTS

I would like to thank my supervisor, Prof. Tahir Mahmood, for the patient guidance, encouragement and advice he has provided throughout my time as his student. I have been extremely lucky to have a supervisor who cared so much about my work, and who responded to my questions and queries so promptly.

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