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A Note on Rightt Abelian Distributive AG-groupoids

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Abstract. A magma that satisfies the left invertive law: $ab \cdot c = cb \cdot a$ is called an AG-groupoid. The concept of right (left) abelian distributive groupoid (RAD resp. LAD) is extended to introduce some new subclasses of an AG-groupoid as right (left) abelian distributive AG-groupoid. The enumerations for these subclasses up to order 6 is provided using a modern computational techniques of GAP and various relations of these new subclasses are investigated with some other existing subclasses of AG-groupoids and other relevant algebraic structures. A manual procedure for the verification of an arbitrary finite AG-groupoid for RAD AG-groupoid is introduced. Various examples and counterexamples are produced with Prover-9 and Mace-4 to strengthen the validity of the produced results.

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1. Introduction

An AG-groupoid Q is a generalization of a commutative semigroup, in which the left invertive law (L.I.L) holds [1].

$$(ab)c = (cb)a. \tag{1.1}$$

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A groupoid is called medial if it satisfies the medial law:

$$ab \cdot cd = ac \cdot bd. \tag{1.2}$$

It is is easy to show that every AG-groupoid is medial. An AG-groupoid is called monoid if it contains a unique left identity. It is easy to prove that very monoid satisfies the paramedial law:

$$ab \cdot cd = db \cdot ca. \tag{1.3}$$

It is also worthwhile to mention that if Q possesses the right identity element then it becomes a semigroup. A groupoid Q is known as left (resp. right) abelian distributive [2], if it satisfies $a \cdot bc = ab \cdot ca$ (resp. $ab \cdot c = ca \cdot bc$). The concepts of these groupoids are extended here to left (resp. right) abelian distributive AG-groupoids. The existence of these abelian distributive AG-groupoids are proved by computationally generated non-associative examples of various finite orders. Further, we also establish their relations with some of the existing subclasses [9, **?**, 14, 15, 22, 25] of AG-groupoids and that with some more useful algebraic structures. AG-groupoids have been enumerated up to order 6 [4] using GAP [3]. We also use the same techniques to enumerate our new subclasses of abelian distributive AG-groupoids. We shall use juxtaposition and the notation "$\cdot$" to avoid repeated use of parenthesis, e.g., $(ab \cdot c)d$ shall denote the same as $((a \cdot b) \cdot c) \cdot d$ likewise $(ab)c$ and $ab \cdot c$ will represent the same element. AG-groupoids have a variety of applications in flocks, geometry, and matrices [1, 7, 16]. Recently, a considerable research has been done in this area and is being investigated as other well established areas of algebra [6, 8, 10, 17, 18, 19, 20]. In the following we give some preliminary concepts and basic definitions of AG-groupoids with their identities that shall be referred in the rest of this article.

By simple application of medial law every AG-groupoid S that satisfies the paramedial law (1.3) also satisfies the following law:

$$ab \cdot cd = dc \cdot ba. \tag{1.4}$$

An AG-groupoid S is called —

(i) — right (resp. left) commutative AG-groupoid, if $a \cdot bc = a \cdot cb$ (resp.$ab \cdot c = ba \cdot c$) holds in S.

(ii) — self-dual AG-groupoid, if $a \cdot bc = c \cdot ba$ holds.

(iii) — left/right distributive (LD) (resp. RD) AG-groupoid, if $a \cdot bc = ab \cdot ac$ (resp.$ab \cdot c = ac \cdot bc$) is satisfied [14].

(iv) — an AG**-groupoid if it holds the identity $a(bc) = b(ac)$.

(v) — flexible AG-groupoid if the law $a(ba) = (ab)a$ is satisfied [2].

(vi) — right/left Bol AG-groupoid; i.e. $a(bc \cdot b) = (ab \cdot c)b/a(b \cdot ac) = a(ba \cdot c)$ [7] and is called Bol AG-groupoid if it is left and right Bol.

(vii) — Moufang AG-groupoid i.e., $ab \cdot ca = (a \cdot bc)a$ [7].

(viii) — Jordan AG-groupoid. i.e., $ab \cdot aa = a(b \cdot aa)$ [7].

(ix) — cyclic associative (CA) i.e., $a(bc) = c(ab)$ [26].

(x) — left Cheban* (resp., right Cheban*) AG-groupoid i.e., $a(bc \cdot d) = ca \cdot bd$ (resp.,$(a \cdot bc)d = ad \cdot cb$) holds [20].

(xi) — left Cheban (resp., right Cheban) AG-groupoid if $a(ab \cdot c) = ba \cdot ac$, (resp., $(a \cdot bc)c = ac \cdot cb$) holds [2].

(xii) — RP- (resp., LP) AG-groupoid if $a(bd \cdot c) = d(ba \cdot c)$ (resp., $(ab \cdot d)c = (ac \cdot d)b$) holds [14].
(xiii) — left transitive if $ab \cdot ac = bc$ holds [2, 7].
(xiv) — slim AG-groupoid if $a(bc) = ac$ holds [2, 28].
(xv) — anti-rectangular AG-groupoid if $(ab)a = b$ holds [2, 24].
(xvi) — idempotent or 2-band (resp., 3-band) AG-groupoid if $a^2 = a$ (resp., $a(aa) = (aa)a = a$).
(xvii) —T^1(resp., T^2) AG-groupoid if $ab = cd \Rightarrow ba = dc$ (resp., $ab = cd \Rightarrow ac = bd$) holds [7].

An AG-groupoid S is called left (resp., right/middle) nuclear square LNS (resp. R/MNS) if $a^2 \cdot bc = a^2b \cdot c$ (resp., $ab \cdot c^2 = a \cdot bc^2/a \cdot b^2c = ab^2 \cdot c$) for all a, b, c in S and is called nuclear square if it is left right and middle nuclear square. Before to put hands on characterization of the RAD AG-groupoids we define and introduce left abelian distributive and the distributive AG-groupoids.

2. Abelian Distributive AG-groupoids

Definition 2.1. *An AG-groupoid S is called left abelian distributive AG-groupoid, abbreviated as LAD if* $\forall\, a, b, c \in S$,

$$a \cdot bc \quad = \quad ab \cdot ca. \tag{2.5}$$

Example 2.2. *Let $Q = \{1, 2, 3, 4\}$ with the Caley's table as given below. Then $(Q, *)$ is an LAD AG-groupoid.*

*	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	2
4	1	1	1	3

Definition 2.3. [14] *An AG-groupoid S is called right abelian distributive (or shortly RAD) AG-groupoid if* $\forall a, b, c \in S$,

$$ab \cdot c \quad = \quad ca \cdot bc. \tag{2.6}$$

Example 2.4. *Let $(Q, *)$ with the Calay's table as under, then Q is an RAD AG-groupoid.*

*	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*b*	*a*

An AG-groupoid Q is called abelian distributive AG-groupoid abbreviated as AD-AG-groupoid if it is both RAD and LAD AG-groupoid.

Example 2.5. *Let $Q = \{1, 2, 3\}$. Then $(Q, *)$ is an associative AD AG-groupoid of order* 3.

*	1	2	3
1	1	1	1
2	1	1	1
3	1	1	2

It is evident from the enumeration of abelian distributive AG-groupoid in Section 5, Table 1 that the number of such non associative AG-groupoids are zero up to order 6. However, the following result not only validates the enumeration for the same subclass but also provides a more authentic result regarding non-existence of the same subclass of any order.

Theorem 2.6. *Every AD-AG-groupoid is associative.*

Proof. Easy. □

2.7. **RAD-AG -Test.** In this section, the idea of Protic et al. [12] is extended here, we discuss a procedure for checking a finite arbitrary AG-groupoid $(G, \cdot)$ to be an RAD-AG-groupoid or not, to do this, for any $a, b \in G$ and any fixed element c of G, we define two new binary operations as follows:

$$a \otimes b = ca \cdot bc, \tag{2.7}$$

$$a \ominus b = ab \cdot c. \tag{2.8}$$

The law $ab \cdot c = ca \cdot bc$ is satisfied if;

$$a \otimes b = a \ominus b. \tag{2.9}$$

To construct the extended table for the operation " $\otimes$ " of any fixed $c \in G$, we re-write the c-row of the " $\cdot$ " table as an index row of the new extended table and multiply its entries by the entries of the c-column of the " $\cdot$ " table to produce the rows of the " $\otimes$ " tables.

Similarly, the extended table for the operation "$\ominus$" of a fixed element $c \in G$ is obtained by multiplying the fixed element $c \in G$ by the entries of the " $\cdot$ " table from the left. The process is repeated for each c in G. If all the tables constructed for the operation " $\otimes$ " and " $\ominus$ " coincide for each respective $c \in G$, then evidently (2.6) satisfies, and the given AG-groupoid is an RAD-AG-groupoid.

Example 2.8. *Verify the following Calay's table of an AG-groupoid* $G = \{a, b, c\}$ *for RAD-AG-groupoid.*

·	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*b*	*b*	*b*

The given table is extended in the way as described above for the purpose to verify it as in the following:

·	a	b	c	a	a	a	a	a	a	b	b	b
a	a	a	a	a	a	a	a	a	a	a	a	a
b	a	a	a	a	a	a	a	a	a	a	a	a
c	b	b	b	a	a	a	a	a	a	a	a	a
		a	a	a		a	a	a		a	a	a
	a	a	a	a	b	a	a	a	c	a	a	a
		a	a	a		a	a	a		a	a	a

It is evident from the constructed extended tables that the tables for the new operation " $\ominus$ " and the operation " $\otimes$ " coincide, thus G is an RAD-AG-groupoid.

2.9. **Enumeration of AD-AG-groupoids.** Enumeration and classification of various mathematical entries is a well worked area of research in finite and pure mathematics. In abstract algebra the classification of algebraic structure is an important pre-requisite for their construction. The classification of finite simple groups is considered as one of the major intellectual achievement of twentieth century. Enumeration results can be obtained by a variety of means like; combinatorial or algebraic consideration. Non-associative structures, quasigroup and loops have been enumerated up to size 11 using combinatorial consideration and bespoke exhaustive generation software. FINDER (Finite domain enumeration) has been used for enumeration of IP loops up to size 13. Associative structures, semigroups and monoids have been enumerated up to size 9 and 10 respectively by constraint satisfaction techniques implemented in the Minion constraint solver with bespoke symmetry breaking provided by the computer algebra system GAP [3]. Distler et al. [4] have enumerated AG-groupoids using the constraint solving techniques developed for semigroups and monoids.

Further, they provided a simple enumeration of the structures by the constraint solver and obtained a further division of the domain into a subclass of AG-groupoids using the computer algebra system GAP and were able to enumerate all AG-groupoids up to isomorphism up to size 6. They also presented enumeration for various other subclasses of AG-groupoids. It is worth mentioning that most of the data presented in [4] has been verified by one of the reviewers of the said article with the help of Mace4 and Isofilter as has been mentioned in the acknowledgment of the said article. All this validate the enumeration and classification results for our LAD and RAD AG-groupoids, as the same technique and relevant data of [4] with different codes has been used for the purpose. Further, all the tables of size up to 3 have been verified manually for our subclasses of AG-groupoids. Table 1, contains the enumerations of these newly introduced subclasses.

3. CHARACTERIZATION OF RAD AG-GROUPOID

We begin our study to investigate the following:

Theorem 3.1. *Each RAD-AG-groupoid Q is right distributive AG-groupoid.*

AG-groupoid**s**$\backslash$**Order**	**3**	**4**	**5**	**6**
Total	**20**	**331**	**31913**	**40104513**
Non-associative RAD AG-groupoids	6	175	21186	34539858
Non-associative LAD AG-groupoids	0	01	27	1106
Non-associative AD AG-groupoids	0	0	0	0
Associative RAD AG-groupoids	6	25	195	5353
Associative LAD AG-groupoids	6	25	195	5353
Associative AD AG-groupoids	6	25	195	5353

TABLE 1. Enumeration for RAD & LAD AG-groupoids

Proof. Let Q be an RAD-AG-groupoid, and let $a, b, c \in Q$. Then by the Identities (2.6) and the medial law, we get

$$\begin{aligned} ab \cdot c &= ca \cdot bc = (bc \cdot c)(a \cdot bc) \text{ by (2.6)} \\ &= (bc \cdot a)(c \cdot bc) = ac \cdot bc \text{ by (1.2) and (2.6)} \\ \Rightarrow ab \cdot c &= ac \cdot bc. \end{aligned}$$

Equivalently, Q is a right distributive AG-groupoid. □

The following counterexample shows that the converse may not be true for the above theorem, thus generally not every RD-AG-groupoid may be RAD-AG-groupoid.

Example 3.2. *Let $(Q, *)$ with the Calay's table below be a groupoid, then it can easily be verified that Q is an RD-AG-groupoid, but is not an RAD-AG-groupoid.*

*	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Theorem 3.3. *Let Q be an RAD AG-groupoid. Then the following hold.*

(i) Q is leftt commutative (LC) AG-groupoid,
(ii) Q is left permutable star (LP) AG-groupoid; i.e. $ab.c = ac.b$,*
(iii) Q is paramedial AG-groupoid.

Proof. Let Q be an RAD AG-groupoid and $a, b, c \in Q$. Then

(i) By (2.6) and medial law, we get

$$ab \cdot c = ca \cdot bc = cb \cdot ac = ba \cdot c.$$

Thus Q is an LC-AG-groupoid.

(ii) Let Q be an RAD AG-groupoid.Then by the medial and left invertive laws, Eq. (2.5), (1.4) and Theorem (3.3 (i)), we get

$$\begin{aligned} ab \cdot c &= cb \cdot a = ac \cdot ba \text{ by (1.1) and (2.6)} \\ &= ab \cdot ca = bc \cdot a = ac \cdot b \text{ by (1.2) and (1.1)} \\ \Rightarrow ab \cdot c &= ac \cdot b. \end{aligned}$$

Hence Q is LP*.

(iii) The Identity (2.6) and Theorem 3.8 reveals that we have

$$\begin{aligned} ab\cdot cd &= (cd\cdot a)(b\cdot cd) = (cd\cdot b)(a\cdot cd) \\ &= (bd\cdot c)(a\cdot cd) = (db\cdot c)(a\cdot cd) = ((a\cdot cd)c)(db) \\ &= (ac\cdot cd)(db) = (db\cdot cd)(ac) = (bc\cdot d)(ac) = (d\cdot bc)(ac) \\ &= (ac\cdot bc)d = (ab\cdot c)d = (ba\cdot c)d = dc\cdot ba = db\cdot ca \\ \Rightarrow ab\cdot cd &= db\cdot ca. \end{aligned}$$

Hence Q is paramedial AG-groupoid. □

It is worth mentioning that the concepts of LD and LAD for AG-groupoids are different. To this end we provide an example of LD-AG-groupoid that is not an LAD AG-groupoid.

Example 3.4. *Let $Q = \{1, 2, 3, 4\}$ with the Caley's table given below. Then it can easily be verified that $(Q, \cdot)$ is an LD-AG-groupoid. However, since $1\cdot(2\cdot 4) \neq (1\cdot 2)\cdot(4\cdot 1)$ thus S is not an LAD AG-groupoid.*

·	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Theorem 3.5. *Let Q be an RAD AG-groupoid. Then Q is a semigroup if any of the following hold.*

(i) Q is left distributive,
(ii) Q is an AG AG-groupoid,*
(iii) Q is T_l^3-AG-groupoid
(iv) Q is T^1-AG-groupoid.

Proof. Let Q be an RAD AG-groupoid and $a, b, c, d \in Q$.

(i) Let Q be RAD AG-groupoid which is also left distributive. Then

$$\begin{aligned} a\cdot bc &= ab\cdot ac = cb\cdot aa\ Q \text{ is RD, and paramedial} \\ &= (aa\cdot b)c = (ba\cdot a)c \text{ by L.I.L} \\ &= (ca\cdot ba), \text{ by L.I.L} \\ &= cb\cdot a = ab\cdot c \text{ by RD, L.I.L} \end{aligned}$$

(ii) Let Q be RAD AG-groupoid which is also an AG*. Then

$$a\cdot bc = ba\cdot c = cb\cdot ac = ca\cdot bc = ab\cdot c.$$

(iii) Let Q be an RAD AG-groupoid which is also T_l^3. Then

$$\begin{aligned} ab \cdot c &= ca \cdot bc, \ Q \text{ is RAD} \\ &= cb \cdot ac = bc \cdot ac \text{ by medial law and LC} \\ &= ba \cdot cc = ab \cdot cc \text{ by medial law and LC} \\ \Rightarrow c \cdot ab &= cc \cdot ab, \ Q \text{ is } T_l^3 \\ &= ca \cdot cb = bc \cdot ac \text{ by medial law and bi-commutative} \\ &= ba \cdot c = ab \cdot c \text{ by RD and LC} \\ &= ac \cdot b = ca \cdot b \text{ by RP* and LC} \\ \Rightarrow bc \cdot a &= b \cdot ca. \end{aligned}$$

(iv) Let Q be an LAD AG-groupoid which is also a T^1. Then

$$\begin{aligned} ab \cdot c &= ba \cdot c = ca \cdot b = bc \cdot ab = cb \cdot ab \\ \Rightarrow c \cdot ab &= ab \cdot cb = ba \cdot cb = ac \cdot b = bc \cdot a \\ \Rightarrow a \cdot bc &= ab \cdot c. \end{aligned}$$

Hence the theorem is proved. □

Refer back to Example 2.4, in fact RAD AG-groupoid may not be commutative as $a = bc \neq cb = b$. Now we give a counterexample to show that left transitive, anti-commutative AG-groupoid and LA-monoid, left/right cancellaitve LA- semigroup, anti-rectangular and idempotent AG-groupoids are not commutative. However, these becomes commutative when combined with an RAD AG-groupoid as proved in the following result.

Example 3.6. *Let $Q = \{1, 2, 3, 4\}$ with the Caley's tables (i) and (ii) given below. Then it can easily be verified that $(Q, \cdot)$ is slim but is not commutative since $1 = 1 \cdot 2 \neq 2 \cdot 1 = 3$. Similarly, $(Q, *)$ is left transitive AG-groupoid but not commutative and $(Q, \circ)$ is anti-rectangular and idempotent but not commutative.*

(i).

$\cdot$	1	2	3	4
1	1	1	1	1
2	3	3	3	3
3	1	1	1	1
4	1	1	1	1

(ii).

$*$	1	2	3	4
1	1	2	3	1
2	3	1	2	3
3	2	3	1	2
4	1	2	3	1

(iii).

$\circ$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Theorem 3.7. *Let Q be an RAD AG-groupoid. Then Q is a commutative semigroup if any of the following hold.*

(i) Q is left transitive,
(ii) Q is (left/right) cancellative AG-groupoid,
(iii) Q is cancellative AG-groupoid,
(iv) Q is anti-commutative AG-groupoid,
(v) Q is anti-rectangular,
(vi) Q is LA-monoid,
(vii) Q is idempotent AG-groupoid,
(viii) Q is 3- band AG-groupoid,
(ix) Q is T^2 AG-groupoid,

(x) *Q is Quasi-cancelllative AG-groupoid, satisfying the two equivalent conditions* [29] $a^2 = ab \,\&\, b^2 = ba \Rightarrow a = b$ *and* $a^2 = ba \,\&\, b^2 = ab \Rightarrow a = b$.

Proof. Let Q be an RAD LA-semigroup and a, b, c be elements of Q.

(i) Let Q be a left transitive AG-groupoid. We prove that Q is commutative.

$$\begin{aligned} bc &= ab \cdot ac = ca \cdot ba \text{ by definition of left transitivity and (1.4)} \\ &= ac \cdot ba = ab \cdot ca \text{ by LC and (1.4)} \\ &= ba \cdot ca = ac \cdot ab \text{ by LC and (1.4)} \\ &= cb. \end{aligned}$$

(ii) Let Q be a right cancellative AG-groupoid. Then $ab = cb \Rightarrow a = b$. Now by the paramedial law and the definition of RC we have,

$$\begin{aligned} ab \cdot c &= ca \cdot bc = ac \cdot bc \text{ by RAD, LC} \\ &= ab \cdot cc = ba \cdot cc \text{ by medial law and LC} \\ &= bc \cdot ac = ba \cdot c \text{ by medial law} \\ \Rightarrow ab \cdot c &= ba \cdot c \\ \Rightarrow ab &= ba \text{ by right cancellativity} \end{aligned}$$

(iii) The result follows by [27, Theorem 1].

(iv) Let Q be an anti-commutative AG-groupoid. Then

$$\begin{aligned} ab \cdot ba &= ba \cdot ba = bb \cdot aa \text{ by LC and medial law} \\ &= ab \cdot ab = ba \cdot ab \text{ by paramedial law and LC} \\ \Rightarrow ab \cdot ba &= ba \cdot ab \\ \Rightarrow ab &= ba \text{ by anti-commutativity} \end{aligned}$$

(v) Assume that Q is anti-rectangular AG-groupoid, then

$$\begin{aligned} ba &= (ab \cdot a)a = aa \cdot ba, \text{ by anti-rectangular and L.I.L} \\ &= (ba \cdot a)a = (aa \cdot b)a, \text{ by left invertive law} \\ &= ab \cdot aa = ba \cdot aa, \text{ by left invertive law and LC} \\ &= (aa \cdot a)b = ab \text{ by left invertive law and anti-rectangular} \end{aligned}$$

(vi) Easy.

(vii) Easy.

(viii) Easy.

(ix) Let Q be T^2-AG-groupoid. Then

$$\begin{aligned} bc \cdot a &= ba \cdot c = ab \cdot c = ac \cdot b \text{ by LP* and LC} \\ \Rightarrow bc \cdot ac &= ab \text{ by } T^2 \\ \Rightarrow ab \cdot c &= ab \text{ by assumption of RAD} \end{aligned} \tag{3.10}$$

Now,

$$\begin{aligned} ca\cdot b &= cb\cdot a = bc\cdot a \text{ by LP* and LC} \\ \Rightarrow ca\cdot bc &= ba \text{ by assumption of } T^2 \\ \Rightarrow ab\cdot c &= ba \text{ by assumption of RAD} \end{aligned} \quad (3.11)$$

Thus by (3.10) and (3.11), Q is commutative.

(x) Assume that Q is quasi-canellative LA-semigrup. Then

$$\begin{aligned} ab\cdot ab &= aa\cdot bb = ba\cdot ba \text{ by medial and paramedial laws} \\ &= ab\cdot ba \text{ by LC} \\ \Rightarrow (ab)^2 &= ab\cdot ba. \end{aligned} \quad (3.12)$$

Similarly,

$$\begin{aligned} ba\cdot ba &= bb\cdot aa = ab\cdot ab \text{ by medial and paramedial laws} \\ &= ba\cdot ab \text{ by LC} \\ \Rightarrow (ba)^2 &= ba\cdot ab. \end{aligned} \quad (3.13)$$

Thus by (3.12) and (3.13) $ab = ba$.

Hence the theorem is proved. □

Theorem 3.8. *Let Q be an RAD AG-groupoid. Then the following are true.*

(i) Q is Moufang AG-groupoid i.e., $ab\cdot ca = (a\cdot bc)a$,
(ii) Q is left nuclear square, ie. $a^2b\cdot c = a^2\cdot bc$,
(iii) Q is right Cheban AG-groupoid i.e., $(a\cdot bc)c = ac\cdot cb$,
(iv) Q is LP-AG-groupoid i.e., $(ab\cdot d)c = (ac\cdot d)b$,
(v) Q is right Cheban AG-groupoid i.e., $(a\cdot bc)d = ad\cdot cb$,*
(vi) Q is rectangular, $ab\cdot cd = ad\cdot cb$.*

Proof. Let Q be an RAD AG-groupoid and $a, b, c, d \in Q$. Then

(i) The Identities LC, LP*, and L.I.L imply that

$$\begin{aligned} ab\cdot ca &= ba\cdot ca = (b\cdot ca)a \text{ by LC, LP*} \\ &= (ca\cdot b)a = (ac\cdot b)a \text{ by LC} \\ &= (bc\cdot a)a = (a\cdot bc)a \text{ by L.I.L} \\ \Rightarrow ab\cdot ca &= (a\cdot bc)a. \end{aligned}$$

Hence Q is Moufang AG-groupoid.

(ii) Q is left nuclear square, ie.$a^2b\cdot c = a^2\cdot bc$.
Using the identity of medial and pramedial and the L.I.L, we have

$$\begin{aligned} a^2b\cdot c &= cb\cdot a^2 = cb\cdot aa \text{ by L.I.L} \\ &= ca\cdot ba = aa\cdot bc = a^2\cdot bc. \text{ by medail and paramedial laws} \end{aligned}$$

Thus Q is left nuclear square AG-groupoid.

(iii) Q is right Cheban AG-groupoid i.e., $(a \cdot bc)c = ac \cdot cb$,

$$\begin{aligned}(a \cdot bc)c &= (bc \cdot a)c = (cb \cdot a)\, c \text{ by LC}\\ &= (bc \cdot a)\, c = (ca)\,(cb) \text{ by LC and LP*}\\ &= ac \cdot cb \text{ by LC}\end{aligned}$$

(iv) We prove that Q is LP AG-groupoid i.e. $(ab \cdot d)c = (ac \cdot d)b$.

$$\begin{aligned}(ab \cdot d)c &= (db \cdot a)c = (da \cdot b)c \text{ by L.I.L and LP*}\\ &= (da \cdot c)b = (ca \cdot d)b. \text{ by LP*, and L.I.L}\\ &= (ac \cdot d)b \text{ by LC}\end{aligned}$$

(v) We show that Q is right Cheban* AG-groupoid. i.e. $(a \cdot bc)d = ad \cdot cb$,

$$\begin{aligned}(a \cdot bc)d &= (bc \cdot a)d = (cb \cdot a)\, d \text{ by LC}\\ &= (a \cdot cb)\, d = ad \cdot cb \text{ by LC and LP*}\end{aligned}$$

(vi) Q is rectangular*, $ab \cdot cd = ad \cdot cb$,

$$\begin{aligned}ab \cdot cd &= ba \cdot cd = da \cdot cb \text{ by LC and parmedial law}\\ &= ad \cdot cb \text{ by LC.} \qquad \square\end{aligned}$$

The following relation among the the RAD and other similar classes is proved by Ahmad et al. [25].

Theorem 3.9. *Any two of the following properties imply the rest.*

(1) Q is an LAD AG-groupoid,
(2) Q is an RAD-AG-groupoid,
(3) Q is an AD-AG-groupoid,
(4) Q is a semigroup.

It is evident from Example 2.2 that RAD AG-groupoid may not be a semigroup as $5(5 \cdot 5) \neq (5 \cdot 5)5$. Now we provide a counterexample to show that right distributive, T^3-AG-groupoid and AG** is not a semigroup. However, these becomes associative when combined with an LAD AG-groupoid.

Example 3.10. *Let $Q = \{1, 2, 3, 4\}$ with the Caley's tables (i) and (ii) given below. Then it can easily be verified that $(Q, \cdot)$ is right distributive and AG** but is not a semigroup since $1 \cdot (1 \cdot 1) \neq (1 \cdot 1) \cdot 1$. Similarly, $(Q, *)$ is T^3-AG-groupoid but not a semigroup as $1 = 1 \cdot (1 \cdot 1) \neq (1 \cdot 1) \cdot 1 = 3$.*

(i)

$\cdot$	1	2	3	4
1	2	2	3	2
2	3	3	3	3
3	3	3	3	3
4	2	2	3	2

(ii)

$*$	1	2	3	4
1	2	1	4	3
2	3	4	1	2
3	4	3	2	1
4	1	2	3	4

Theorem 3.11. *Let Q be an RAD AG-groupoid. Then it is right distributive if it is Stein, i.e., $a \cdot bc = bc \cdot a$.*

Proof. Let Q be an RAD AG-groupoid such that Q also satisfies the property of Stein $a \cdot bc = bc \cdot a$. Then

$$\begin{aligned} ab \cdot ac &= ba \cdot ac = (b \cdot ac)a \text{ by LC and LP*} \\ &= (a \cdot ac)b = (ac \cdot a)b \text{ by L.I.L and LC} \\ &= (ac \cdot b)a = (ca \cdot b)a \text{ by LP* and LC} \\ &= ab \cdot ca = ba \cdot ca \text{ by L.I.L and LC} \\ &= bc \cdot a = a \cdot bc \text{ by RD and Stein} \\ \Rightarrow ab \cdot ac &= a \cdot bc. \end{aligned}$$

Hence Q is left distributive. □

Theorem 3.12. *Let Q be a slim AG-groupoid. Then it is an RAD.*

Proof. Let Q be slim AG-groupoid. Then

$$\begin{aligned} ca \cdot bc &= (bc \cdot a)c = (ac \cdot b)c \text{ by LC and paramedial} \\ &= cb \cdot ac = (ac \cdot b)c \text{ by L.I.L and LC} \\ &= ab \cdot c \text{ by slim property} \\ \Rightarrow ca \cdot bc &= ab \cdot c. \end{aligned}$$

Hence Q is right abelian distributive. □

Proposition 3.13. [26] *Every CA AG-groupoid is left and right nuclear square and that Bol* $a(bc \cdot d) = (ab \cdot c)d$.*

Theorem 3.14. *Let Q be a right distributive AG-groupoid. Then the following hold if Q is CA-LA semigroup.*

(i) Q is left commutative (LC),
(ii) Q is right abelian distributive.

Proof. Let Q be a right distributive AG-groupoid that satisfies the CA property. Then

(i) Q is left commutative (LC):

$$\begin{aligned} ab \cdot c &= ac \cdot bc = xy \cdot z^2 \text{ by RD and medial law} \\ &= a \cdot bc^2 = c^2 \cdot ab \text{ by right nuclear square and CA} \\ &= c^2 a \cdot b = ba \cdot c^2 \text{ by left nuclear square and CA} \\ &= ba \cdot cc = bc \cdot ac = ba \cdot c \text{ by medial law and RD} \\ \Rightarrow ab \cdot c &= ba \cdot c. \end{aligned}$$

Hence Q is left commutaitve.

(ii) Q is right abelian distributive:

$$\begin{aligned}
ab\cdot ca &= (ca\cdot b)a = (ba\cdot c)a \text{ by RD and L.I.L}\\
&= ac\cdot ba = a(ac\cdot b) \text{ by L.I.L and CA}\\
&= a(ab\cdot cb) = a(ac\cdot bb) \text{ by RD and medial law}\\
&= a(ac\cdot b^2) = a(a\cdot cb^2) \text{ by right nuclear square}\\
&= cb^2\cdot a^2 = (a^2b^2)c \text{ by CA and left inveritve law}\\
&= (ab\cdot ab)c = (aa\cdot b)c \text{ by medial law and RD}\\
&= cb\cdot a^2 = cb\cdot aa = ca\cdot ba \text{ by L.I.L, RD and medial law}\\
&= cb\cdot a = bc\cdot a \text{ by RD and LC part (i)}\\
\Rightarrow ab\cdot ca &= bc\cdot a.
\end{aligned}$$

Hence Q is right abelian distributive. □

Example 3.15. *Let $Q = \{1, 2, 3, 4\}$ with the Caley's tables (i) given below. Then it can easily be checked that $(Q, \cdot)$ is RD that is not right abelian distributive since $2 = 0\cdot 1 = (0\cdot 0)1 \neq (1\cdot 0)(0\cdot 1) = 3\cdot 2 = 1$.*

$\cdot$	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

Theorem 3.16. *Let Q be an RAD AG-groupoid. Then Q is middle nuclear square if it is CA (hence right nuclear sqaure) and thus nuclear square.*

Proof. Let Q be an RAD AG-groupoid. Since Q is RAD so is left nuclear square by Theorem 3.8. Further, assume that Q is CA (right nuclear square). We show that Q is middle nuclear square. To this end we use that Q is cyclic associative (CA), left commutative (LC), paramedial, and that $c\cdot ab^2 = b(a\cdot bc)$ holds in Q.

We first show that $c\cdot ab^2 = b(a\cdot bc)\forall a, b, c$ in Q.

$$c\cdot ab^2 = b^2\cdot ca = bb\cdot ca \text{ by CA}$$
$$= bc\cdot ba = b(a\cdot bc) \text{ by medial law and CA} \tag{3.14}$$
$$\Rightarrow c\cdot ab^2 = b(a\cdot bc). \tag{3.15}$$

Now we prove that Q is middle nuclear square.

$$\begin{aligned}
a\cdot b^2c &= c\cdot ab^2 = b(a\cdot bc) \text{ by CA and (3.14)}\\
&= bc\cdot ba = b^2\cdot ca \text{ by CA and medial law}\\
&= b^2c\cdot a = cb^2\cdot a \text{ by LNS and LC}\\
&= ab^2\cdot c \text{ by CA}
\end{aligned}$$

The nuclear square property is followed by Proposition 3.13 and the fact that RAD is left nuclear square. □

Theorem 3.17. *Let Q be an CA-RAD AG-groupoid. Then the following are true.*

(i) Q *is left_Cheban i.e.* $a(ab \cdot c) = ba \cdot ac$*,*
(ii) Q *is left_Cheban_star i.e.* $a(bc \cdot d) = ca \cdot bd$*,*
(iii) Q *is RP i.e.* $a(bd \cdot c) = d(ba \cdot c)$*.*

Proof. Let Q be an CA-RAD AG-groupoid. Then the following are true.

(i) Q is left_Cheban i.e. $a(ab \cdot c) = ba \cdot ac$,

$$\begin{aligned} a(ab \cdot c) &= a(cb \cdot a) = a(ab \cdot c) \text{ by LIL} \\ &= c(a \cdot ab) \text{ by CA} \\ &= ab \cdot ca = ba \cdot ca) \text{ by CA and LC} \\ &= ac \cdot ab = ca \cdot ab \text{ by bi-commutative LC} \\ &= ba \cdot ac \text{ by bi-commutativity} \\ \Rightarrow a(ab \cdot c) &= ba \cdot ac. \end{aligned}$$

(ii) Q is left_Cheban_star i.e. $a(bc \cdot d) = ca \cdot bd$,

$$\begin{aligned} a(bc \cdot d) &= a(dc \cdot b) = b(a \cdot dc) \text{ by L.I.L and CA} \\ &= dc \cdot ba \text{ by CA} \\ &= ac \cdot bd = ca \cdot bd \text{ by paramedial law and LC} \\ \Rightarrow a(bc \cdot d) &= ca \cdot bd. \end{aligned}$$

(iii) Q is RP i.e. $a(bd \cdot c) = d(ba \cdot c)$.

$$\begin{aligned} a(bd \cdot c) &= a(db \cdot c) = a(cb \cdot d) \text{ by LC and L.I.L} \\ &= d(a \cdot cb) = d(b \cdot ac) \text{ by CA} \\ &= d(c \cdot ba) = ba \cdot dc \text{ by CA} \\ &= cd \cdot ab = dc \cdot ab \text{ by bi-commutavity} \\ &= cd \cdot ab = bd \cdot ac \text{ by LC and paramedial law} \\ &= db \cdot ac = ca \cdot bd \text{ by LC and bi-commutavity} \\ &= d(ca \cdot b) = d(ba \cdot c) \text{ by CA and L.I.L} \\ \Rightarrow a(bd \cdot c) &= d(ba \cdot c). \end{aligned}$$

Hence the result follows. □

Right abelian distributive AG-groupoid has various properties with the right alternative. The combination of these leads to a locally associative AG-groupoid that has associative powers and shall be discussed in detail in the next section. Here we note that in this case it becomes a nuclear square. From Theorem 3.8 it is evident that RAD is left nuclear square but not the right or middle nuclear square as depicted in the following example table (i). Similarly, it is evident that RA is also not nuclear square as depicted in table (ii). The following properties of the combination of the two subclasses is quite interesting. It shows that the square of elements commutes with other elements. Similarly the product of elements of $Q \times Q$ with the elements and their squares in Q for an RAD-RA-AG-groupoid. Further, it has been investigated that for right alternative AG-groupoids the conditions of right and middle nuclear square are equivalent [11].

(i).

·	0	1	2	3
0	1	1	1	1
1	2	2	2	2
2	2	2	2	2
3	1	1	1	1

(ii).

·	0	1	2	3
0	2	2	3	3
1	3	1	3	3
2	3	3	3	3
3	3	3	3	3

Theorem 3.18. *Let Q be a right alternative RAD AG-groupoid. Then for all $a, b, c \in Q$ the following are true.*

(i) $aa \cdot b = b \cdot aa$,
(ii) $ab \cdot c^2 = ab \cdot c$,
(iii) $ab \cdot ab = ba^2$,
(iv) $ab \cdot ac = c(ba^2)$,
(v) *Q is right nuclear square,*
(vi) *Q is middle nuclear square,*
(vii) *Q is nuclear square.*

Proof. Let Q be a right alternative RAD AG-groupoid. Then for all $a, b, c \in Q$.

(i) LHS $= a^2b = aa \cdot b = ba \cdot a = b \cdot aa = ba^2 =$ RHS by L.I.L and RA
(ii) LHS $= ab \cdot cc = ac \cdot bc = ab \cdot c =$ RHS by medial law and RD
(iii) LHS $= ab \cdot ab = aa \cdot b = a^2b = ba^2 =$ RHS by LD and part (1) above
(iv) LHS $= ab \cdot ac =$

$$\begin{aligned} &= ab \cdot ac = a^2 \cdot bc = ab \cdot c \text{ by medial law and LNS} \\ &= ba^2 \cdot c = (ab \cdot ab)c \text{ by LC and part 3 above} \\ &= (ab)^2c = c(ab)^2 = c(ab \cdot ab) = c \cdot ba^2 = \text{RHS by part 1 and part 3} \end{aligned}$$

(v) Q is right nuclear square i.e. $a \cdot bc^2 = ab \cdot c^2$,

$$\begin{aligned} ab \cdot c &= ab \cdot c^2 \text{by part (ii) above} \\ &= c^2b \cdot a = c^2 \cdot ba \text{ by L.I.L and left nuclear square} \\ &= a \cdot bc^2 \text{by part (iv) as } a^2 \cdot bc = c \cdot ba^2 \end{aligned}$$

(vi) Q is middle nuclear square i.e. $ab^2 \cdot c = a \cdot b^2c$,

$$\begin{aligned} ab^2 \cdot c &= cb^2 \cdot a = b^2c \cdot a \text{ by L.I.L and left nuclear square} \\ &= ac \cdot b^2 = \text{ by L.I.L and right nuclear square} \\ &= a \cdot cb^2 = a \cdot b^2c \text{ by right nuclear square part (i) above.} \end{aligned}$$

(vii) Q is nuclear square. Follows by Theorem 3.8 and above.

Hence the theorem is proved. □

Conjecture 3.19. *Every RAD-AG-groupoid is middle nuclear square if it is right nuclear square AG-groupoid.*

4. Decomposition of Locally Associative RAD AG-groupoids

First we show by an example that locally associative RAD does not guarantee associative powers. The following table (i) is locally associative but is not power associative as

$aa^{m+1} = a^{m+1}a$ not holds for $m = 2$. For instance $a(a(aa)) = (a(aa))a$ not holds for $a = 0$.

(i)

$\cdot$	0	1	2	3
0	1	2	1	1
1	2	2	2	2
2	2	2	2	2
3	1	1	2	2

(ii)

$\cdot$	0	1	2	3
0	2	2	2	3
1	3	2	2	2
2	2	2	2	2
3	2	2	2	2

Alternatively, we discuss decomposition of right alternative that is locally associative but not have associative powers. However with RAD it has associative powers as can be seen in table (ii) above.

We recall by Theorem 3.3 and 3.8 that for any a, b, c, d in an RAD AG-groupoid Q, the following are true.

$$ab \cdot cd = db \cdot ca. \tag{4.16}$$

$$ab \cdot c = ac \cdot b. \tag{4.17}$$

The identity for right alternative AG-groupoid is given as

$$ab \cdot b = a \cdot bb. \tag{4.18}$$

An AG-groupoid Q is called a locally associative AG-groupoid if $(aa)a = a(aa)$ for all a in Q [5]. A locally associative AG-groupoid Q satisfying the identity (2.5) is called a locally associative RAD AG-groupoid. It is clear from 4.18 that every right alternative is locally associative. decomposition for various subclasses of AG-groupoids is considered by different researchers [**?**, 21, 23]. We introduce a "relation η in a right alternative/locally associative RAD AG-groupoid Q as follows, for any integer $m > 0$, we say that $a\eta b$ if and only if $ab^m = b^{m+1}$ and $ba^m = a^{m+1}$ for all a, b in Q. We first prove that Q has associative powers and then show that the relation η is congruence". From now on by an RAD AG-groupoid shall mean a riAGght alternative RAD AG-groupoid otherwise stated else.

Lemma 4.1. *Every RAD AG-groupoid Q has associative powers, i.e., $aa^{n+1} = a^{n+1}a$ for all $a \in Q$.*

Proof. Let Q be an RAD AG-groupoid, then for any $a \in Q$, we have $a^1 = a$ and $a^{m+1} = a^m a$ where $m > 0$. Now, the identity

$$aa^{m+1} = a^{m+1}a, \tag{4.19}$$

is true for $m = 1$ and $m = 2$. Further suppose that (4.19) holds for $m = k - 1$, that is $aa^k = a^k a$. Then by (1.1, 2.6) LC, right alternativity, RD, right nuclear square and the supposition, we have

$$\begin{aligned} a^{k+1}a &= (a^k a)a = a^k \cdot aa = a^{k-1}a \cdot aa \\ &= (aa \cdot a)a^{k-1} = (a \cdot aa)a^{k-1} = a(aa \cdot a^{k-1}) \\ &= a(a^{k-1}a \cdot a) = a(a^k a) = aa^{k+1}. \end{aligned}$$

Hence by induction it follows that $aa^{m+1} = a^{m+1}a$. □

Lemma 4.2. *Let Q be an RAD AG-groupoid and $a, b \in Q$, then $(ab)^m = a^m b^m$ for any integer $m \geq 1$ and $(ab)^m = a^m b^m$ for $m \geq 2$.*

Proof. Obviously for $m = 1$, the result is true as $(ab)^1 = a^1 b^1 = ab$. Assume it is true for $m = k$, $(ab)^k = a^k b^k$. We show it is true for $m = k + 1$, by identity (1.4), we have

$$(ab)^{k+1} = (ab)^k (ab) = (a^k b^k)(ab) = (a^k a)(b^k b) = a^{k+1} b^{k+1}.$$

Hence it is true for each $m \geq 1$. If $m \geq 2$, then using the identities (4.16, 1.4) and Lemma 4.1, we get

$$\begin{aligned} (ab)^m &= a^m b^m = (a^{m-1} a)(b^{m-1} b) = (aa^{m-1})(bb^{m-1}) = \\ &= (b^{m-1} a^{m-1})(ba) = (b^{m-1} b)(a^{m-1} a) = b^m a^m. \end{aligned}$$

Hence the result follows. □

Lemma 4.3. *In an RAD AG-groupoid Q, $a^r a^s = a^{r+s}\, \forall a \in Q$ and positive integers r, s.*

Proof. By Lemma 4.1, we have $aa^s = a^{s+1}$, then result holds for $r = 1$. Assume that the result is true for $r = t$, that is $a^t a^s = a^{t+s}$. Then by the identities (1.1) and the Lemma 4.1, we have

$$\begin{aligned} a^{t+1} a^s &= (a^t a) a^s = (a^s a) a^t = (aa^s) a^t \\ &= (a^t a^s) a = a^{t+s} a = a^{t+s+1} \\ \Rightarrow a^{t+1} a^s &= a^{t+s+1}. \end{aligned}$$

Hence by mathematical induction on r, the result follows. □

Proposition 4.4. *In an RAD AG-groupoid Q, $(a^r)^s = a^{rs}\, \forall a \in Q$ and some positive integers r, s.*

Proof. For $r = 1$, it is true because $(a^1)^s = a^s$. Assume it is true for $r = t$, $(a^t)^s = a^{ts}$. For $r = t + 1$. Then by Lemma 4.2 and 4.3, we have

$$(a^{t+1})^s = (a^t a)^s = (a^t)^s a^s = a^{ts} a^s = a^{ts+s} = a^{(t+1)s}.$$

Hence the result is true for every $r \geq 1$. □

Lemma 4.5. *Let Q be an RAD AG-groupoid. Then $a^s b^r = b^r a^s\ \forall\, a, b$ in Q, and $r, s > 1$.*

Proof. By the Identities (1.4, 4.16) and Lemma 4.1, we have

$$\begin{aligned} a^s b^r &= (a^{s-1} a)(b^{r-1} b) = (a^{s-1} b^{r-1})(ab) \\ &= (bb^{r-1})(aa^{s-1}) = (b^{r-1} b)(a^{s-1} a) = b^r a^s. \end{aligned}$$

Thus the result is proved. □

Theorem 4.6. *If Q is RAD AG-groupoid, if $ab^r = b^{r+1}$ and $ba^s = a^{s+1}$ for $a, b \in Q$, and any positive integers r, s, then $a\eta b$.*

Proof. Without loss of generality assume that $s > r$. For $r = 1$, put $b^{s-1} b^0 = b^{s-1}$. Then by identity (4.16) and Lemmas 4.1, 4.3, we get

$$\begin{aligned} b^{s-r} b^{r+1} &= b^{s-r}(ab^r) = (b^s b^{-r})(ab^r) = \\ &= (b^r b^{-r})(ab^s) = (b^{r-r})(ab^s) = ab^s. \end{aligned}$$

Thus $ab^r = b^{r+1}$, similarly $ba^s = a^{s+1}$and so $a\eta b$. □

Next, we prove that the relation η is congruence on Q. A relation is called congruence if it is reflexive, symmetric, transitive and compatible.

Theorem 4.7. *The relation "η" is congruence on an RAD AG-groupoid Q.*

Proof. Obviously, η is reflexive and symmetric. We now show that η is transitive.

Let $a\eta b$ and $b\eta c$. Then for any $a, b, c \in Q$ there exist positive integers s, r such that $ab^s = b^{s+1}$, $ba^s = a^{s+1}$ and $bc^r = c^{r+1}$, $cb^r = b^{r+1}$. Let $k = (s+1)(r+1) - 1$, that is $k = s(r+1) + r$, such that $r, s > 1$. Then using proposition 4.4, Lemmas 4.3, 4.5 and identities (1.1, 4.16), we get

$$\begin{aligned}
ac^k &= ac^{s(r+1)+r} = a(c^{s(r+1)} \cdot c^r) = a(c^{r+1})^s c^r) = a((bc^r)^s c^r) = \\
&= a(b^s c^{rs} \cdot c^r) = a(c^r c^{rs} \cdot b^s) = (a \cdot c^r c^{rs})(b^s a) = (ab^s)(c^r c^{rs} \cdot a) = \\
&= a(b^s \cdot c^r c^{rs}) = a(c^r \cdot b^s c^{rs}) = c^r(a \cdot b^s c^{rs}) = c^r(ab^s \cdot c^{rs} a) = \\
&= c^r(ac^{rs} \cdot b^s a) = c^r(a \cdot c^{rs} b^s) = c^r(c^{rs} \cdot ab^s) = c^r(c^{rs} b^{s+1}) = \\
&= c^r c^{rs} \cdot b^{s+1} c^r = c^r b^{s+1} \cdot c^{rs} c^r = c^r(b^{s+1} c^{rs}) = b^{s+1}(c^r c^{rs}) = \\
&= b^{s+1} c^{r(s+1)} = (bc^r)^{s+1} = (c^{r+1})^{s+1} = c^{s(r+1)+r+1} = c^{k+1}.
\end{aligned}$$

Similarly, $ca^k = a^{k+1}$. Thus η is an equivalence on Q.

Next, we show that η is compatible. To this end we assume that $a\eta b$ such that for some positive s,

$$ab^s = b^{s+1} \quad \text{and} \quad ba^s = a^{s+1}.$$

Let $c \in Q$. Then by the identity (1.4) and Lemmas 4.1 and 4.2 we get

$$(ac)(bc)^s = (ac)(b^s c^s) = (ab^s)(cc^s) = (b^{s+1})(c^s c) = b^{s+1} c^{s+1} = (bc)^{s+1},$$

and

$$(bc)(ac)^s = (bc)(a^s c^s) = (ba^s)(cc^s) = (a^{s+1})(c^s c) = a^{s+1} c^{s+1} = (ac)^{s+1}.$$

So, $ac\eta bc$, Similarly, $ca\eta cb$. Hence η is a congruence on Q. □

In the next theorem we prove that the above relation is seperative on Q. A relation "η" on an RAD AG-groupoid Q is separative if $ab\eta a^2$ and $ab\eta b^2$ implies that $a\eta b$.

Theorem 4.8. *The relation "η" on an RAD AG-groupoid Q is separative.*

Proof. Let $a, b \in Q$, $ab\eta a^2$ and $ab\eta b^2$. Then by the definition of η there exists positive integers r and s such that

$$(ab)(a^2)^r = (a^2)^{r+1} = a^{2r+2} \quad , \quad a^2(ab)^r = (ab)^{r+1}$$

$$(ab)(b^2)^s = (b^2)^{s+1} \quad , \quad b^2(ab)^s = (ab)^{s+1}.$$

Now, by using the Proposition 4.4, Identities (1.1, 4.16) and Lemma 4.1, and bi-commutativity, (4.17), nuclear square properties3.18 we get,

$$\begin{aligned}
a^{2r+2} &= (ab)(a^2)^r = (ab)(a^r a^r) = (a^r b \cdot a^r a) \\
&= (a^r a^r \cdot ba) = ab \cdot a^r a^r = ab \cdot a^{2r} = (a^{2r})ab \\
&= a^{2r} a \cdot b = aa^{2r} \cdot b = ab \cdot a^{2r} \\
&= ba \cdot a^r = ba^{2r+1}.
\end{aligned}$$

That is, $ba^j = a^{j+1}$ where $j = 2r + 1$.

Similarly, it can be shown that $ab^k = b^{k+1}$,where $k = 2m + 1$. Thus, by Theorem 4.6, $a\eta b$.

Similarly, it can be shown that $ab^k = b^{k+1}$,where $k = 2m + 1$. Thus, by Theorem 4.6, $a\eta b$. Hence the relation η is separative on Q. □

5. CONCLUSION

The concept of left (right) abelian distributive groupoid (LAD resp. RAD) is extended to introduce the subclasses of an AG-groupoid as left (right) abelian distributive AG-groupoid. These classes have been enumerated up to order 6 using the computational techniques of GAP. A numerous relations of RAD AG-groupoids are investigated with other existing subclasses of AG-groupoids and with some other related algebraic structures. A manual procedure for the verification of an arbitrary finite AG-groupoid for RAD as subclass of AG-groupoids is introduced. Furthermore, examples and counterexamples are produced with Prover-9 and Mace-4 to strengthen the validity of the produced results. Moreover, a right alternative RAD AG-groupoid is decomposed by some congruences and a separative congruence is introduced.

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REFERENCES

[1] Amanullah, M. Rashad, I. Ahmad, *Abel-Grassmann groupoids of modulo matrices*, Mehran University Research Journal of Engineering and Technology, **35,**No.1, (2016) 63-70.

[2] I. Ahmad, and M. Rashad, *some general properties of Left abelian AG-groupoids*, Punjab Univ. j. math. to appear (2019).

[3] J. Denes, A.D. Keedwell "*Latin Squares and their applications*" 1st Edition, Academic Press Inc. New York

[4] A. Distler, M. Shah and V. Sorge, *Enumeration of AG-groupoids,* Lecture Notes in Computer Science, Volume 6824(2011) 1-14. (1974).

[5] W. A. Dudek and R.Gigoń, *Completely inverse AG**-groupoids*, Semigroup Forum **87,** (2013) 201-229 (2013).

[6] W. A. Dudek, M. Khan, N. Khan, *Characterizations of intra-regular Abel-Grassmann's groupoids*, J. Intelligent and Fuzzy Systems **27** (2014) 2915-2925.

[7] GAP: Groups Algorithm and Programming, Version 4.4.12, 2008, (2012).

[8] M. Iqbal and I. Ahmad and M. I. Ali, *A note on CA-AG-groupoids*, British J. Math. comp. Sci. **12,**No.5(2016) 1-16.

[9] M. Iqbal and I. Ahmad, *Ideals in CA-AG-groupoids*, Indian J. Pure Appl. Math. **49,**No. 2 (2018) 265-284.

[10] I. Ahmad, Iftikhar Ahmad and M. Rashad, *A Study of anti-commutativity in AG-Groupoids*, Punjab Univ. j. math. **48,** No.1 (2016) 99-109.

[11] M.A. Kazim and M. Naseerudin, *On almost semigroups*, The Alig. Bull. Math., 2, 1−7, (1972).

[12] Mushtaq Q. and Yusuf S.M., *On Locally Associative AG-groupoid*, J. Nat. Sci. Math., XIX(1), 57–62 (1979).

[13] P. V. Protic and N. Stevanovic, *AG-test and some general properties of Abel-Grassmann's groupoids*, PU. M. A, **6** (1995) 371-383.

[14] Mushtaq Q, Iqbal, *Decomposition of locally associative LA-semigroup*, Semigroup Forum, **41** (1990) 155-164

[15] [22] P. V. Proti´c and M. Boˇzinovi´c, *Some congruences on an AG**-groupoid*, Filomat **9,** No. 3 (1995) 879-886.

[16] S. Rahman, I. Ahmad, M. Iqbal and Amanullah , *A note on left abelian distributive AG-groupoids*, Punjab Univ. j. math. **52**, No.1 (2000)47-63.

[17] S. Rahman, *Characterization of Abel Grassmann's groupoids by the left abelian distributive property*, M.Phil thesis, Department of Mathematics, University of Malakand, Pakistan, (2017).

[18] M. Rashad, I. Ahmad, M. Shah and AB. Saeid, *Bi-commutative AG-groupoids, Enumeration of Bi-Commutative–AG-groupoids,* Journal of Siberian Federal University. Mathematics & Physics, **13**No.3 314–330 DOI: 10.17516/1997-1397-2020-13-3-314-330 (2020).

[19] M. Rashad, I. Ahmad, Amanullah and M. Shah, *On relations between right alternative and nuclear square AG-groupoids*, Int. Math. Forum, **8,** No.5 (2013) 237-243.

[20] M. Rashad, *Investigation and classification of new subclasses of AG-groupoids*, PhD thesis, University of Malakand, Pakistan, (2015).

[21] M. Rashad and I. Ahmad, *A note on unar LA-semigroup*, Punjab Univ. j. math. **50,** No.3 (2018) 113-121.

[22] [27] M. Rashad, I. Ahmad, Amanullah and M. Shah, *A Study on Cheban Abel-Grassmann's Groupoids*, Punjab Univ. j. math. **51,** No. 2 (2019) 79-90.

[23] A. Rauf, *Characterization of slim Abel Grassmann's groupoids*, M.Phil thesis, Department of Mathematics, University of Malakand, Pakistan (2017)

[24] M. Shah, T. Shah and A. Ali, *On the Cancellativity of AG-groupoids,* Int. Math. Forum, **6,** No. 44 (2011) 2187 - 2194.

[25] M. Shah, *A theoretical and computational investigation of AG-groups,* PhD thesis, Quaid-e-Azam University Islamabad, (2013).

[26] M. Shah, I. Ahmad, and A. Ali, *On introduction of new classes of AG-groupoids*, Res. J. Recent Sci., **2,**No.1 (2013) 67-70.

[27] Xiaohong Zhang, Yingcang MA and Peng Yu, *On two conjectures of Abel Grassmann's groupoids*, https://doi.org/10.3390/sym11060816 Symmetry, **11,** No.6 (2019) 816.