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Vibration frequency analysis of three-layered cylinder shaped shell with FGM middle layer under effect of various volume fraction laws

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Abstract.: In this study, vibration frequency analysis of three layered functionally graded material (FGM) cylinder-shaped shell is studied with FGM central layer whereas the internal and external layers are of the same isotropic materials. Sander's shell theory is applied for strain and curvature-displacement relationships. The Rayleigh Ritz method is employed to attain the shell frequency equation. Influence on natural frequencies (NFs) is observed for various volume fraction laws. The characteristic beam functions are used to estimate the dependence of axial modal functions. Results are obtained for thickness to radius ratios and length to radius ratios for different edge conditions. The validity of this method is checked for numerous results found in the open literature.

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Key Words: cylindrical shell, Sander's shell theory, isotropic material, functionally graded material, Rayleigh-Ritz method, natural frequencies..

1. Introduction

Vibration of cylinder shaped shells is a general field of research in mechanical dynamics. These shells from composed of various kinds of materials. Loy et al. [1] examined the fundamental frequencies of circular shaped shells by the generalized differential quadrature method (DQM). Loy et al. [2] investigated vibrations of FGM cylinder shaped shells. These shells were fabricated by nickel and stainless steel. They used Love's shell theory for shell dynamical equations. The Rayleigh-Ritz procedure was employed to obtain the governing shell equations. Naeem and Sharma [3] explored the behavior of vibration frequencies for cylindrical shells with different end conditions. Ritz polynomial functions were utilized to estimate the axial modal dependence. Pradhan et al. [4] analyzed vibration of FGM cylinder shaped shell for different end conditions. They determined impact of volume fractions on the shell natural frequencies. Zhang et al. [5] examined vibrations of cylindrical shells with fluid by employing wave propagation approach. Chen et al. [6] investigated the three-dimensional vibration analysis of fluid-filled orthotropic FGM cylindrical shell' for simply supported edge conditions. They studied variation in natural frequencies for various physical parameters. Patel et al. [7] analyzed vibration frequencies of functionally graded cylinder shaped shells by using finite element method. Higher order approximation theory was used by them. Flugge's shell theory and a semi-reverse procedure were utilized by Li and Batra [8] to examine the axial buckling of three-layered cylindrical shell with FGM middle layer for simply-supported boundary conditions at both ends. Zhang et al. [9] presented vibration behavior of cylinder shaped shell by employing the local adaptive DQM. They used Goldenveizer-Novozhilov shell theory to obtain the shell frequency equations. Vibration analysis for FGM cylindrical shell under three volume fraction laws (VFLs) was studied by Arshad et al. [10]. This analysis was based on Love's shell theory. These VFLs organized the material arrangements in the shell radial direction. Li Xuebin [11] scrutinized the vibration behavior of cylinder-shaped shell by using Flugge's shell theory. Wave propagation technique was employed by him to observe impacts of irregularity of waves in cylindrical shells. Ansari et al. [12] studied vibration frequencies of FGM cylindrical shell depended on Sanders' shell theory via allied analytical technique under a variety of end point conditions. Iqbal et al. [13] used wave propagation approach to examine vibration of FGM cylinder shaped shells. The axial modal dependence was estimated by characteristic beam functions. Arshad et al. [14-15] investigated vibration of bi-layered cylindrical shells (CSs) depended on Love's shell theory by employing Ritz formulation. First layer was organized from FGM and second was structured from isotropic material. Further they did work on two-layered FGM cylindrical shell where both layers are were of FGM materials. Naeem et al. [16] examined vibration of cylindrical shell by employing the generalized DQM. They verified the present validity of the technique by a few comparisons of the results. Vel [17] worked on exact elasticity solutions for the vibrations of FGM cylinder shaped shells under isotropic end point conditions. Isvandzibaei and Moarrefzadeh [18] determined vibration analysis for two kinds of FG cylindrical shells by Rayleigh-Ritz formulation under simply supported edge conditions. They observed impact of arrangements of essential materials (stainless steel and nickel) on shell frequencies. Naeem et al. [19] presented vibration study of immersed thin FGM cylindrical shells for compressed fluid. The fluid impacts were checked with the use of acoustic wave equation. They used wave propagation approach and Hankel function of second kind in relationship with fluid loaded term to formulate the eigenvalue problem. Vibration frequencies of immersed CSs which were based on elastic foundations was analyzed by Shah et al. [20]. These shells were of isotropic nature and immersed in fluid. Shell motion equations were determined by applying the wave propagation approach with Love's shell theory. Vibration analysis of three-layered FGM cylindrical shell depended on Loves shell theory under different end point conditions have been investigated by Naeem et al. [21]. Sofiyev and Kuruoglu [22] studied the instability of three-layered FG cylindrical shell. They employed

Donnell's shell theory and Galerkin technique to investigate the time dependent periodic axial compressive loads. Influence of a number of VFLs for various edge conditions were determined by them. Ghamkhar et al. [23-24] studied the effect of FGM central layer thickness for vibration analysis of three-layered CSs by employing Ritz formulation. Further they determined vibration frequencies of FGM three-layered CSs with effect of ring supports. A number of volume fraction laws were applied to study this vibration analysis. The influence of one ring support along shell length was investigated for a number of edge conditions.

2. MATHEMATICAL FORMULATION

Here a cylindrical shell is considered for vibration study with geometric parameters viz.length L, radius R and thickness h shown in the Fig. 1. An orthogonal cylindrical coordinate system (x, θ, y) is fixed at the mid surface of the shell. Here x, θ and z lie in the axial, angular and radial directions of the shell respectively. Shell deformation displacements represented by the functions $u_1(x, \theta, z)$ and $u_2(x, \theta z)$ lie in the longitudinal, tangential and transverse directions of the shell respectively.

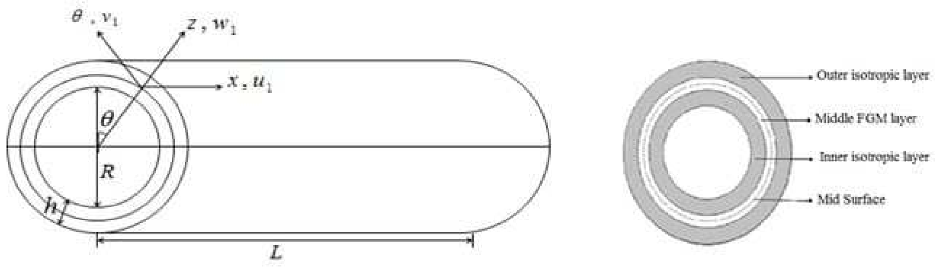

FIGURE 1. Coordinate System and three layered cylinder shaped Shell Geometry.

The strain energy signified by $\Im$ for a cylinder shaped shell expressed as:

$$\Im = \frac{1}{2}\int_0^l \int_0^{2\pi} K^T [S] K R d\theta dx \tag{2. 1}$$

Where

$$K = e_1, e_2, \gamma, K_1, K_2, 2\tau \tag{2. 2}$$

where e_1, e_2, γ and K_1, K_2, τ denote the strains and curvatures reference surfaces respectively. Here T denotes the matrix transpose. For shell motion equations, strain and curvature displacement relationships are assumed from Sanders' shell theory and described as:

$$e_1, e_2, \gamma = \left\{ \frac{\partial u_1}{\partial x}, \frac{1}{R}\left(\frac{\partial v_1}{\partial \theta} + w_1\right), \frac{\partial v_1}{\partial x} + \frac{1}{R}\left(\frac{\partial u_1}{\partial \theta}\right) \right\} \tag{2. 3}$$

$$K_1, K_2, \tau = \left\{-\frac{\partial^2 w_1}{\partial x^2}, -\frac{1}{R^2}\left(\frac{\partial^2 w_1}{\partial \theta^2} - \frac{\partial v_1}{\partial \theta}\right), -\frac{2}{R}\left(\frac{\partial^2}{\partial x \partial \theta} - \frac{3}{4}\frac{\partial v_1}{\partial x} + \frac{1}{4}\frac{\partial u_1}{\partial x}\right)\right\} \tag{2. 4}$$

and [S] can be expressed as

$$[S] = \begin{bmatrix} a_{11} & a_{12} & 0 & b_{11} & b_{12} & 0 \\ a_{12} & a_{22} & 0 & b_{12} & b_{22} & 0 \\ 0 & 0 & a_{66} & 0 & 0 & b_{66} \\ b_{11} & b_{12} & 0 & d_{11} & d_{12} & 0 \\ b_{12} & b_{22} & 0 & d_{12} & d_{22} & 0 \\ 0 & 0 & b_{66} & 0 & 0 & d_{66} \end{bmatrix} \tag{2. 5}$$

where a_{ij}'s symbolize the extensional stiffness, b_{ij}'s the bending stiffness.and d_{ij}'s the bending shiftness.(i,j=1,2 and 6) . These material coefficients are related with rigid moduli and the shell thickness variable. Their relationships are defined as:

$$a_(ij), b_(ij), d_(ij) \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2) dz \tag{2. 6}$$

where $Q_(ij)$'sare the reduced stiffness for isotropic materials and stated as in Loy et al. [1]:

$$Q_{11} = Q_{22} = E(1-u^2) =, Q_{12} = \mu E(1-\mu^2)^{-1}, Q_{66} == 0.55E(1+\mu)^{-1} \tag{2. 7}$$

Here Young's modulus has been denoted E by μ and signifies the Poisson ratio. The stiffness b_{ij}'s=0 for homogenous cylinder shaped shells and $b'_{ij}s \neq 0$ for in-homogeneous FGM cylinder shaped shells and their values depend on the material distribution. $Q'_{ij}s$'s are functions of the physical properties: Young's modulus and Poisson's ratio of shell materials. Using the Eqs. (2. 2) and (2. 5) in the strain energy relation (2. 1),$\Im$ taken the following form:

$$\Im = \frac{1}{2}\int_0^L \int_0^{2\pi} \left\{a_{11}e_1^2 + a_{22}e_2^2 + 2a_{12}e_1e_2 + a_{66}\gamma^2\right.$$
$$+2b_{11}e_1K_1 + 2b_{12}e_1K_2 + 2b_{12}e_2K_1 + 2b_{22}e_2K_2$$
$$\left.+b_{66}\gamma\tau + d_{11}K_1^2 + d_{22}K_2^2 + 2d_{12}K_1K_2 + 4d_{66}\tau^2\right\} R d\theta dx \tag{2. 8}$$

Substituting the strain-displacement and curvature displacement relationships from expressions (2. 3) and (2. 4) into the above expression (2. 8), we obtain the following form of the shell strain energy as:

$$\Im = \frac{1}{2}\int_0^{2\pi}\int_0^L a_{11}\left(\frac{\partial u_1}{\partial x}\right)^2 + \frac{a_{22}}{R^2}\left(\frac{\partial v_1}{\partial \theta + w_1}\right)^2 + \frac{2a_{12}}{R}\frac{\partial u_1}{\partial x}\left(\frac{\partial v_1}{\partial \theta} + w_1\right)^2$$
$$+a_{66}\left(\frac{\partial v_1}{\partial x} + \frac{1}{R}\frac{\partial u_1}{\partial \theta}\right)^2 - 2b_{11}\left(\frac{\partial u_1}{\partial x}\right)\left(\frac{\partial^2 w_1}{\partial x^2}\right)$$
$$-\frac{2b_{11}}{R^2}\left(\frac{\partial u_1}{\partial x}\right)\left(\frac{\partial^2 w_1}{\partial \theta^2} - \frac{\partial v_1}{\partial \theta}\right) - \frac{2b_{12}}{R}\left(\frac{\partial u_1}{\partial x}\right)\left(\frac{\partial^2 w_1}{\partial x^2} + w_1\left(\frac{\partial^2 w_1}{\partial x^2}\right)\right) \tag{2. 9}$$

The cylindrical shell kinetic energy is symbolized by Γ and is expressed as:

$$\Gamma = \frac{1}{2}\int_0^{2\pi}\int_0^L \left[\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial v_1}{\partial t}\right)^2 + \left(\frac{\partial w_1}{\partial t}\right)^2\right] \tag{2. 10}$$

where time variable has been designated by t and mass density is represented by ρ . Mass density per unit length is denoted by $\rho_t$and it is described as

$$\rho_t = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho dz \tag{2. 11}$$

The Lagrange energy functional for a cylinder-shaped shell is represented byΠ and is formulated by difference between the kinetic and strain energies of the shell. It is written in the following expression:

$$\Pi = \Gamma - \Im \tag{2. 12}$$

3. Numerical Procedure

Various numerical techniques are employed to solve shell problems. Energy variational procedures are considered suitable for these problems. Main two types of such procedures are Rayleigh-Ritz method and Galerkin method. Here the Rayleigh-Ritz procedure is utilized to investigate vibration characteristics of cylinder shaped shells. To apply this method, the displacement fields are separated and assumed in the followings:

$$\begin{aligned} u_1(x,\theta,t) &= x_m U_1(x)\cos(n\theta) sin\omega t \\ v_1(x,\theta,t) &= y_m V_1(x)\sin(n\theta) cos\omega t \\ w_1(x,\theta,t) &= z_m W_1(x)\cos(n\theta) sin\omega t \end{aligned} \tag{3. 13}$$

where x_m, y_m and z_m signify the amplitudes of vibration in the x,θ and z direction.The axial and circumferential wave modes have been signified by m and n respectively, ω stands for the shell angular vibration frequency. Functions $U_1(x), V_1(x), W_1(x)$ stand for the axial modal dependence.

$$U_1 = \frac{d\vartheta(x)}{dx}, V_1(x) = W_1(x) = \vartheta(x) \tag{3. 14}$$

where the axial modal function has been denoted by $\vartheta(x)$ which fulfills boundary condition prescribed at the two shell ends. Various functions are adopted for this shell modal dependence like polynomial functions, trigonometric and beam functions. In practice, $\vartheta(x)$ is taken as the beam function. For classical modal dependence, modal functions and variables which are given as:

$$\vartheta(x) = \beta_1\cosh(\Upsilon_m x) + \beta_2\cos(\Upsilon_m x) - \sigma_m(\beta_3 sinh(\Upsilon_m)x) + (\beta_4 sin(\Upsilon_m)x) \tag{3. 15}$$

Here values of $\beta_j, (j = 1, 2, 3, 4)$vary in accordance with the type of boundary conditions applied on shell edges. When the differential equation is solved for a specific edge condition, a transcendental is obtained and its roots are denoted by Υ_m and the parameters σ_m depend on the value of Υ_m.
Firstly the expressions for the shell strain and kinetic energies from (2. 9) and (2. 10) are substituted into the Lagrange functional (2. 12) along with the modal displacement functions from the expression (3. 13) and by applying the principle of minimum energy, we get $\prod_{max}$ into the following form

$$\prod_{max} = \frac{\pi hLR}{2}[R^2\omega^2\rho_t \int_0^1 \left(\beta^2(x_mU_1)^2 + \beta^2(y_mV_1)^2 + \beta^2(z_mW_1)^2\right) dX$$
$$-\int_0^1 \alpha^2\beta^2 a_{11}\left(\frac{dU_1}{dX}\right)^2 + a_{22}(-n\beta y_mV_1 + z_mW_1)^2$$
$$+2\alpha\beta a_{12}\left(x_m\frac{dU_1}{dX}(-n\beta y_mV_1 + z_mW_1)\right) + a_{66}(\alpha\beta y_m\frac{dV_1}{dX} + n\beta x_mU_1)$$
$$-2\alpha^3\beta^2 b_{11}\left(x_m\frac{dU_1}{dx}\right)\left(z_m^2\frac{d^2W_1}{dX^2}\right) - 2\alpha\beta^2 b_{12}\left(x_m\frac{dU_1}{dX}\right)\left(-n^2z_mW_1 + n\beta y_mV_1\right)$$
$$-2\alpha\beta^2 b_{12}(-n\beta y_mV_1 + z_mW_1)\left(z_m^2\frac{d^2W_1}{dX^2}\right)$$
$$-2\beta b_{22}(-n\beta y_mV_1 + z_mW_1)(-n^2z_mW_1 + n\beta y_mV_1)$$
$$-4\beta b_{66}\left(\alpha\beta y_m\frac{dV_1}{dX} + n\beta x_mU_1\right)\left(n\alpha z_m\frac{dW_1}{dX} - \frac{3\alpha\beta y_m}{4}\frac{dV_1}{dX} + \frac{n\beta}{4}x_mU_1\right)$$
$$+\alpha^4\beta^2 d_{11}\left(z_m^2\frac{d^2W_1}{dX^2}\right)^2 + \beta^2 d_{22}(-n^2z_mW_1 + n\beta y_mV_1)^2 + 2\alpha^2\beta^2 d_{12}(z_m^2\frac{d^2W_1}{dX^2})$$
$$\times(-n^2z_mW_1 + n\beta y_mV_1) + 4d_{66}(n\alpha z_m\frac{dW_1}{dX} - \frac{3\alpha\beta y_m}{4})\frac{dV_1}{dX} + \frac{n\beta}{4}x_mU_1)^2]dX \quad (3.\ 16)$$

To apply the Rayleigh'Ritz procedure, $\prod_{max}$ is minimized with regard to the vibration amplitudes x_m, y_m and z_m,we get a system of linear algebraic simultaneous equations in x_m, y_m and z_m, which follows as:

$$\frac{\partial \prod_{max}}{\partial x_m} = \frac{\partial \prod_{max}}{\partial y_m} = \frac{\partial \prod_{max}}{\partial z_m} = 0 \quad (3.\ 17)$$

After proper arrangement of terms, the resulting system of equations is written in matrix form as:

$$\left\{[c] - \Delta^2 = R^2\omega^2\rho_t\right\} \quad (3.\ 18)$$

8) The system generates an eigenvalue problem comprising of the eigen-frequencies and eigen-mode shapes. Here

$$\Delta^2 = R^2\omega^2\rho_t \quad (3.\ 19)$$

[C]and [M] are stiffness and mass matrices of the cylindrical shell respectively, and

$$X^T = [x_m, y_m, z_m]. \tag{3. 20}$$

With the help of MATLAB software, the solution of the eigenvalue equation (3. 18) is obtained to get shell natural frequencies.

4. Functionally graded materials

In this three layered cylindrical shell, the middle layer is constructed by FGMs and isotropic is used for internal and external layers as shown in Fig 1. Here the stiffness moduli are improved as:

$$\begin{aligned} a_{ij} &= a_{ij}^{(ISO)} + a_{ij}^{(FGM)} + a_{ij}^{(ISO)} \\ b_{ij} &= b_{ij}^{(ISO)} + b_{ij}^{(FGM)} + b_{ij}^{(ISO)} \\ d_{ij} &= d_{ij}^{(ISO)} + d_{ij}^{(FGM)} + d_{ij}^{(ISO)} \end{aligned} \tag{4. 21}$$

where i,j=1, 2, 6 and superscript(ISO) denotes the isotropic internal and external layers and(FGM) denotes the central FGM layer. The FGMs contain two necessary materials. These materials are stainless steel and nickel. The material parameters for stainless steel material are: $E_2, \nu_2, \rho_2$and for nickel material are: E_1, ν_1, ρ_1 . The thickness of internal and external layer is presumed to be h/7. Then the actual material quantities for FGM layer are given as:

$$E_f = [E_1 - E_2]\left(\frac{14z + 5h}{10h}\right)^N + E_2 \tag{4. 22}$$

$$\nu_f = [\nu_1 - \nu_2]\left(\frac{14z + 5h}{10h}\right)^N + \nu_2 \tag{4. 23}$$

$$\rho_f = [\rho_1 - \rho_2]\left(\frac{14z + 5h}{10h}\right)^N + \rho_2 \tag{4. 24}$$

Material properties for central FGM layer vary from $z = \frac{-5h}{14}$ to $z = \frac{5h}{14}$.From the above relations, the effective material properties become $E_f = E_2, \nu_f = \nu_2$ and $\rho_f = \rho_2$, at $z = \frac{-5h}{14}$ and for $z = \frac{5h}{14}$ material properties are $E_f = E_1, \nu_f = \nu_1$ and $\rho_f = \rho_1$ The shell is contained only stainless steel material at $z = \frac{-5h}{14}$ and consisted of only nickel material at $z = \frac{5h}{14}$ The distribution of materials in a FGM shell is controlled by several volume fraction laws. Three volume fraction laws are expressed in mathematical form. If z symbolizes the basic shell thickness variable then the volume fraction law V_f of a FGM is formulated as:

$$V_f = \left(\frac{14z + 5h}{10h}\right)^N \tag{4. 25}$$

where h represents the shell thickness and N denotes the power law proponent which may take values from zero to infinity. A volume fraction law formulated by Arshad et al. [10] as:

$$V_f = 1 - exp^{-\left(\frac{14z+5h}{10h}\right)^N} \tag{4. 26}$$

And the material properties are written as:

$$E_f = [E_1 - E_2]\left[1 - exp^{\left(\frac{14z+5h}{10h}\right)^N}\right] + E_2 \tag{4. 27}$$

$$\nu_f = [\nu_1 - \nu_2]\left[1 - e^{-\left(\frac{14z+5h}{10h}\right)^N}\right] + \nu_2 \tag{4. 28}$$

$$\rho_f = [\rho_1 - \rho_2]\left[1 - e^{-\left(\frac{14z+5h}{10h}\right)^N}\right] + \rho_2 \tag{4. 29}$$

Trigonometric volume fraction law for a FGM cylinder-shaped shell is stated as:

$$V_{f_1} = sin^2\left[\left(\frac{14z+5h}{10h}\right)^N\right], V_{f_2} = cos^2\left[\left(\frac{14z+5h}{10h}\right)^N\right] \tag{4. 30}$$

Here

$$V_{f_1} + V_{f_2} = 1$$

The material parameters for FG cylindrical shell are written as:

$$E_f = [E_1 - E_2]sin^2\left[\left(\frac{14z+5h}{10h}\right)^N\right] + E_2 \tag{4. 31}$$

$$V_f = [V_1 - V_2]sin^2\left[\left(\frac{14z+5h}{10h}\right)^N\right] + V_2 \tag{4. 32}$$

$$\rho_f = [\rho_1 - \rho_2]sin^2\left[\left(\frac{14z+5h}{10h}\right)^N\right] + \rho_2 \tag{4. 33}$$

5. Results and discussion

For authenticity of the current work, results for simply supported - simply supported $(ss-ss)$ and clamped - clamped$(c-c)$cylindrical shells are compared with others available in the literature. Frequency parameters are compared with those presented in Zhang et al. [9]. This comparison showed in Table 1 and Table 2.Comparison of natural frequencies(Hz) with those available in Loy & Lam [1] for clamped-free $(c-f)$isotropic cylindrical shell is displayed in the Table 3. It can be observed that the current results are in good agreement with those acquired by different techniques.

Types of three layered FGM cylindrical shell by exchanging the FG essential materials are presented in Table 4. where z_1, z_2 and z_3 signify the Aluminum, Stainless Steel and Nickel material respectively. Material properties for the above materials are presented in reference [2] and [4].

Table 5 displays the natural frequencies (Hz) versus for type I shell with four boundary conditions; $ss-ss, c-c, c-f$ and clamped-simply supported $(c-s$) Nature frequencies are obtained with three volume fraction Laws; polynomial, exponential and trigonometric. It is noticed that NFs (Hz) for clamped-clamped edge conditions are maximum as compared

TABLE 1. Frequency parameter $\delta = \omega R\sqrt{(1-\nu^2)\rho/E}$ comparison for simply-supported isotropic cylinder shaped shell. $(\nu = 0.3, m = 1, h = 0.05, R = 1, L = 20)$

	n			
	1	2	3	4
Zhang *et al* [9]	0.0161	0.03927	0.10981	0.21028
Present	0.0161	0.03927	0.10981	0.21028
Difference	0.006	0.001	0.001	0

TABLE 2. Comparison of frequency parameter $\delta = \omega R\sqrt{(1-\nu^2)\rho/E}$ for an isotropic cylindrical shell with clamped edge conditions. $(\nu = 0.3, m = 1, h = 0.05, R = 1, L = 20)$

	N			
	1	2	3	4
Zhang *et al* [9]	0.03285	0.04064	0.10997	0.21032
Present	0.0344	0.04077	0.11001	0.21038
Difference %	4.7	0.33	0.03	0.02

TABLE 3. Comparison of frequency parameter $\delta = \omega R\sqrt{(1-\nu^2)\rho/E}$ $c-f$ cylindrical shell. $(m = 1\nu = 0.28, , h = 63.5mm, R = 1.63mm, L = 502mm)$

	N				
	2	3	4	5	6
Loy & Lam [1]	319.5	769.9	1465.7	2366.9	3479
Present	319.52	769.86	1465.73	2366.93 3470	

TABLE 4. Configurations of shell types

Types of she	Internal Layer	Central Layer	External Layer
Type I	z_1	z_2/z_3	z_1
Type II	z_1	z_3/z_2	z_1

with other edge conditions. Nature frequencies are decreased for $n = 1$ to $n = 2$ then start to increase for $n = 3$ to onwards. They have same behavior for three volume fraction laws. Nature frequencies under polynomial volume fraction law are greater than those which are obtained with exponential and trigonometric volume fraction laws. In trigonometric volume fraction law case, nature frequencies are minimum for $n = 1, 2$ as compare to other two laws and for n=3 to onwards they are greater than those which are attained with exponential volume fraction law.

Table 6 shows the behavior of nature frequencies versus n with three volume fraction laws for type-II cylindrical shell. Natural frequencies (Hz) for shell type-II are less than those for shell type-I. Nature frequencies with polynomial volume fraction law are minimum as compared to other two volume fraction laws. Nature frequencies (Hz) with trigonometric volume fraction law are maximum for $n = 1, 2$ and for $n = 3$ to onwards NFs with exponential volume fraction law are maximum.

Effects of volume fraction laws on NFs are changed with the increase of n. Effects of volume fraction laws for Type-I Type-II cylindrical shell are opposite.

TABLE 5. Vibrations of natural frequencies (Hz) for shell Type I versus (n)$(= 20, N = R = m =, h = 0.01)$

n	$ss-ss$	$c-c$	$c-f$	cs
		Polynomial volume fraction law		
1	13.4124	28.6769	5.1109	20.7003
2	7.8167	11.8892	6.6724	9.4717
3	18.4309	18.94	18.3	18.6141
4	35.1014	35.2031	35.0487	35.1409
5	56.7103	56.7469	56.6684	56.7274
6	83.1631	83.1847	8.1231	83.1751
7	114.4417	114.4587	114.4019	114.4522
8	150.5401	150.5556	150.5002	150.5501
9	191.456	191.4709	191.4159	191.4585
10	237.1881	237.2028	237.1479	237.1979
		Exponential volume fraction law		
1	13.349	28.5415	5.0867	20.6025
2	7.7492	11.8149	6.6044	9.4026
3	18.2378	18.748	18.1068	18.4213
4	34.7308	34.83327	34.6783	34.7703
5	56.111	56.1476	56.0694	56.128
6	82.284	82.3055	82.2444	82.2959
7	113.2379	113.2488	113.1924	113.2423
8	128.9486	148.964	148.9091	148.985
9	189.4319	189.4467	189.3922	189.4417
10	234.6806	234.6951	234.6407	234.6902
		Trigonometric volume fraction law		
1	13.3033	28.4436	5.0867	20.5318
2	7.7577	11.7975	6.6230	9.3993
3	18.2947	18.8001	18.1648	18.4765
4	34.8421	34.9432	34.7898	34.8814
5	56.2914	56.3278	56.2498	56.3084
6	82.5489	82.5703	82.5092	82.5608
7	113.5964	113.6134	113.5569	113.6068
8	149.4282	149.4437	149.2886	149.4382
9	190.042	190.0568	190.0022	190.0517
10	235.4364	235.4609	235.33964	235.4461

Table 7 and Table 8 represent the Natural frequencies (Hz) versus ratios with three VFL for shell type-I $\&$ II respectively. Natural frequencies (Hz) are decreased with increase in ratios. At for shell type-I, natural frequencies with EVFL and TVFL are decreased 0.6% and 0.7% from the NFs with PVFL respectively. Then for$L/R = 10, 20, 30, 40, 50$,natural frequencies with EVFL and TVFL are decreased 1% and 0.7% from the NFs with PVFL respectively. Now for shell type-II at , the NFs with EVFL are increased 0.6% and for $L/R = 10, 20, 30, 40, 50$, these are increased 1% from the NFs with PVFL. Natural frequencies with TVFL are increased 0.8% for all edge conditions and for all L/R ratios as compared with the NFs under PVFL.

TABLE 6. Vibrations of natural frequencies (Hz) for shell Type II versus (n) $L = 20, m = R = N = 1, h = 0.01$

n	$ss - ss$	$c - c$	$c - f$	cs
		Polynomial volume fraction law		
1	13.412	28.6566	5.1019	20.6871
2	7.8126	11.8806	6.6659	9.4659
3	18.4136	18.9231	18.2824	18.597
4	35.0677	35.1696	35.0149	35.1072
5	56.6556	56.6923	56.6138	56.6727
6	83.0829	83.1045	8.043	83.0948
7	114.3312	114.3483	114.2915	114.3417
8	150.3948	150.4104	150.55	150.4048
9	191.2712	191.2862	191.2313	191.2811
10	236.9593	236.974	236.9192	26.9691
		Exponential volume fraction law		
1	13.4765	28.7936	5.1263	20.1858
2	7.8802	11.9554	6.7338	9.53352
3	18.6062	19.1146	18.4749	18.7892
4	35.4371	35.5389	35.2841	35.4767
5	57.2529	57.2897	57.2108	57.2701
6	83.9590	83.9807	53.9188	83.9711
7	115.5371	115.5542	115.4969	115.5476
8	151.9810	151.9967	151.9408	151.9911
9	193.2886	193.3036	193.2482	193.2985
10	239.4585	239.4733	239.417	239.4684
		Trigonometric volume fraction law		
1	13.5237	28.8946	5.14433	20.8588
2	7.8728	11.9742	6.7164	9.5398
3	18.5526	19.0659	18.4203	18.7374
4	35.33323	35.4349	35.279	35.3721
5	57.0830	57.1200	57.0409	57.1002
6	83.1097	83.7314	83.6694	83.7217
7	115.1937	115.2109	115.1537	115.2043
8	151.5294	151.5451	151.4893	151.5395
9	192.7141	192.7292	192.6738	192.7240
10	238.7468	238.7616	238.7064	238.7567

TABLE 7. Vibrations ofNFs (Hz) for Type I shell against (L/R) $(m = 1, n = 3, h = 0.01, N = 1)$

L/R	$ss - ss$	$c - c$	$c - f$	cs
		Polynomial volume fraction law		
5	36.8567	66.4695	22.1105	51.3224
10	20.1803	26.0434	18.5822	22.5091
20	18.4309	18.9401	1833001	18.6141
30	18.3128	18.4198	18.290	18.3517
40	18.2882	18.3237	18.2740	18.3014
50	18.2798	18.2750	18.2721	18.2856
		Exponential volume fraction law		
5	36.6473	66.1704	21.9207	51.0730
10	19.9895	2.8541	18.3891	22.3203
20	18.2378	18.7480	18.1068	18.4213
30	18.1197	18.2270	18.0859	18.1586
40	18.0951	08.1305	18.0809	18.1083
50	18.0866	18.1019	18.0790	18.0925
		Trigonometric volume fraction law		
5	36.5848	65.9807	21.9475	50.9445
10	20.0311	25.8512	18.4450	223428
20	18.2947	18.8001	18.1648	18.4765
30	18.1775	18.2838	18.1441	18.2167
40	18.531	18.188	18.1391	18.1662
50	18.1447	18.1598	18.1372	18.1505

6. CONCLUSIONS

In this work, frequency analysis of three layered FGM cylinder shaped shell with effect of different volume fraction laws has been done. The internal and external layers of shell

TABLE 8. Vibrations of NFs (Hz) for Type II shell against (L/R) $(m = 1, n = 3, h = 0.01, N = 1)$

L/R	$ss-ss$	$c-c$	$c-f$	cs
		Polynomial volume fraction law		
5	36.8882	66.5355	22.0958	51.3612
10	20.1688	26.0370	18.5649	22.4998
20	18.4136	18.9231	18.2824	18.5970
30	18.2952	18.4024	18.2614	18.3341
40	18.2706	18.3061	18.2564	18.2837
50	18.2621	18.1773	18.2546	18.2681
		Exponential volume fraction law		
5	37.0994	66.8381	22.2856	51.6133
10	20.3594	26.2269	18.774	22.6888
20	18.6062	19.1146	18.474	18.7892
30	18.4878	18.5947	18.4539	18.5266
40	18.4631	18.4985	18.4489	18.4762
50	18.4546	18.4698	18.4469	18.4605
		Trigonometric volume fraction law		
5	37.1662	67.0350	22.2623	51.7475
10	20.3212	26.2332	18.7050	22.6696
20	18.5526	19.0659	08.4203	18.7374
30	18.4333	18.5412	18.3992	18.4725
40	18.4084	18.4442	18.3941	18.4217
50	18.3999	18.4153	18.3922	18.4052

TABLE 9. Vibrations of NFs (Hz) for Type I shell against (h/R) $(m = 1, n = 3, L = 5, R = 1, N = 1)$

L/R	$ss-ss$	$c-c$	$c-f$	cs
		Polynomial volume fraction law		
0.001	31.6449	63.6643	12.2544	47.6673
0.02	48.4245	74.3994	38.9295	61.1259
0.05	100.1659	115.6461	93.2609	107.0556
0.1	192.5161	203.2648	185.1386	197.43
0.5	906.8766	926.1353	889.1769	916.1126
		Exponential volume fraction law		
0.001	31.1968	63.367	12.1955	47.4445
0.02	49.0521	74.0164	38.5451	60.7551
0.05	99.1934	114.7568	92.2789	106.1207
0.1	190.4934	201.3361	183.1391	195.4435
0.5	896.0648	915.3367	878.6115	905.2815
		Trigonometric volume fraction law		
0.001	31.6485	63.6485	12.2359	47.6494
0.02	49.4589	74.5301	38.9038	61.1995
0.05	100.1598	115.8339	933.1933	107.1308
0.1	192.4305	203.4122	185.0081	198.4391
0.5	906.1589	925.8633	888.9012	915.5236

are fabricated by isotropic material and the central layer is of FGM. Sander's theory is used for strain and curvature displacement relationships. To solve the current problem Rayleigh Ritz method is employed. Variation of natural frequencies is investigated for four edge conditions. It is concluded that the material distribution governed by the volume fraction laws has little effect $(< 1\%)$ on vibration frequency. However, Natural frequencies are examined with effect of three VFLs. It is noticed that natural frequencies becomes maximum with the increase in thickness to radius ratios. These are decreased with the increase of L/R ratios. It is also observed that natural frequencies (Hz) with three volume fraction laws have little $(\leq 1\%)$ difference. Their values are very close to each other.

Author contributions. All the author contribute equally in calculating the results and writing the paper. All authors read and approved the final manuscript.

TABLE 10. Vibrations of NFs (Hz) for Type II shell against (h/R) $(n = 3, L = 5, R = N = 1)$

L/R	$ss-ss$	$c-c$	$c-f$	cs
		Polynomial volume fraction law		
0.001	31.6485	63.6485	12.2359	47.6494
0.02	49.4589	74.53301	38.9038	61.1995
0.05	100.1598	115.8339	93.1933	107.1308
0.1	192.4305	203.4122	185.0081	198.4391
0.5	906.1587	925.53..	555.6012	915.5236
		Exponential volume fraction law		
0.001	31.7985	63.9492	12.2953	47.8748
0.02	49.8323	74.9163	39.2878	61.5728
0.05	101.1324	116.7250	94.1735	108.0666
0..1	194.4520	205.3409	187.0045	199.4251
0.5	916.9833	936.6757	899.477	326.3688
		Trigonometric volume fraction law		
0.001	31.9086	64.1715	12.3363	48.0411
0.02	49.8160	75.0470	39.1938	61.6312
0.05	100.2218	116.5924	93.8861	107.8694
0.1	193.8298	204.7849	186.3726	198.8307
0.5	912.6673	923.2424	894.9835	922.0206

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