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A Note on Left Abelian Distributive LA-semigroups

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Abstract. A groupoid with the left invertive law is an LA-semigroup or an Abel-Grassmann's groupoid (AG-groupoid). This in general is a non-associative structure that lies between a groupoid and a commutative semigroup. In this note, the significance of the left Abelian distributivie (LAD) LA-semigroup is considered and investigated as a subclass. Various relations with some other known subclasses are established and explored. A hard level problem suggested for LAD-LA-semigroup to be self-dual [29] is solved. Moreover, the notion of ideals is introduced and characterized for the subclass. Several examples and counterexamples generated with the modern tools of Mace-4 and GAP are produced to improve the authenticity of investigated results.

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1. Introduction

A left almost semigroup or an LA-semigroup is a non-associative and non-commutative structure in general, that generalizes a commutative semigroup. This structure was introduced by Kazim and Naseeruddin in 1972 [1]. The alternative name of AG-groupoid for LA-semigroup is used by Stevanovic and Protic [2] with the reference of a well reputed book [3], consisted of important identities that has been published in 1974. Certain basic

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results of the structure of LA-semigroups are investigated by Mushtaq *et al.* [4] in their research work likewise; an LA-semigroup with right identity is a commutative monoid, every left cancellative LA-semigroup is right cancellative; every right cancellative LA-monoid is left cancellative.

Kepka *et al.* [5] have discussed some non-associative structures on cancellative groupoids. According to Jezek *et al.* [6] medial quasigroup's image is homomorphic on every medial groupoid. They also gave a discussion on paramedial cancellative groupoids of equational theory [7]. It is pertinent to note that LA-semigroups can be generalized by using paramedial groupoid. Protic *et al.* [8] made several innovation in LA-semigroup to optimize the concept of AG-3-band.

A new period of LA-semigroup started in 2006 when Mushtaq *et al.* [9] defined ideals in LA-semigroup. Shah *et al.* [10] introduced the Bol*-LA-semigroup and paramedial LA-semigroup in 2012 and investigated various properties and computed their enumeration up to order 6. In 2013 Rashad *et al.* [11] established relations between the nuclear square LA-semigroup and the alternative LA-semigroup and produced various results.

In LA-semigroup the left invertive law: $(ab)c = (cb)a$ holds [1]. We shall use juxtaposition and the notation "·" to avoid frequent use of parenthesis, e.g., $(ub \cdot c)d$ will denote the same as $((u \cdot b) \cdot c) \cdot d$ likewise $(pq)r$ and $pq \cdot r$ shall represent the same element. A groupoid Q is known as left (resp., right) abelian distributive, if it satisfies $a \cdot bc = ab \cdot ca$ (resp., $ab \cdot c = ca \cdot bc$) [12]. The concept of left abelian distributive groupoid is extended here to left abelian distributive LA-semigroup. The existence of this abelian distributive LA-semigroup is proved by computationally generated non-associative examples of various finite order. Further, we also establish its relation with some of the already known subclasses [13, 14, 15] of LA-semigroups and with other well known algebraic structures. LA-semigroups have been enumerated up to a considerable higher order 6 using GAP [16] by Distler *et al.* [17]. We also use the same techniques to enumerate our new subclass of left abelian distributive LA-semigroup. Table 1, contains the enumerations of this new subclass of LA-semigroup. An LA-semigroup Q is called monoid if it contains a unique left identity [9]. It is easy to prove that very monoid satisfies the paramedial property. It may also be worthwhile to mention that if Q possesses the right identity element then it becomes a commutative semigroup. An LA-semigroup Q is called medial if it satisfies the identity, $ab \cdot cd = ac \cdot bd$. It is easy to show that every LA-semigroup is medial. LA-semigroup has vast applications in the theory of flocks, geometry and matrices [1, 18, 19]. Recently, a considerable research has been done in this area and is being investigated like other well established areas of algebra [20, 21, 22, 23]. In the following we give some preliminary concepts and basic definitions with their identities that shall be referred in the rest of this note.

2. PRELIMINARIES

A groupoid Q is said to be an LA-semigroup [2] if the left invertive law (L.I.L) is also satisfied by Q, i.e., $(pq)r = (rq)p$ for all p, q, r in Q. A groupoid Q is said to be medial, if $pq \cdot rs = pr \cdot qs$, for all p, q, r, s holds in Q. Every LA-semigroup also satisfies the medial law (M.L) [6]. Similarly, a groupoid Q is called paramedial if the identity $pq \cdot rs = sq \cdot rp$, holds for all p, q, r, s in Q. Let Q be an LA-semigroup and $p, q, r, s \in Q$, then Q is called ...

(i) ... AG* if $(pq)\,r = q\,(pr)$ [24].
(ii) ... AG** if $p\,(qr) = q\,(pr)$ [10].
(iii) ... LA-band if $qq = q\,\forall q \in Q$, i.e, if every element is idempotent [25], a commutative LA-band is also called an LA-semilattice.
(iv) ... left distributive (LD) (resp., right distributive (RD)) if $p(qr) = pq \cdot pr$ (resp., $(pq)r = pr \cdot qr$) [26].
(v) ... Type-1-LA-semigroup (in short T^1-LA-semigroup) if $pq = sr \Rightarrow qp = rs$[10].
(vi) ... Bol* if $p\,(qr \cdot s) = (pq \cdot r)\,s$ [27].
(vii) ... left commute (LC) (resp., right commute (RC)) if $(pq)\,r = (qp)\,r$ (resp., $p\,(qr) = p\,(rq)$).
(viii) ... self-dual LA-semigroup if $p(qr) = r(qp)$ [19, 28].
(ix) ... left nuclear square (resp., right nuclear square/middle nuclear square) if $\left(p^2q\right)r = p^2\,(qr)$ (resp., $(pq)\,r^2 = p\left(qr^2\right)$/ $\left(pq^2\right)r = p\left(q^2r\right)$) [13].
(x) ... nuclear square LA-semigroup if it is left, right and middle nuclear square [13].
(xi) ... LA-monoid, if it contains left identity, i.e., if there exists $e \in Q$ such that $eq = q$ $\forall\, q$ in Q.
(xii) ... Stein LA-semigroup, if $p(qr) = (qr)p$.
(xiii) ... left (resp., right) cancellative if $pq = pr$ (resp., $qp = rp$) $\Rightarrow q = r$ [29] and is cancellative, if it is both left as well as right cancellative [29].
(xiv) ... left (resp., right) alternative if $(qq)s = q(qs)$ (resp., $(qs)s = q(ss)$).

A subset L of an LA-semigroup Q is called left ideal (resp., right ideal) if $QL \subseteq L$ (resp., $LQ \subseteq L$). A subset L of an LA-semigroup Q is called ideal if it is both left and right ideal.

The following known facts about different subclasses are available in the literature of LA-semigroup.

Proposition 2.1. [19] *A self-dual LA*-semigroup *having left identity is commutative monoid.*

Proposition 2.2. [13] *Every AG*-groupoid is paramedial.*

Proposition 2.3. [13] *Every paramedial LA*-semigroup *is left nuclear square.*

Proposition 2.4. [19] *The subsequent conditions are equivalent for LA*-semigroup Q*:*
(i) Q *is left* cancellative*,*
(ii) Q *is right* cancellative*,*
(iii) Q *is* cancellative*.*

Proposition 2.5. [29] *Every* cancellative *left nuclear square LA*-semigroup *is paramedial.*

3. Left Abelian distributive-LA-semigroups

The left abelian distributive LA-semigroup abbreviated by LAD-LA-semigroup has been introduced by Rashad [29] in his PhD thesis. We further investigate LAD-LA-semigroup as a subclass of LA-semigroup and investigate the highlighted suggested hard result of [29] wherein it has been suggested that LAD-LA-semigroup may be self-dual. Various other relations of LAD-LA-semigroup are also studied here in this note.

Definition 3.1. [30] *A left abelian distributive LA*-semigroup Q *(or shortly LAD-LA*-semigroup*) is one which satisfies the identity* $q(rs) = (qr)(sq)$ *for all* q, r, s *in* Q.

For existence of such LA-semigroup, an example is provided as follows.

Example 3.2. *Let $Q = \{p, q, r, s, t\}$. Then it is quite easy to show that $(Q, \cdot)$ is a non-associative LAD-LA*-semigroup *of lowest order.*

$\cdot$	p	q	r	s
p	p	p	p	p
q	p	p	p	p
r	p	p	p	q
s	p	p	p	r

3.3. **Enumeration of LAD-LA-semigroups.** By using GAP [16], LA-semigroups are enumerated by Distler *et al.* [17] up to order 6. We use the same techniques and tools with different codes for enumeration of the LAD-LA-semigroups. We further categorize LAD-LA-semigroups into non-commutative, associative, and non-associative LAD-LA-semigroups as given in the following.

Order	3	4	5	6
Total LA-semigroups	20	331	31913	40104513
Associative	12	62	446	7510
Non-associative	8	269	31467	40097003
Total LAD-LA-semigroups	0	4	107	4886
Non-associative	0	1	27	1106
Associative	0	3	80	3780
Associative and non-commutative	0	4	107	4886

Table 1: Enumeration of LAD-LA-semigroups of various orders

3.4. **Relations of LAD-LA-semigroup with other subclasses.** In this section, several relations of LAD-LA-semigroup with other subclasses of LA-semigroup namely; the Stein, right permutable (RP), and the CA-LA-semigroup and with the semigroup are explored. We prove that CA, semigroup and Stein are LAD LA-semigroups but in general, the converse may not be true.

Example 3.5. *LAD-LA*-semigroup *that is neither an AG* nor a Stein AG-groupoid.*

$\cdot$	p	q	r	s
p	q	r	r	r
q	s	r	r	r
r	r	r	r	r
s	r	r	r	r

Lemma 3.6. [29] *Every LAD-LA*-semigroup *is (i) RC-LA*-semigroup , *(ii) LD-LA*-semigroup, *(iii) paramedial, (iv) left nuclear square, (v) LP-LA*-semigroup.

It was pointed out in [29] that LAD-LA-semigroup may be self-dual. However, it was hard enough to prove or disprove. The same result is proved in the following theorem.

Theorem 3.7. *Let Q be an LAD-LA*-semigroup*. Then for all p, q, r, s in Q, the following hold.*

(i) $(pq \cdot r)s = (sp)(rq)$,
(ii) $(pq \cdot r)r = r(pq)$,
(iii) $pp \cdot qr = p \cdot qr$,
(iv) $(p \cdot qr)(qr) = (qr)(p(p \cdot qr))$,
(v) $(pq \cdot qr) = q(p \cdot rp)$,
(vi) $p(q \cdot rq) = p(qr)$,
(vii) $(pq)(qr) = q(pr)$,
(viii) $(pq)(r(r \cdot pq)) = (pq)(rr)$,
(ix) $(p \cdot qr)(qr) = (qr)(pp)$,
(x) $p(q \cdot qr) = p \cdot qr$,
(xi) $p \cdot qq = q \cdot pp$,
(xii) $p(q \cdot rr) = p \cdot qr$,
(xiii) $(pq)(rr) = p(qr)$,
(xiv) $p(qr) = r(pq) = r(qp)$ *i.e., Q is self-dual.*

Proof. Let Q be an LAD-LA-semigroup and $p, q, r, s \in Q$.

(i) To prove the identity $(pq \cdot r)s = (sp)(rq)$, using left invertive law and the medial law we obtain

$$(pq \cdot r)s = (sr)(pq) = (sp)(rq)$$
$$\Rightarrow (pq \cdot r)s = (sp)(rq).$$

(ii) To prove the identity $(pq \cdot r)r = r(pq)$, using Lemma 3.6 and L.I.L, we get

$$r(pq) = (rp)(rq) = (rq \cdot p)r = (pq \cdot r)r$$
$$\Rightarrow r(pq) = (pq \cdot r)r.$$

(iii) To prove the identity $pp \cdot qr = p \cdot qr$, using (ii) and L.I.L we obtain

$$\text{RHS } = p \cdot qr = (qr \cdot p)p = (pp)(qr) = \text{LHS}.$$

(iv) To prove the identity $(p \cdot qr)(qr) = (qr)(p(p \cdot qr))$, by M.L, (iii) Lemma 3.6 and (ii) we get

$$\begin{aligned}(qr)(p(p \cdot qr)) &= (qr)(p(pq \cdot pr)) = (qr \cdot p)((qr)(pq \cdot pr)) \\ &= (qr \cdot p)((qr)(pp \cdot qr)) = (qr \cdot p)((qr)(p \cdot qr)) \\ &= (qr \cdot p)((p \cdot qr)(qr)) = (qr \cdot (p \cdot qr))(p \cdot qr) = (p \cdot qr)(qr)) \\ \Rightarrow (p \cdot qr)(qr) &= (qr)(p(p \cdot qr)).\end{aligned}$$

(v) To prove the identity $(pq)(qr) = q(p \cdot rp)$, by (ii), Lemma 3.6, and M.L we obtain

$$\begin{aligned}(pq)(qr) &= (qr \cdot pq)(pq) = (qr \cdot qp)(pq) = (q \cdot rp)(pq) \\ &= (qp)(rp \cdot q) = (qp)(q \cdot rp) = q(p \cdot rp) \\ \Rightarrow (pq)(qr) &= q(p \cdot rp).\end{aligned}$$

(vi) To prove the identity $p(q \cdot rq) = p(qr)$, by (iii), Lemma 3.6, paramedial law and (v) we get

$$\begin{aligned} p(qr) &= (pp)(qr) = (pp)(rq) = (qp)(rp) \\ &= (qp)(pr) = p(q \cdot rq) \\ \Rightarrow & p(q \cdot rq) = p(qr). \end{aligned}$$

(vii) To prove the identity $(pq)(qr) = q(pr)$, by (v) and (vi) we have

$$\begin{aligned} (pq)(qr) &= q(p \cdot rp) = q(pr) \\ \Rightarrow & (pq)(qr) = q(pr). \end{aligned}$$

(viii) To prove the identity $(pq)(r(r \cdot pq)) = (pq)(rr)$, by (iv), (ii), L.I.L and (iii) we have

$$\begin{aligned} (pq)(r(r \cdot pq)) &= (r \cdot pq)(pq) = ((pq \cdot r)r)(pq) \\ &= (rr \cdot pq)(pq) = (pq \cdot pq)(rr) = (pq)(rr) \\ \Rightarrow (pq)(r(r \cdot pq)) &= (pq)(rr). \end{aligned}$$

(ix) To prove the identity $(p \cdot qr)(qr) = (qr)(pp)$, using (iv) and (viii) we get

$$\begin{aligned} (p \cdot qr)(qr) &= (qr)(p(p \cdot qr)) = (qr)(pp) \\ \Rightarrow & (p(qr))(qr) = (qr)(pp). \end{aligned}$$

(x) To prove the identity $p(q \cdot qr) = p \cdot qr$, by Lemma 3.6 and (vi) we have

$$\begin{aligned} p(q \cdot qr) &= p(q \cdot rq) = p(qr) \\ \Rightarrow & p(q \cdot qr) = p \cdot qr. \end{aligned}$$

(xi) To prove the identity $p(qq) = q(pp)$, by Lemma 3.6 and (vii) we get

$$\begin{aligned} p(qq) &= (pq)(pq) = (pq)(qp) = q(pp) \\ \Rightarrow & p(qq) = q(pp). \end{aligned}$$

(xii) To prove the identity $p(q \cdot rr) = p(qr)$, by Lemma 3.6, L.I.L, (vi) we have

$$\begin{aligned} p(q \cdot rr)) &= p(rr \cdot q) = p(qr \cdot r) \\ &= p(r \cdot qr) = p(rq) = p(qr) \\ \Rightarrow & p(q \cdot rr) = p(qr). \end{aligned}$$

(xiii) To prove the identity $(pq)(rr) = p(qr)$, by M.L, Lemma 3.6, (vii), (vi), L.I.L, (v), (ix), (xi), (iii) we obtain

$$\begin{aligned}
(pq)(rr) &= (pr)(qr) = (pr)(rq) = r(pq) = r(p \cdot qp) = r(p \cdot pq) \\
&= (rp)(r \cdot pq) = (rp)(pq \cdot r) = (rp)(rq \cdot p) = (rp)(p \cdot rq) \\
&= (rp)(p \cdot qr) = p(r(qr \cdot r)) = p(r(rr \cdot q)) = p((rr \cdot q)r) \\
&= p(rq \cdot rr) = p((r \cdot rq)(rq)) = p((rq \cdot rq)r) = p((rr \cdot qq)r) \\
&= p((r \cdot qq)(rr)) = p((q \cdot rr)(rr)) = p((rr)(q \cdot rr)) = p(r(q \cdot rr)) \\
&= p((q \cdot rr)r) = p((r \cdot rr)q) = p(q(r \cdot rr)) = p((qr)(q \cdot rr)) \\
&= p((qr)(rr \cdot q)) = p((qr)(qr \cdot r)) = p((qr)(r \cdot qr)) = p(qr \cdot r) \\
&= p(r \cdot qr) = p(rq) = p(qr) \\
\Rightarrow (pq)(rr) &= p(qr).
\end{aligned}$$

(xiv) To prove the identity $p(qr) = r(pq)$, by (xiii), Lemma 3.6, (vii) and M.L we have

$$\begin{aligned}
\text{RHS} &= r(pq) = (rp)(qq) = (rq)(pq) = (rq)(qp) = q(rp) \\
&= (qr)(pp) = (qp)(rp) = (qp)(pr) = p(qr) = \text{LHS} \\
&\Rightarrow r(pq) = p(qr).
\end{aligned}$$

Thus $p(qr) = r(pq) = r(qp)$.

Equivalently, Q is self-dual and hence the theorem is proved. □

Theorem 3.8. *Every LAD-LA-*semigroup *is an AG**.*

Proof. Let Q be an LAD-LA-semigroup and let $p, q, r \in Q$. Then by Theorem 3.7 (xiv) and Lemma 3.6

$$p(qr) = q(rp) = q(pr) \Rightarrow p(qr) = q(pr).$$

Thus Q is an AG**. □

The following counterexample depicts that every LAD-LA-semigroup is not associative.

Example 3.9. *Let* $R = \{r, s, t, u\}$. *Then* $(R, \cdot)$ *is an LAD LA-*semigroup *which is not a* semigroup *as* $(r \cdot r)r = u \neq t = r(r \cdot r)$.

$\cdot$	r	s	t	u
r	s	t	t	t
s	u	t	t	t
t	t	t	t	t
u	t	t	t	t

Theorem 3.10. *For each of the following an LAD LA-*semigroup Q *is a* semigroup*;*

(a). Q *is AG*-groupoid,*

(b). Q *is Stein LA-*semigroup.

Proof. (a). Let Q be an AG*-groupoid and $p, q, r \in Q$, then by L.I.L, AG*, LAD and M.L

$$\begin{aligned}(pq)\,r &= (rq)\,p = q\,(rp) = (qr)\,(pq) = (pq \cdot r)\,q = (rq \cdot p)\,q \\ &= p\,(rq \cdot q) = (p \cdot rq)\,(qp) = (qp \cdot rq)\,p \\ &= (q \cdot pr)\,p = (pr)\,(qp) = (pq)\,(rp) = p\,(qr) \\ \Rightarrow (pq)\,r &= p\,(qr)\,.\end{aligned}$$

That is $p\,(qr) = (pq)\,r$. Hence Q is a semigroup.

(b). Let Q be Stein and let $p, q, r \in Q$, then by LAD, M.L, Stein and L.I.L

$$p(qr) = (pq)(rp) = (pr)\,(qp) = p\,(rq) = (rq)\,p = (pq)\,r.$$

Thus, $p\,(qr) = (pq)\,r$. Hence Q is a semigroup. □

It may also be noted that not every semigroup is an LAD-LA-semigroup as depicted in the next example.

Example 3.11. *Let $Q = \{q, r, s, t\}$. Then $(Q, \cdot)$ is a* semigroup*, that is not an LAD-LA-*semigroup.

$\cdot$	q	r	s	t
q	r	q	q	q
r	q	r	r	r
s	q	r	r	r
t	q	r	r	r

Since $q(q \cdot q) = q \neq r = (q \cdot q)(q \cdot q)$, Q is not an LAD.

3.12. **Relations of LAD-LA-semigroup with AG* and AG**-groupoids.** Here, we shall describe some relations among the LAD-LA-semigroup with the well-worked subclasses AG* and AG**-groupoids. The produced examples in this section show that neither LAD-LA-semigroup is AG*-groupoid nor an AG**-groupoid. We further compare a self-dual LA-semigroup with both the LAD LA-semigroup and an AG**-groupoid.

Example 3.5 shows that LAD-LA-semigroup is not AG*-groupoid. Furthermore, the converse also may not be true as provided below.

Example 3.13. *The table as given below is an AG*-groupoid but is not LAD as $r(r \cdot r) = r \neq s = (r \cdot r)(r \cdot r)$.*

$\cdot$	r	s	t	u
r	s	r	r	r
s	r	s	s	s
t	r	s	s	s
u	r	s	s	s

Lemma 3.14. *Every LAD-AG*-groupoid Q is CA-LA*-semigroup.

Proof. Let $p, q, r \in Q$, then by AG*, LAD and medial law;

$$\begin{aligned}p\,(qr) &= q\,(pr) = (qp)\,(rq) = (qr)\,(pq) = q\,(rp) \\ &= r\,(qp) = (rq)\,(pr) = (rp)\,(qr) \\ \Rightarrow p\,(qr) &= r\,(pq)\,.\end{aligned}$$

Thus Q is a CA-LA-semigroup. □

Since each CA-LA-semigroup is Bol* [31, Theorem 1], every Bol*-LA-semigroup is paramedial [31, Lemma 9] and every paramedial is left nuclear square [31]. Further, every CA-LA-semigroup is right nuclear square [31, Theorem 3], thus from Theorem 3.14 we have the following;

Corollary 3.15. *Every LAD-AG*-groupoid is (i) Bol*, (ii) paramedial, (iii) left nuclear square, (iv) right nuclear square.*

Proposition 3.16. [31] *Every Bol*-LA-band is commutative.*

The combination of Proposition 3.16 and Theorem 3.14, and the fact that a commutative LA-semigroup is always associative, evidently produces the following results.

Corollary 3.17. *LAD-AG*-band is commutative.*

Corollary 3.18. *Every LAD-AG*-band is semilattice.*

Example 3.19. *The following LAD-LA*-semigroup *of lowest order is not* T^1*, as* $p \cdot p = r = p \cdot q$*, but* $p \cdot p = r \neq s = q \cdot p$*.*

$\cdot$	p	q	r	s
p	r	r	r	r
q	s	r	r	r
r	r	r	r	r
s	r	r	r	r

Theorem 3.20. *Left* cancellative *LAD is* T^1*-LA*-semigroup.

Proof. Let p, q, r, s and x be elements of a left cancellative LAD-LA-semigroup Q such that x is cancellative in Q. Now let $pq = rs$, we have to show that $qp = sr$. By the properties of LAD, L.I.L and the assumption;

$$\begin{aligned} x(qp) &= (xq)(px) = (px \cdot q)x = (qx \cdot p)x \\ &= (xp)(qx) = x(pq) = x(rs) = (xr)(sx) \\ &= (sx \cdot r)x = (rx \cdot s)x = (xs)(rx) \\ \Rightarrow x(qp) &= x(sr). \end{aligned}$$

Thus Q is a T^1-LA-semigroup. □

Example 3.21. T^1*-LA*-semigroup *of order* 4 *that is not LAD, as* $q(qq) \neq (qq)(qq)$*.*

$\cdot$	q	r	s	t
q	r	q	q	q
r	q	r	r	r
s	q	r	r	r
t	q	r	r	r

Theorem 3.22. *Every AG*-band is associative.*

Proof. Let Q be an AG*-band and $p, q, r \in Q$, then by AG-band, medial and L.I.L and AG*;

$$\begin{aligned} p\,(qr) &= (pp)\,(qr) = (pq)\,(pr) = (pr \cdot q)\,p = (qr \cdot p)\,p = (qr \cdot pp)\,p \\ &= (qp \cdot rp)\,p = (p \cdot rp)\,(qp) = (rp)\,(p \cdot qp) = (rp)\,(pp \cdot qp) \\ &= (rp)\,(pq \cdot pp) = (rp)\,(pq \cdot p) = ((pq \cdot p)\,p)\,r \\ &= ((pp)\,(pq))\,r = (p \cdot pq)\,r = (r \cdot pq)\,p = (pq)(rp) \\ \Rightarrow p\,(qr) &= (pq)\,(rp)\,. \end{aligned}$$

Thus Q is an LAD-LA-semigroup and hence by Corollary 3.18 Q is associative. □

Next, we show by an example that the converse implication may not be true.

Example 3.23. *LAD-LA*-semigroup Q *with an idempotent element* r. *Clearly* Q *is not an AG*-band as* $(p \cdot p)p = s \neq r = p(p \cdot p)$ *and* $q \cdot q = r \neq q$.

$\cdot$	p	q	r	s
p	q	r	r	r
q	s	r	r	r
r	r	r	r	r
s	r	r	r	r

Example 3.24. *Let* $Q = \{p, q, r, s, t, u\}$. *Then* Q *with the following table represents a* semigroup. *As* $(p \cdot q)\,p = u \neq q = q\,(p \cdot p)$, *thus is not an AG*.*

$\cdot$	p	q	r	s	t	u
p	q	s	q	q	u	q
q	t	q	q	q	q	q
r	q	q	q	q	q	q
s	u	q	q	q	q	q
t	q	q	q	q	q	q
u	q	q	q	q	q	q

Theorem 3.25. *Every LAD-LA-semigroup* Q *is nuclear square.*

Proof. By Lemma 3.6 every LAD-LA-semigroup is paramedial and by Theorem 2.3 every paramedial is left nuclear square.

Next, we prove that LAD is right nuclear square. For this let $p, q, r \in Q$, then by the LAD, M.L, AG*, left nuclear square, self-dual and RC properties

$$\begin{aligned} p(qr^2) &= (pq)(r^2p) = (pr^2)(qp) = p(r^2q) = (r^2p)q \\ &= r^2(pq) = q(r^2p) = q(pr^2) = (pq)r^2 \\ \Rightarrow p(qr^2) &= (pq)r^2. \end{aligned}$$

Thus Q is a right nuclear square.

Finally, let Q be an LAD-LA-semigroup and $p, q, r \in Q$, then by self-dual, LAD, M.L, AG*, left nuclear square

$$\begin{aligned}(pq^2)r &= r(pq^2) = (rp)(q^2r) = (rq^2)(pr) = r(q^2p)\\ &= (q^2r)p = q^2(rp) = (rq^2)p = (pq^2)r\\ \Rightarrow (pq^2)r &= (pq^2)r.\end{aligned}$$

Thus Q is middle nuclear square. Hence the theorem is proved. □

Proposition 3.26. *Every associative LAD-LA-semigroup is an AG*-groupoid.*

Proof. Let p, q, r be elements of an LAD-semigroup Q, then by L.I.L, M.L, semigroup and LAD properties;

$$\begin{aligned}(pq)\,r &= (rq)\,p = r\,(qp) = (rq)\,(pr) = (rp)\,(qr)\\ &= r\,(pq) = (rp)\,q = (qp)\,r = q(pr)\\ \Rightarrow (pq)\,r &= q\,(pr)\,.\end{aligned}$$

Thus Q is an AG*. □

The following example reflects that a direct relation between AG* and Stein LA-semigroups does not exist. Further, it is proved that LAD-Stein becomes an AG*.

Example 3.27. *We list here two tables of order 6 and 5 respectively.* (i) *AG*-groupoid that is not a Stein LA*-semigroup as $p(p \cdot q) = t \neq u = (p \cdot q)p$ and,
(ii) *Stein LA*-semigroup *that is not an AG* as* $(p \cdot p)q = t \neq s = p(p \cdot q)$.

(i)

$\cdot$	p	q	r	s	t	u
p	r	s	t	t	t	t
q	r	s	u	u	t	t
r	t	t	t	t	t	t
s	u	u	t	t	t	t
t	t	t	t	t	t	t
u	t	t	t	t	t	t

(ii)

$\cdot$	p	q	r	s	t
p	r	r	s	t	t
q	s	s	t	t	t
r	s	t	t	t	t
s	t	t	t	t	t
t	t	t	t	t	t

Theorem 3.28. *An LAD-Stein LA-semigroup is always an AG*-groupoid.*

Proof. Suppose Q is an LAD-Stein and $p, q, r \in Q$, then by Stein, LAD, M.L, L.I.L

$$\begin{aligned}(pq)\,r &= r\,(pq) = (rp)\,(qr) = (rq)\,(pr) = r\,(qp) = (qp)\,r\\ &= (rp)\,q = q\,(rp) = (qr)\,(pq) = (qp)\,(rq) = q(pr)\\ \Rightarrow (pq)\,r &= q\,(pr)\,.\end{aligned}$$

Thus Q is an AG*-groupoid. □

Example 3.29. *LAD-LA*-semigroup *given in Example 3.5 is an AG**, but it is not a* semigroup *since* $(p \cdot p)p \neq p(p \cdot p)$.

3.30. **Relations between LAD and alternative LA-semigroups.** Here, a counterexample is produced to show that not every LAD-LA-semigroup may necessarily be a left or a right alternative LA-semigroup.

Example 3.31. *Let* $Q = \{p, q, r, s\}$*.Then* $(Q, \cdot)$ *is an LAD-LA*-semigroup*. However, it is not left alternative LA*-semigroup *as* $(p \cdot p)p = s \neq r = p(p \cdot p)$*. Similarly, it is not right alternative.*

$\cdot$	p	q	r	s
p	q	r	r	r
q	s	r	r	r
r	r	r	r	r
s	r	r	r	r

Theorem 3.32. *For an LAD-LA*-semigroup Q *with a left* cancellative *element each of the following is true.*
(i) Q *is left alternative,*
(ii) Q *is right alternative.*

Proof. Let x be a left cancellative element in Q and $p, q, r, s \in Q$.
(i) By the given properties;

$$\begin{aligned} x\,(pp \cdot q) &= x\,(qp \cdot p) = (x \cdot qp)\,(px) = (xp)\,(qp \cdot x) = x\,(p \cdot qp) \\ &= x\,(pq \cdot pp) = x\,(pp \cdot qp) = x\,(p \cdot pq) \\ \Rightarrow pp \cdot q &= p \cdot pq \text{ by left cancellativity of } x. \end{aligned}$$

Thus Q is a left alternative.
(ii) By assumption and the given condition of x,

$$\begin{aligned} x^2\,(q \cdot pp) &= \left(x^2 q\right)\left(pp \cdot x^2\right) = \left(x^2 \cdot pp\right)\left(qx^2\right) = x^2\,(pp \cdot q) \\ &= (xx)\,(pp \cdot q) = (x \cdot pp)\,(xq) = (xq \cdot pp)\,x \\ &= ((pp \cdot q)\,x)\,x = (xx)\,(pp \cdot q) = x^2\,(qp \cdot p) \\ \Rightarrow (q \cdot pp) &= (qp \cdot p)\,. \end{aligned}$$

Thus Q is a right alternative LA-semigroup. □

Corollary 3.33. *Every LAD-LA*-semigroup *having a* cancellative *element is an alternative LA*-semigroup.

Theorem 3.34. *Every LAD-AG*-groupoid is Stein.*

Proof. Let Q be LAD-AG* and $p, q, r \in Q$, then by LAD, M.L and by the properties of AG* we have;

$$\begin{aligned} (qr)\,p &= r(qp) = (rq)(pr) = (rp)(qr) = r(pq) = p(rq) \\ &= (pr)(qp) = (pq)(rp) = p(qr). \end{aligned}$$

Thus $(qr)p = p(qr)$ and equivalently Q is a Stein LA-semigroup. □

Theorem 3.35. *LAD-AG-band is a commutative* semigroup.

Proof. Let Q be an LAD-LA-band and $p, q \in Q$, then by LA-band, M.L, paramedial law

$$pq = p\,(qq) = (pp)\,(qq) = (qp)\,(qp) = (qq)\,(pp) \Rightarrow pq = qp.$$

Thus Q is commutative and hence a semigroup. □

3.36. **Relation between LAD-LA-semigroup and semigroup.** Example 3.31 clearly pictures that not every LAD-LA-semigroup may be a semigroup. However, it becomes possible under certain conditions as highlighted in the following theorem.

Theorem 3.37. *For each of the following an LAD-LA*-semigroup Q *is a commutative* semigroup.
(i) Q *has a right* cancellative *element,*
(ii) Q *has a left identity.*

Proof. (i) Let Q be an LAD-LA-semigroup having a right cancellative element x and $p, q, r \in Q$. Then by M.L, paramedial law, LAD and right cancellativity of x;

$$\begin{aligned} (pq)\,x^2 &= (pq)\,(xx) = (px)\,(qx) = (xx)\,(qp) = x^2\,(qp) \\ &= \left(x^2 q\right)\left(px^2\right) = \left(x^2 p\right)\left(qx^2\right) = x^2\,(pq) = (xx)\,(pq) \\ &= (qx)\,(px) = (qp)\,(xx) = (qp)\,x^2 \\ \Rightarrow pq &= qp. \end{aligned}$$

Thus Q is commutative and hence a semigroup.
(ii) Let e be the left identity of Q and $p, q, r \in Q$. Then by LAD, M.L

$$pq = e(pq) = (ep)(qe) = (eq)(pe) = e(qp) \Rightarrow pq = qp.$$

Hence Q is commutative and thus a semigroup. □

Theorem 3.38. *Every AG**-band is an LAD-LA*-semigroup.

Proof. Let Q be AG**-band and $p, q, r \in Q$, then by the properties of AG-band, M.L, L.I.L, AG**, paramedial law

$$\begin{aligned} p\,(qr) &= (pp)\,(qr) = (pq)\,(pr) = (pr \cdot q)\,p = (qr \cdot p)\,p \\ &= (qr \cdot pp)\,p = (qp \cdot rp)\,p = (p \cdot rp)\,(qp) = (pp \cdot rp)\,(qp) \\ &= ((rp \cdot p)\,p)\,(qp) = ((pp \cdot r)\,p)\,(qp) = (qp \cdot p)\,(pp \cdot r) \\ &= (rp)\,(pp \cdot qp) = (rp)\,(pq \cdot pp) = (pq)\,(rp \cdot pp) \\ &= (pq)\,(rp \cdot p) = (pq)\,(pp \cdot r) = (pq)\,(pp \cdot rr) \\ &= (pq)\,(pr \cdot pr) = (pq)\,(rr \cdot pp) = (pq)\,(rp) \\ \Rightarrow p\,(qr) &= (pq)\,(rp)\,. \end{aligned}$$

Thus Q is an LAD-LA-semigroup. □

Theorem 3.39. *LAD-LA*-semigroup *is right commute if it is AG**.*

Proof. Let Q be an LAD-LA-semigroup satisfying the AG** property and $p, q, r \in Q$, then by the assumption and M.L we have;

$$p(qr) = q(pr) = (qp)(rq) = (qr)(pq) = q(rp)$$
$$= r(qp) = (rq)(pr) = (rp)(qr) = r(pq).$$

Thus $p(qr) = p(rq)$. Hence Q is right commute. □

3.40. **Ideals in LAD-LA-semigroups.** We shall introduce ideals in LAD-LA semigropus in this section and shall include various examples to show that their existence. We recall the following definition,

Definition 3.41. *Let Q be an LA*-semigroup*, a subset B of Q is called left (resp., right) ideal if $QB \subseteq B$ (resp., $BQ \subseteq B$) and B is an ideal if it is both a left and right ideal.*

Next, we shall provide some examples to demonstrate that more than one ideals may be occurred in an LAD-LA-semigroup. In this note, we shall also illustrate that neither left ideal deduce right ideal nor right ideal speculate left ideal for an LAD-LA-semigroup. We show that some of the subsets act as left ideal but not a right ideal and vice versa for an LAD-LA-semigroup. Keeping continue, we shall also provide examples to show that various subsets of an LAD-LA-semigroup are even ideals but some of them are neither left nor right ideal.

Example 3.42. *Let Q =$\{p, q, r, s, t\}$. Then $(Q, \cdot)$ is an LAD-LA*-semigroup*.*

$\cdot$	p	q	r	s	t
p	q	r	r	r	q
q	s	r	r	r	s
r	r	r	r	r	r
s	r	r	r	r	r
t	q	r	r	r	q

If $B_1 = \{q, r\}$, then $QB_1 = \{r\} \subseteq B_1$. Hence B_1 is a left ideal of Q. Again as $B_1Q = \{q, r, s\} \nsubseteq B_1$, B_1 is not right ideal of Q.

If $B_2 = \{r, s\}$, then B_2 is an ideal of Q, i.e. $B_2Q = \{r\} \subseteq B_2$. Again $QB_2 = \{r\} \subseteq B_2$.

If $B_3 = \{q, r, s, t\}$, then $QB_3 = \{q, r, s\} \subseteq B_3$, hence B_3 is left ideal of Q. Again $B_3Q = \{q, r, s\} \subseteq B_3$, thus B_3 is also a right ideal and hence an ideal of Q.

If $B_4 = \{p, q\}$, then $QB = \{q, r, s\} \nsubseteq B_4$ hence B_4 is not left ideal of Q. Again $B_4Q = \{q, r, s\} \nsubseteq B_4$, thus B_4 is neither a right nor a left ideal of Q.

$B_1 = \{q, r\}, B_2 = \{r, s\}, B_3 = \{q, s\}, B_4 = \{q, r, s\}, B_5 = \{p, q, s\}, B_6 = \{q, r, s, t\}$ are left ideals of Q i.e. $B_1$is left ideal but not right ideal of Q, B_2 is left and right ideal, B_3 is left but not a right ideal, B_4 is neither left nor a right ideal, B_5 is left but not a right ideal and B_6 is an ideal of Q.

It can easily be deduced that if a right identity element is contained in an LA-semigroup, it becomes a commutative semigroup, further, we have also proved in Theorem 3.37 that if a left identity is present in an LAD-LA-semigroup, it becomes a semigroup, hence in this scenario, the left and right ideals make a relation.

Theorem 3.43. *Let p be a fixed element of an LAD-LA-*semigroup *Q with right identity e. Then pQ is an ideal of Q.*

Proof. Let Q be an LAD-LA-semigroup with right identity e and p be a fixed element of Q, then for any $x, y \in Q$ and by L.I.L, M.L, LAD

$$\begin{aligned}(pQ)Q &= \underset{x,y\in Q}{\cup}(px)y = (yx)p = (yx)(pe) = (yp)(xe) \\ &= (xe \cdot p)y = (xp)y = (xp)(ye) = (xy)(pe) \\ &= (pe \cdot y)x = (py)x = (py)(xe) = (py \cdot x)(e \cdot py) \\ &= (py \cdot e)(x \cdot py) = (py)(x \cdot py) = (py)(xe \cdot py) \\ &= (p \cdot xe)(y \cdot py) = (px)(y \cdot py) = (px)(ye \cdot py) \\ &= (px)(y \cdot ep) = (py)(x \cdot ep) = (py)(xe \cdot px) \\ &= (py)(x \cdot px) = (px)(y \cdot px) = (px \cdot e)(y \cdot px) \\ &= (px)(ey) = (pe)(xy) = \underset{x,y\in Q}{\cup}p(xy) \subseteq pQ\end{aligned}$$

$$\Rightarrow (pQ)Q \subseteq pQ.$$

Thus pQ is a right ideal of Q. Again by M.L, L.I.L

$$\begin{aligned}Q(pQ) &= \underset{x,y\in Q}{\cup}x(py) = (xe)(py) = (xp)(ey) = (ey \cdot p)x \\ &= (py \cdot e)x = (py)x = (py)(xe) = (xe \cdot y)p \\ &= (xe \cdot ye)p = (xy \cdot ee)p = (xy \cdot e)p \\ &= (pe)xy = \underset{x,y\in Q}{\cup}p(xy) \subseteq pQ\end{aligned}$$

$$\Rightarrow Q(pQ) \subseteq pQ.$$

This proves that pQ is a left ideal of Q. □

Theorem 3.44. *Let p be a fixed element of an LAD-LA-*semigroup *Q with right identity e and B be a left ideal of Q. Then pB is a right ideal of Q.*

Proof. Since B is left ideal of Q, $QB \subseteq B$.

Now, let $(pb)g \in (pB)Q$, then by L.I.L, M.L, and the assumption of LAD

$$\begin{aligned}(pB)Q &= \bigcup_{g\in Q, b\in B}(pb)g = (gb)p = (gb)(pe) = (gp)(be)\\ &= (be\cdot p)g = (bp)g = (bp)(ge) = (bg)(pe)\\ &= (pe\cdot g)b = (pg)b = (pg)(be) = (pg\cdot b)(e\cdot pg)\\ &= (pg\cdot e)(b\cdot pg) = (pg)(b\cdot pg) = (pg)(be\cdot pg)\\ &= (p\cdot be)(g\cdot pg) = (pb)(g\cdot pg) = (pb)(ge\cdot pg)\\ &= (pb)(g\cdot ep) = (pg)(b\cdot ep) = (pg)(be\cdot pb)\\ &= (pg)(b\cdot pb) = (pb)(g\cdot pb) = (pb\cdot e)(g\cdot pb)\\ &= (pb)(eg) = (pe)(bg) = p(bg) = (pb)(gp)\\ &= (pg)(bp) = p(gb) = \bigcup_{g\in Q, b\in B}p(gb) \subseteq p(QB) \subseteq pB\end{aligned}$$

$$\Rightarrow (pB)Q \subseteq pB.$$

This proves that pB is a right ideal of Q. □

Theorem 3.45. *Let B be a right ideal of an LAD-LA-*semigroup *Q with a left identity e. Then pB is a left ideal of Q for all p in Q.*

Proof. Let $p \in Q$ and B be a right ideal of LAD-LA-semigroup Q. Then by definition $BQ \subseteq B$. Now, by M.L, L.I.L and the assumption of LAD

$$\begin{aligned}Q(pB) &= \bigcup_{g\in Q, b\in B}g(pb) = (ge)(pb) = (gp)(eb) = (eb\cdot p)g\\ &= (pb\cdot e)g = (pb)g = (pb)(ge) = (ge\cdot b)p\\ &= (ge\cdot be)p = (gb\cdot ee)p = (gb\cdot e)p = (pe)gb\\ &= p(gb) = (pg)(bp) = (pb)(gp) = \bigcup_{g\in Q, b\in B}p(bg) \subseteq p(BQ) \subseteq pB\end{aligned}$$

$$\Rightarrow Q(pB) \subseteq pB.$$

Hence, pB is a left ideal of Q. □

3.46. **Conclusion.** A new class of LA-semigroups, LAD-LA-semigroup is investigated. Various relationship among different subclasses of LA-semigroup and LAD-LA-semigroup are established. A suggested hard level problem is resolved that LAD-LA-semigroup is self-dual. Ideals are defined and investigated that could be extended and further investigated for other kinds of ideals like: prime ideals, semi-prime and bi-ideals and many more.

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