Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol.52(2)(2020) pp 57-72

Some Iterative Methods to Solve Nonlinear Equations Having Faster Convergence

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Received: 29 May, 2019 / Accepted: 27 December, 2019 / Published online: 01 February, 2020

Abstract. This paper presents modified Potra and Pták method to solve nonlinear equations having single variable using McDougall and Wotherspoon scheme. Also two iterative methods are obtained as variants of Potra and Pták method by merging iterations of secant method and the method given by Amat and Bascular. The convergence order of each newly obtained method is higher than that of Potra and Pták method. Finally, some examples are demonstrated to know performances of these methods and to compare these among themselves and other existing methods.

2000 Mathematics Subject Classification: 65H05
Keywords: Convergence order, Newton's method, Predictor-corrector method, Nonlinear equations, Hybrid methods.

1. Introduction

Single variable nonlinear equations appear in almost all areas of mathematical as well as physical sciences. However, analytic solutions of such equations are almost impossible or the process of finding such solutions may be tedious. In such situation, numerical methods are employed to get solution of nonlinear equation

$$f(x) = 0 \tag{1. 1}$$

The two commonly used such classical methods are Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1. 2}$$

and secant method

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \tag{1. 3}$$

The method (1.2) converges quadratically and convergence order of the method (1.3) is 1.618 [2]. Newton's method requires first derivative of the function but sometime finding the derivative of the function becomes complicated. To avoid this difficulty, we can use Steffensen's method [7]

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)} \tag{1. 4}$$

The convergence order of this method is same as Newton's method and is obtained by replacing $f'(x)$ by the ratio $\frac{f(x_n+f(x_n))-f(x_n)}{f(x_n)}$ in the Newton's method (1. 1). In literature, several methods have appeared as variants of these methods using different techniques. Some of them are found in [1], [3] -[19] and references therein. Here, it is not possible to describe the development of all methods. However, we will mention the work of those authors which motivated us to carry out this investigations.

In the work of Weerakoon and Fernando [19], they used the technique of numerical integration to improve convergence order of Newton's method. They used Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\, dt \tag{1. 5}$$

and approximated the integral by trapezoidal rule that is

$$\int_{x_n}^{x} f'(t)\, dt = \frac{(x - x_n)}{2}[f'(x) + f'(x_n)]. \tag{1. 6}$$

Then they obtained the variant of Newton's method which is given by the formula

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n^*)}, \tag{1. 7}$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$.

Recently in 2014, McDaugall and Wotherspoon [14] obtained a modified Newton's method using a different strategy. Their method is as follows:
If x_0 is the initial approximation, then

$$x_0^* = x_0 \tag{1. 8}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'[\frac{1}{2}(x_0 + x_0^*)]}. \tag{1. 9}$$

Subsequently for $n \geq 1$, the iterations can be obtained as

$$x_n^* = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1} + x_{n-1}^*)]} \tag{1. 10}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]}. \tag{1. 11}$$

It was proved that the last method is of order $1 + \sqrt{2}$.

In [17], Potra and Pták proposed the iterative method

$$x_{n+1} = x_n - \frac{f\left[x_n - \frac{f(x_n)}{f'(x_n)}\right] + f(x_n)}{f'(x_n)} \tag{1. 12}$$

to solve nonlinear equations with single variable as a improved Newton's method. The convergence order of this method is three which can be shown easily by converting the iteration (1. 12) to a fixed point problem $\Phi(x) = x$, where

$$\Phi(x) = x - \frac{f\left[x - \frac{f(x)}{f'(x)}\right] + f(x)}{f'(x)}. \tag{1. 13}$$

For the third order convergence, it is sufficient to show that at the fixed point of Φ, the derivative of Φ' and Φ'' vanish; this can be verified by elementary calculations . At first, we modify this method using McDougall and Wotherspoon scheme [14] and obtain new method having order of convergence 3.5615. Again, we construct hybrid methods by amalgamating the iterations of method (1.4) with existing method (1.3), and Amat and Busquier method [1], respectively, and obtain two new methods having order of convergence 4 and 6. Finally, we observe some numerical examples to compare the effectiveness of newly obtained methods and similar existing methods.

2. THE METHOD AND ITS CONVERGENCE RESULT

Here, we propose the method obtained by modifying the method (1. 12) given by Potra and Pták using McDougall and Wotherspoon's predictor-corrector method [14]. The method is given below:
If x_0 is the initial approximation, then

$$x_0^* = x_0 \tag{2. 14}$$

$$\begin{aligned} x_1 &= x_0 - \frac{f\left[x_0 - \frac{f(x_0)}{f'\left(\frac{x_0+x_0^*}{2}\right)}\right] + f(x_0)}{f'\left(\frac{x_0+x_0^*}{2}\right)} \\ &= x_0 - \frac{f\left[x_0 - \frac{f(x_0)}{f'(x_0)}\right] + f(x_0)}{f'(x_0)}. \end{aligned} \tag{2. 15}$$

Subsequently, for $n \geq 1$, the iterations can be calculated as follows:

$$x_n^* = x_n - \frac{f\left[x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)}\right] + f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)}, \tag{2. 16}$$

$$x_{n+1} = x_n - \frac{f\left[x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}\right] + f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}. \tag{2. 17}$$

For the convergence of the method (2. 14)-(2. 17), we prove the result given below:

Theorem 2.1. *Let α be a simple zero of a function f which has enough number of continuous derivatives in a neighborhood of α. If x_0 is sufficiently close to α, then the method (2. 14)-(2. 17) is convergent and has the convergence order* 3.5615.

Proof. Suppose e_n and e_n^* denote, respectively, the errors in the terms x_n and x_n^*. Also, we denote $c_j = \dfrac{f^j(\alpha)}{j!f'(\alpha)}$, $j = 2, 3, 4...$, which are constants. Then from (2. 14), $x_0^* = x_0$ implies $e_0^* = e_0$. We now proceed to calculate the error e_1 in x_1. By using Taylor series expansion and binomial expansion, we get

$$\begin{aligned}
x_0 - \frac{f(x_0)}{f'(x_0)} &= \alpha + e_0 - \frac{f(\alpha + e_0)}{f'(\alpha + e_0)} \\
&= \alpha + e_0 - \frac{f'(\alpha)[e_0 + c_2e_0^2 + c_3e_0^3 + O(e_0^4)]}{f'(\alpha)[1 + 2c_2e_0 + 3c_3e_0^2 + 4c_4e_0^3 + O(e_0^4)]} \\
&= \alpha + e_0 - [e_0 + c_2e_0^2 + c_3e_0^3 + O(e_0^4)][1 + 2c_2e_0 + 3c_3e_0^2 + 4c_4e_0^3 + O(e_0^4)]^{-1} \\
&= \alpha + c_2e_0^2 + (2c_3 - 2c_2^2)e_0^3 + O(e_0^4),
\end{aligned}$$

so that after some calculations, we get

$$f\left(x_0 - \frac{f(x_0)}{f'(x_0)}\right) = f'(\alpha)[c_2e_0^2 + (2c_3 - 2c_2^2)e_0^3 + c_2^3e_0^4 + O(e_0^5)],$$

$$f\left(x_0 - \frac{f(x_0)}{f'(x_0)}\right) + f(x_0) = f'(\alpha)[e_0 + 2c_2e_0^2 + 3c_3e_0^3 - 2c_2^2e_0^3 + c_2^3e_0^4 + O(e_0^5)]$$

and

$$\begin{aligned}
\frac{f\left(x_0 - \frac{f(x_0)}{f'(x_0)}\right) + f(x_0)}{f'(x_0)} &= [e_0 + 2c_2e_0^2 + 3c_3e_0^3 - 2c_2^2e_0^3 + c_2^3e_0^4 + O(e_0^5)] \\
&\quad [1 + 2c_2e_0 + 3c_3e_0^2 + 4c_4e_0^3 + O(e_0^4)]^{-1} \\
&= e_0 - 2c_2^2e_0^3 + O(e_0^4).
\end{aligned}$$

Hence from (2. 15),

$$\alpha + e_1 = \alpha + e_0 - e_0 + 2c_2^2e_0^3 + O(e_0^4)$$

i.e.,

$$e_1 = ae_0^3, \tag{2. 18}$$

where $a = 2c_2^2$ and we have neglected the higher power of e_n. Again, from (2. 16)

$$x_1^* = x_1 - \frac{f\left[x_1 - \dfrac{f(x_1)}{f'(x_0)}\right] + f(x_1)}{f'(x_0)}. \tag{2. 19}$$

Here,

$$\begin{aligned}
f(x_1) &= f(\alpha + e_1) \\
&= f'(\alpha)[e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)], \\
f'(x_0) &= f'(\alpha + e_0) \\
&= 1 + 2c_2e_0 + 3c_3e_0^2 + 4c_4e_0^3 + O(e_0^4).
\end{aligned}$$

Also,

$$f\left[x_1 - \frac{f(x_1)}{f'(x_0)}\right] = f\left[\alpha + e_1 - \frac{e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)}{1 + 2c_2e_0 + 3c_3e_0^2 + O(e_0^3)}\right].$$

After some calculation, we get

$$f\left[x_1 - \frac{f(x_1)}{f'(x_0)}\right] = f'(\alpha)[2c_2e_0e_1 + 3c_3{e_0}^2e_1 - 4c_2^2e_0^2e_1 + O(e_0^6)]$$

so that

$$f\left[x_1 - \frac{f(x_1)}{f'(x_0)}\right] + f(x_1) = f'(\alpha)[e_1 + 2c_2e_0e_1 + 3c_3e_0^2e_1 - 4c_2^2e_0^2e_1 + \cdots]$$

and

$$\begin{aligned}\frac{f\left[x_1 - \frac{f(x_1)}{f'(x_0)}\right] + f(x_1)}{f'(x_0)} &= [e_1 + 2c_2e_0e_1 + 3c_3e_0^2e_1 - 4c_2^2e_0^2e_1 + \cdots] \\ &\quad [1 + 2c_2e_0 + 3c_3e_0^2 + \cdots]^{-1} \\ &= e_1 - 4c_2^2e_0^2e_1 + O(e_0^4). \end{aligned} \tag{2. 20}$$

From (2. 19), the error e_1^* in x_1^* can be calculated as

$$\begin{aligned} e_1^* &= e_1 - [e_1 - 4c_2^2e_0^2e_1 + O(e_0^4)] \\ &= 4c_2^2e_0^2e_1 + O(e_0^4) \\ &= abe_0^5, \end{aligned} \tag{2. 21}$$

where $b = 4c_2^2$ and we have neglected the higher power terms of e_0.

Next, we compute the error e_2 in x_2. Now,

$$\begin{aligned} \frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)} &= \frac{f'(\alpha)[e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)]}{f'\left(\alpha + \frac{e_1+e_2}{2}\right)} \\ &= \frac{e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)}{1 + c_2e_1 + c_2e_1^* + \frac{3}{4}c_3e_1^2 + O(e_1^3)} \\ &= e_1 + \frac{1}{4}c_3e_1^3 - c_2e_1e_1^* - c_2^2e_1^2e_1^* + \cdots \end{aligned}$$

so that

$$x_1 - \frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)} = \alpha - \frac{1}{4}c_3e_1^3 + c_2e_1e_1^* + c_2^2e_1^2e_1^*,$$

where the higher power terms are neglected. Thus

$$f\left(x_1 - \frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}\right) = f'(\alpha)[c_2e_1e_1^* + c_2^2e_1^2e_1^* - \frac{1}{4}c_3e_1^3]$$

and

$$f\left(x_1-\frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}\right)+f(x_1)=e_1f'(\alpha)(1+c_2e_1+c_3e_1^2+c_2e_1^*+c_2^2e_1e_1^*-\frac{1}{4}c_3e_1^2+\cdots).$$

Also,

$$\frac{f\left(x_1-\frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}\right)+f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}=e_1-\frac{3}{2}c_3e_1^2e_1^*+\cdots.$$

From (2. 17),

$$x_2=x_1-\frac{f\left(x_1-\frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}\right)+f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}$$

Thus, substituting the values, we get

$$\begin{aligned}\alpha+e_2&=\alpha+e_1-(e_1-\frac{3}{2}c_3e_1^2e_1^*+\cdots)\\ \Rightarrow e_2&=\frac{3}{2}c_3e_1^2e_1^*+O(e_0^{12})\\ \therefore\qquad e_2&=\frac{3}{2}c_3e_1^2e_1^*=a^3bce_0^{11},\end{aligned}$$

where $c=\frac{3}{2}c_3$. In fact, it can be worked out for $n\geq 1$, that the following relation holds:

$$e_{n+1}=ce_n^2e_n^*. \tag{2. 22}$$

In order to compute e_{n+1} explicitly, we need e_n^*. We already find e_1^*. We now compute e_2^*. From (2. 16)

$$x_2^*=x_2-\frac{f\left[x_2-\frac{f(x_2)}{f'\left(\frac{x_1+x_1^*}{2}\right)}\right]+f(x_2)}{f'\left(\frac{x_1+x_1^*}{2}\right)}.$$

Using similar process as above, the error e_2^* in x_2^* can be calculated as

$$e_2^*=de_1^2e_2,$$

where $d=c_2^2$. Again it can be checked that, in general, for $n\geq 2$, the following relation holds:

$$e_n^*=de_{n-1}^2e_n. \tag{2. 23}$$

From (2. 22) and (2. 23), it is clear that the errors e_n^* and e_{n+1}, respectively, in x_n^* and x_{n+1} for $n\geq 2$ in the method (2. 14)-(2. 17) satisfy the following recursion formula:

$$e_n^*=de_{n-1}^2e_n \tag{2. 24}$$

$$e_{n+1}=ce_n^2e_n^*. \tag{2. 25}$$

To find convergence order of this method, we find a relation in the form

$$e_{n+1} = Ae_n^p, \tag{2. 26}$$

where A is some constant. Thus,

$$e_n = Ae_{n-1}^p \qquad \text{or} \qquad e_{n-1} = A^{-\frac{1}{p}} e_n^{\frac{1}{p}}. \tag{2. 27}$$

From (2. 24), (2. 25), (2. 26) and (2. 27),

$$Ae_n^p = ce_n^2 e_n^* = ce_n^2 de_{n-1}^2 e_n = cde_n^2 A^{\frac{-2}{p}} e_n^{\frac{2}{p}} e_n = cdA^{\frac{-2}{p}} e_n^{(3+\frac{2}{p})}.$$

Equating the power of e_n,

$$p = 3 + \frac{2}{p}$$

$$\text{or,} \qquad p^2 - 3p - 2 = 0$$

$$\text{or,} \qquad p = \frac{3 \pm \sqrt{17}}{2}.$$

Taking positive value, $p = 3.5615$. Thus, the method (2. 14)-(2. 17) is convergent with order 3.5615. □

3. HYBRID METHODS AND THEIR CONVERGENCE ANALYSIS

Here, our aim is to introduce new iterative methods whose convergence order are higher than that of method (1. 12) given by Potra and Pták in [17]. For this, we suggest the method where iterations of method (1. 12) and secant method (1. 3) are performed alternately. The method is given below:

$$x_{n+1} = \overline{x_n} - \frac{\overline{x_n} - x_n}{f(\overline{x_n}) - f(x_n)} f(\overline{x_n}), \tag{3. 28}$$

$$\text{where} \qquad \overline{x_n} = x_n - \frac{f(x_n) + f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}{f'(x_n)}. \tag{3. 29}$$

Again, we recall the following derivative free method presented by the Amat and Basquier in [1]:

$$x_{n+1} = x_n - A_n^{-1} f(x_n), \tag{3. 30}$$

$$\text{where} \qquad A_n = [y_n, x_n; f] = \frac{f(x_n) - f(y_n)}{x_n - y_n},$$

$$y_n = x_n + \delta_n(x_{n-1} - x_n), \qquad \delta_n \le |O(e_n)^{\frac{3}{2}}|.$$

This is the second order method. They obtained this method by modifying classical secant method (1. 3). We shall prove that if we use the iterates alternatively from the method

(1. 12) and this method, the resulting method will be sixth order convergence for suitable value of δ_n. The proposed method is given below:

$$x_{n+1} = \overline{x_n} - A_n^{-1} f(\overline{x_n}), \tag{3. 31}$$

where $$A_n = [y_n, \overline{x_n}; f] = \frac{f(\overline{x_n}) - f(y_n)}{\overline{x_n} - y_n},$$

$$y_n = \overline{x_n} + \delta_n (x_n - \overline{x_n}), \qquad \delta_n \leq |O(e_n)^{\frac{3}{2}}|$$

and $$\overline{x_n} = x_n - \frac{f(x_n) + f\left(x_n - \dfrac{f(x_n)}{f'(x_n)}\right)}{f'(x_n)}.$$

Let us prove the following convergence result on above mentioned method (3. 28)-(3. 29).

Theorem 3.1. *Let α be a simple zero of a function f which has enough number of continuous derivatives in a neighborhood of α. If x_0 is sufficiently close to α, then the proposed method (3. 28)-(3. 29) has convergence order at least 4.*

Proof. Assume e_n and $\overline{e_n}$ denote the errors in x_n and $\overline{x_n}$, respectively, that is, $x_n = \alpha + e_n$ and $\overline{x_n} = \alpha + \overline{e_n}$. Denote $c_j = \dfrac{f^j(\alpha)}{j! f'(\alpha)}$. If we give a little attention on the proof of Theorem 2.1, it is clear that the error equation of (3. 29) is given by

$$\begin{aligned} \overline{e_n} &= 2c_2^2 e_n^3 + O(e_n^4) \\ &= A e_n^3 + O(e_n^4), \qquad \text{where } A = 2c_2^2. \end{aligned} \tag{3. 32}$$

Here,

$$\begin{aligned} \overline{x_n} - x_n &= (\alpha + \overline{e_n}) - (\alpha + e_n) \\ &= \overline{e_n} - e_n \\ &= A e_n^3 - e_n + O(e_n^4) \end{aligned}$$

Using Taylor's expansion, we get

$$\begin{aligned} f(x_n) &= f(\alpha + e_n) \\ &= f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)] \end{aligned}$$

and using (3. 32), we obtain

$$\begin{aligned} f(\overline{x_n}) &= f(\alpha + \overline{e_n}) \\ &= f'(\alpha)[A e_n^3 + O(e_n^6)]. \end{aligned}$$

Thus, we get

$$\begin{aligned} f(\overline{x_n}) - f(x_n) &= f'(\alpha)[A e_n^3 + O(e_n^6)] - f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)] \\ &= -f'(\alpha) e_n [1 + c_2 e_n + (c_3 - A) e_n^2 + O(e_n^3)] \end{aligned}$$

and

$$\begin{aligned}\frac{(\overline{x_n}-x_n)f(\overline{x_n})}{f(\overline{x_n})-f(x_n)} &= \frac{[Ae_n^3-e_n+O(e_n^4)]f'(\alpha)[Ae_n^3+O(e_n^6)]}{-f'(\alpha)e_n[1+c_2e_n+(c_3-A)e_n^2+O(e_n^3)]}\\ &= [Ae_n^3+O(e_n^5)][1+c_2e_n+(c_3-A)e_n^2+O(e_n^3)]^{-1}\\ &= [Ae_n^3-Ac_2e_n^4+O(e_n^5)].\end{aligned}$$

Thus, the error equation in (3. 28) is given by

$$\begin{aligned}e_{n+1} &= \overline{e_n}-Ae_n^3+Ac_2e_n^4+O(e_n^5)]\\ &= Ae_n^3+O(e_n^4)-Ae_n^3+Ac_2e_n^4+O(e_n^5)\\ &= \lambda e_n^4+O(e_n^5),\end{aligned}$$

where λ is some constant. Thus, convergence order of the method (3. 28)-(3. 29) is at least 4 and the theorem is proved. □

Let us prove the following convergence result of method (3. 31):

Theorem 3.2. *Let α be a simple zero of a function f which has enough number of continuous derivatives in a neighborhood of α. If x_0 is sufficiently close to α, then the method (3. 31) has convergence order at least 5.5 and it becomes 6 for the suitable choice of δ_n.*

Proof. From Theorem 3.1, the error $\overline{e_n}$ in $\overline{x_n}$ is given by

$$\overline{e_n} = 2c_2^2e_n^3+O(e_n^4).$$

Since

$$y_n = \overline{x_n}+\delta_n(x_n-\overline{x_n}) = \overline{x_n}+a_n, \qquad a_n = \delta_n(x_n-\overline{x_n}),$$

we have by Taylor expansion

$$f(y_n) = f(\overline{x_n}+a_n) = f(\overline{x_n})+a_nf'(\overline{x_n})+\frac{a_n^2}{2}f''(\overline{x_n})+\cdots .$$

After some calculation, we get

$$A_n = f'(\overline{x_n})+O(a_n),$$

and the method (3. 31) can be expressed as

$$x_{n+1} = \overline{x_n}-\frac{f(\overline{x_n})}{f'(\overline{x_n})}+O(a_n).$$

Thus, we have

$$x_{n+1} = \overline{x_n}-\frac{f(\overline{x_n})}{f'(\overline{x_n})}+O(\delta_n|x_n-\overline{x_n}|), \tag{3. 33}$$

From (3. 33), we have

$$
\begin{aligned}
e_{n+1} &= \overline{e_n} - \frac{f(\alpha+\overline{e_n})}{f'(\alpha+\overline{e_n})} + O(\delta_n|\alpha+e_n-\alpha-\overline{e_n}|) \\
&= \overline{e_n} - \left(\overline{e_n} + c_2\overline{e_n}^2 + c_3\overline{e_n}^3 + O(\overline{e_n}^4)\right)\left(1+2c_2\overline{e_n}+3c_3\overline{e_n}^2+O(\overline{e_n}^3)\right)^{-1} + O(\delta_n|e_n-\overline{e_n}|) \\
&= \overline{e_n} - (\overline{e_n} + c_2\overline{e_n}^2 - 2c_2\overline{e_n}^2 + O(\overline{e_n}^3) + O(\delta_n|e_n-\overline{e_n}|) \\
&= c_2\overline{e_n}^2 + O(\delta_n|e_n-\overline{e_n}|) + \cdots \\
&= c_2(2c_2^2e_n^3)^2 + O(\delta_n|e_n-2c_2^2e_n^3|) + \cdots \\
&= 4c_2^5e_n^6 + O(\delta_n|e_n-2c_2^2e_n^3|) + \cdots . \qquad (3.\ 34)
\end{aligned}
$$

Since $\delta_n \leq |O(\overline{e_n})^{\frac{3}{2}}|$, and so if we assume that $\delta_n \leq |O(\overline{e_n})^2|$, that is, $\delta_n \leq |O(e_n)^6|$, then method (3. 31)is at least sixth order convergence. But, if we take $\delta_n = |O(\overline{e_n})^{\frac{3}{2}}|$, the convergence order becomes 5.5. □

Remark 3.3. If the solution of nonlinear equation is unknown, scheme which we use to find δ_n is the same as Amat and Basquier proposed in [1] . This scheme is

$$
\begin{aligned}
\delta_0 &= O(10^{-k}) \leq O(e_0)^{\frac{3}{2}} \\
\delta_n &= O(\delta_0^{2^n}),
\end{aligned}
$$

where k is an integer such that

$$O(10^{-k}) \leq |f(\alpha) - f(x_0)| = |f(x_0)| \leq O(|\alpha - x_0|).$$

4. Numerical Results

Here, three numerical examples are presented to show the efficiency of methods obtained in previous sections. We compare these methods with existing Potra and Pták (PP) method (1. 12), Weerakoon and Fernando (WF) method [19], McDougall and Wotherspoon(MW) method (1. 8)-(1. 11) and Newton's method. To perform the numerical calculation, we use Matlab Software and stopping criteria $|x_{n+1} - x_n| < (10)^{-12}$ or $|f(x_{n+1}| < (10)^{-14}$.

Example 4.1. We apply methods (2. 14)-(2. 17) and (3. 28)-(3. 29) on the nonlinear equation

$$3x + \sin x - e^x = 0. \qquad (4.\ 35)$$

To determine appropriate initial approximation of root, let us draw the graph.

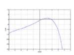

FIGURE 1. Graph of $f(x) = 3x + \sin x - e^x$.

Figure 1 shows that the equation (4.1) has a simple roots in $(1, 2)$ and another simple root in $(0, 1)$. Taking initial guess $x_0 = 3$, Table 1 displays the iterations of existing Potra and Pták method (1. 12), Weerakoon and Fernando method [19], McDougall and Wotherspoon method (1. 8)-(1. 11), and our methods (2. 14)-(2. 15) and (3. 28)-(3. 29).

Example 4.2. Again, we apply methods (2. 14)-(2. 17) and (3. 28)-(3. 29) on equation

$$f(x) = x^3 + 2x^2 - 3x - 1 = 0. \quad (4.\ 36)$$

From the Intermediate Value Theorem (also known as Bolzano's theorem) [2], one of the root of this equation lies in $(0, 2)$ since $f(0)f(2) < 0$. Taking initial guess $x_0 = 2$, Table 2 displays the iterations of some existing methods and our methods.

Example 4.3. Finally, we apply methods (3. 28)-(3. 29) and (3. 31) on the nonlinear equation

$$(x - 2)^{23} - 1 = 0 \quad (4.\ 37)$$

By inspection of above equation, it is clear that $x = 3$ is the root of this equation. Taking initi guess $x_0 = 4$. Table 3 displays the iterations of Potra and Pták method (1. 12), and our methods (3. 28)-(3. 29) and (3. 31). Also Table 3 demonstrates that the convergence rate can be improved by choosing the suitable value of δ_n.

TABLE 1. Comparison of distinct methods.

Method	n	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
MW method	1	2.394517490417379	0.605482509582621	3.099858826809431
	2	1.993751342267486	0.400766148149893	0.449894328099718
	3	1.894136065604574	0.099615276662912	0.016215768321299
	4	1.890031757384865	0.004104308219709	0.000007977482077
	5	1.890029729252003	0.000002028132863	0.000000000000068
	6	1.890029729251985	0.000000000000017	0.000000000000000
PP method	1	2.223022716747521	0.776977283252479	1.771401528396967
	2	1.918429998489068	0.304592718258453	0.114786469963655
	3	1.890068371118427	0.028361627370641	0.000151999733143
	4	1.890029729252092	0.000038641866335	0.000000000000419
	5	1.890029729251985	0.000000000000107	0.000000000000000
WF method	1	2.182401798156115	0.817598201843885	1.501646202703620
	2	1.905217119872699	0.277184678283415	0.060614766751669
	3	1.890033324087735	0.015183795784964	0.000014139991158
	4	1.890029729251985	0.000003594835750	0.000000000000000
Present method (2. 14) -(2. 17)	1	2.223022716747521	0.776977283252479	1.771401528396967
	2	1.903254578217390	0.319768138530131	0.052683015289491
	3	1.890030006989384	0.013224571228006	0.000001092453759
	4	1.890029729251985	0.000000277737399	0.000000000000000
Present method (3. 28) -(3. 29)	1	2.072980588934883	0.927019411065118	0.853003949652775
	2	1.891066539031407	0.181914049903475	0.004082261437855
	3	1.890029729254035	0.001036809777372	0.000000000008063
	4	1.890029729251985	0.000000000002050	0.000000000000000

TABLE 2. Comparison of distinct methods.

Method	n	x_n	$\|x_n - x_{n-1}\|$	$\|f(x_n)\|$
MW method	1	1.470588235294118	0.529411764705882	2.093832688784858
	2	1.226889909604565	0.243698325689552	0.176634866528456
	3	1.198883162389074	0.028006747215492	0.001171937455065
	4	1.198691244588557	0.000191917800517	0.000000006548349
	5	1.198691243515997	0.000000001072560	0.000000000000000
PP method	1	1.347421606542067	0.652578393457933	1.035129693410847
	2	1.202167869692565	0.145253736849502	0.021293690797169
	3	1.198691313272115	0.003476556420450	0.000000425885336
	4	1.198691243515997	0.000000069756118	0.000000000000001
WF method	1	1.317412413069151	0.682587586930849	0.805382355879769
	2	1.199882716602041	0.117529696467110	0.007282302633090
	3	1.198691245071338	0.001191471530703	0.000000009495896
	4	1.198691243515997	0.000000001555341	0.000000000000000
Present method (2. 14) -(2. 17)	1	1.347421606542067	0.652578393457933	1.035129693410847
	2	1.199673166705735	0.147748439836332	0.006000378396200
	3	1.198691243536719	0.000981923169016	0.000000000126513
	4	1.198691243515997	0.000000000020722	0.000000000000000
Present method (3. 28) -(3. 29)	1	1.262611277391092	0.737388722608908	0.413379428130185
	2	1.198711091370871	0.063900186020221	0.000121180247031
	3	1.198691243515997	0.000019847854874	0.000000000000000

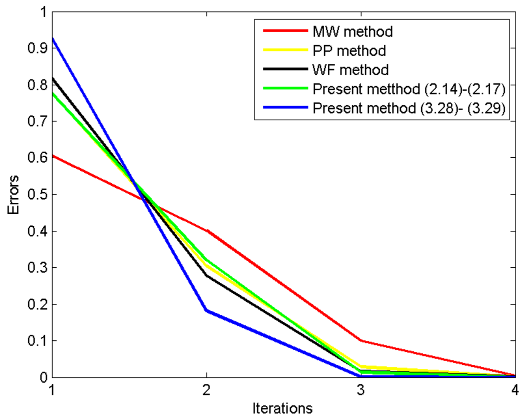

FIGURE 2. Graphs of the errors of different methods up to four iterations using Table 1.

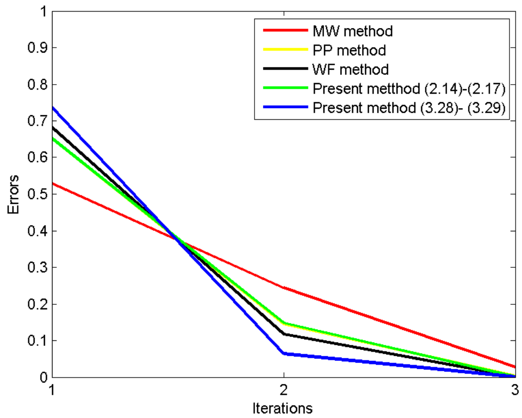

FIGURE 3. Graphs of the errors of different methods up to three iterations using Table 2.

5. CONCLUSION

From the observation of above comparison tables and graphs of different numerical methods, we conclude that our newly introduced methods modified from Potra and Pták method using different techniques can easily compete with McDougall and Wotherspoon method, Potra and Pták method (1. 12), and Weerakoon and Fernando method [19]. In each iteration, we need to calculate one more function in methods (2. 14)-(2. 17) and (3. 28)-(3. 29), and two more functions in method (3. 31) than Potra and Pták method.

TABLE 3. Comparison of distinct methods.

n	PP method	Present method (3. 28)-(3. 29)	Present method (3. 31) $\delta_n = (3 - x_n)^{3/2}$	Present method (3. 31) $\delta_n = (3 - x_n)^2$
1	3.881762281757005	3.843143104308642	3.814658265096324	3.729238036112728
2	3.770514691012868	3.698588407251381	3.645089902139495	3.526770236835179
3	3.665844113356311	3.565371837885375	3.484739002121267	3.361258490695884
4	3.567362235341004	3.442608817238754	3.338026931042994	3.220194565600152
5	3.474705145072179	3.329507542251746	3.205117622709691	3.099435117528323
6	3.387535965688113	3.225479138393579	3.088517260188958	3.014759658594771
7	3.305558823234900	3.130819006660631	3.010352168613209	3.000003318880189
8	3.228575744377471	3.050435204423297	3.000000483378891	
9	3.156705026488715	3.004892673718495	3.000000000000000	
10	3.091180490016622	3.000001327332550		
11	3.036856948205285	3.000000000000000		
12	3.005553045154960			
13	3.000036207524484			
14	3.000000000011477			
15	3.000000000000000			

Nevertheless, this cost is nominal in the comparison of order of convergence of the methods introduced in this paper.

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