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A New k-Fractional Integral Operators and their Applications

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Abstract.: In this paper, we present a new k-fractional integral oper-ators involving parameters γ, λ analogous to the Riemann-Liouville k-fractional integral. This new fractional integral operators dependent on an exponential function of arbitrary exponent in the kernel of the integral. We prove, certain basic properties such as semi group property, commu-tative law and boundedness for new fractional integral operators. Also, we discuss Chebyshev type inequalities and some k-fractional integral in-equalities corresponding to the new k-fractional integral operators.

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1. INTRODUCTION

For the past four decades or so, the subject of fractional calculus (in other words, calculus of integrals and derivatives of any arbitrary real or complex order) has obtained remarkable popularity and importance, owing mainly to its revealed applications in a notable number of diverse and widespread fields of science and engineering. Accordingly, a strikingly large number of papers and monographs involving the fractional calculus have been provided (see, e.g., [6], [3], [8], [12] and [13]). The fractional concept of derivatives mostly defined by means of fractional integral. But the idea of integration and derivative of non-integer arrangement is motivated, in 1965 asked by L'Hospital inside his letter toward Leibniz through the question "What is the derivative of $\frac{d^n f}{dx^n}$ for order $n = \frac{1}{2}$". Also some inequlaities like Chebyshev type inequalities, midpoint type and Hermite-Hadamard inequalities has been examined for generalized functions of fractional integral operators

(see[15]and[1]. However, the k-Riemann-Liouville fractional integral involving (x-t) kernel has been investigated by many authors to solve fractional and differential inequalities (see[7],[5],[11], [14] and [10]).
Riemann-Liouville [9] considering continuous function z about order $\gamma > 0$ over the interval $[a, u]$ defined as,

$$I_{a+}^{\gamma}\left(z(u)\right) = \frac{1}{\Gamma\left(\gamma\right)} \int_{a}^{u} \left(u - v\right)^{\gamma - 1} z(v)dv,\ u > a,$$

.
prompted by the Cauchy integral formula

$$\int_{a}^{u} \int_{a}^{u} \int_{a}^{u} ... \int_{a}^{u} z(v)dv_1 dv_2 dv_3 ... dv_n = \frac{1}{\Gamma\left(n\right)} \int_{a}^{u} \left(u - v\right)^{n-1} z(v)dv,\ u > a.$$

Later, Hadamard [4] defined fractional integral about order $\gamma > 0$ over the interval $[a, u]$ for a continuous function z and given as,

$$I_{a+}^{\gamma} z(u) = \frac{1}{\Gamma\left(\gamma\right)} \int_{a}^{u} \frac{1}{v} \left(log \frac{u}{v}\right)^{\gamma - 1} z(v)dv.$$

This is established by the generalization of the integral.

$$\int_{a}^{u} \frac{dv_1}{v_1} \int_{a}^{v_1} \frac{dv_2}{v_2} ... \int_{a}^{v_{n-1}} \frac{f\left(v_n\right)}{v_n} dv_n = \frac{1}{\Gamma\left(n\right)} \int_{a}^{u} \frac{1}{v} \left(log \frac{u}{v}\right)^{n-1} z(v)dv.$$

In the recent year, in [2] Diaz and Pariguan have introduced an extended form of the classical Gamma and Beta functions defined as,

$$B_k(u, v) = \frac{1}{k} \int_{0}^{1} t^{\frac{u}{k} - 1} (1 - t)^{\frac{v}{k} - 1} dt \tag{1. 1}$$

and

$$\Gamma_k(x) = \int_{0}^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt \tag{1. 2}$$

respectively.
Where, Γ_k is the Euler Gamma k-functions. For $k = 1$, it becomes classical Euler Gamma function and Beta function.

$$\Gamma_k(u) = \lim_{m \to \infty} \frac{m! k^m (mk)^{\frac{u}{k} - 1}}{(u)_{m,k}}. \tag{1. 3}$$

Where, $(u)_{k,m} = \prod_{i=0}^{m-1}(u+ik)$ is called to be Pochhammer k-symbol for the factorial functions. Some basic properties of Gamma and Beta k-extended form are defined as

$$B_k(u,v) = \frac{1}{k}B(\frac{u}{k},\frac{v}{k}) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}, \tag{1. 4}$$

also

$$\Gamma_k(r+k) = r\Gamma_k(u), \tag{1. 5}$$

$$\Gamma(r) = \lim_{k\to 1}\Gamma_k(r) \tag{1. 6}$$

$$\Gamma_k(r) = k^{\frac{r}{k}-1}\Gamma_k\left(\frac{r}{k}\right). \tag{1. 7}$$

Later, by using above definitions, [7] have introduced the k-Riemann Liouville fractional Integral, which is defined as

$$_kI^{\gamma}(z(u)) = \frac{1}{k\Gamma_k(\gamma)}\int_a^u (u-v)^{\frac{\gamma}{k}-1}z(v)dv,\ u>a.$$

For $k=1$, it becomes classical definition of the Riemann-Liouville fractional integral.

2. A New k-Fractional Integral Operators

In this paper, we want to present a new k-fractional integral operators which modify all of the defined k-fractional integrals and k-Riemann Liouville fractional integral in the sense of kernel. Also, the paper focuses to explore the behaviour of new fractional integral operators through different inequalities and some basic properties, when use the sum of two parameters as exponent which is also become a single parameter as exponent for $\lambda = 0$. The new fractional integral operators illustrated by some theorems and examples, is defined as:

2.1. Definition. Let z be a continuous function on the interval (a,u) and $\lambda \geq 0$. Then a new k-fractional integral operators of order $\gamma > 0$, of z is defined as

$$_kI_{a^+}^{\gamma,\lambda}z(u) = \frac{1}{k\Gamma_k(\gamma)}\int_a^u \left(e^{u+\lambda}-e^{v+\lambda}\right)^{\frac{\gamma}{k}-1}e^{v+\lambda}z(v)dv \tag{2. 8}$$

which, is based on the relation

$$\int_a^u e_1^{v+\lambda}dv_1\int_a^{v_1}e_2^{v+\lambda}dv_2 \ldots \int_a^{v_{n-1}}e_n^{v+\lambda}z(v_n)dv_n = \frac{1}{\Gamma(n)}\int_a^u \left(e^{u+\lambda}-e^{v+\lambda}\right)^{n-1}e^{v+\lambda}z(v)dv.$$

For $k=1$, (2. 8) becomes in standard form as

$$I_{a^+}^{\gamma,\lambda}z(u) = \frac{1}{\Gamma(\gamma)}\int_a^u \left(e^{u+\lambda}-e^{v+\lambda}\right)^{\gamma-1}e^{v+\lambda}z(v)dv.$$

Now, we prove the new fractional integral operators is well defined:

Theorem 2.1. *Let $g \in L_1[c,d], c > 0$. Then for $\gamma > 0 and\ \lambda, k > 0$ ${}_kI_a^{\gamma} g(u)$ almost exist everywhere on the interval $[c,d]$ and ${}_kI_a^{\gamma,\lambda} g(u) \in L_1[c,d]$.*

Proof: Let $Q : \Delta = [c,d] \times [c,d] \to R$ by defining $Q(u,v) = \left[\left(e^{u+\lambda} - e^{v+\lambda}\right)^{\frac{\gamma}{k}-1} e^{v+\lambda}\right]_+$ or,

$$Q(u,v) = \begin{cases} \left(e^{u+\lambda} - e^{v+\lambda}\right)^{\frac{\gamma}{k}-1} e^{v+\lambda} & c \le v \le u \le d \\ 0 & c \le u \le v \le d. \end{cases}$$

Thus, as Q is measurable on Δ, we get

$$\begin{aligned} \int_c^d Q(u,v)dv = \int_c^u Q(u,v)dv + \int_u^d Q(u,v)dv &= \int_c^u \left(e^{u+\lambda} - e^{v+\lambda}\right)^{\frac{\gamma}{k}-1} e^{v+\lambda} dv, \\ &= \frac{k}{\gamma} \left(e^{u+\lambda} - e^{c+\lambda}\right)^{\frac{\gamma}{k}}. \end{aligned}$$

By using the repetition of the integral, we obtain

$$\begin{aligned} \int_c^d \left[Q(u,v)|g(u)|dv\right] du &= \int_c^d |g(u)| \left[\int_c^d Q(u,v)dv\right] du \\ &= \frac{k}{\gamma} \int_c^d \left(e^{u+\lambda} - e^{c+\lambda}\right)^{\frac{\gamma}{k}} |g(u)| du, \\ &\le \frac{k}{\gamma} \left(e^{d+\lambda} - e^{c+\lambda}\right)^{\frac{\gamma}{k}} \int_c^d |g(u)| du. \end{aligned}$$

That is

$$\begin{aligned} \int_c^d \left[Q(u,v)|g(u)|dv\right] du &= \int_c^d |g(u)| \left[\int_c^d Q(u,v)dv\right] du \\ &\le \frac{k}{\gamma} \left(e^{d+\lambda} - e^{c+\lambda}\right)^{\frac{\gamma}{k}} ||g(u)||_{L_1[c,d]} < \infty. \end{aligned}$$

Therefor, by Tonelli's theorem the function $P : \Delta \to R$ is integrable up to Δ such that $P(u,v) = Q(u,v)g(u)$. Hence by using Fubini's theorem $\int_c^d Q(u,v)g(u)du$ is an integrable function on $[c,d]$. It means that ${}_kI^{\gamma} g(u)$ is integrable on $[c,d]$.

Theorem 2.2. *Let z be continuous on (a,u) and also, let $\gamma, \beta > 0$ $\lambda \ge 0$. Then for all u*

$${}_kI^{\gamma,\lambda}\left({}_kI^{\beta,\lambda} z(u)\right) = {}_kI^{(\gamma+\beta),\lambda} z(u) = {}_kI^{\beta,\lambda}\left({}_kI^{\gamma,\lambda} z(u)\right).$$

Proof: According to the definition of new k-fractional integral and using Dirichlet's formula, we have

$${}_kI^{\gamma,\lambda}({}_kI^{\beta,\lambda} z(u)) = \frac{1}{k\Gamma_k(\gamma)} \int_a^u \left(e^{u+\lambda} - e^{v+\lambda}\right)^{\frac{\gamma}{k}-1} e^{v+\lambda} \left({}_kI^{\beta,\lambda} z(v)\right) dv$$

and

$$= \frac{1}{k\Gamma_k(\gamma)} \int\limits_a^u \left(e^{u+\lambda} - e^{v+\lambda}\right)^{\frac{\gamma}{k}-1} e^{v+\lambda} \left[\frac{1}{k\Gamma_k(\beta)} \int\limits_a^v \left(e^{v+\lambda} - e^{t+\lambda}\right)^{\frac{\beta}{k}-1} e^{t+\lambda} z(t) dt \right] dv.$$

$$= \frac{1}{k^2\Gamma_k(\gamma)\Gamma_k(\beta)} \int\limits_a^u e^{t+\lambda} z(t) \left[\int\limits_t^u \left(e^{u+\lambda} - e^{v+\lambda}\right)^{\frac{\gamma}{k}-1} \left(e^{v+\lambda} - e^{t+\lambda}\right)^{\frac{\beta}{k}-1} e^{v+\lambda} dv \right] dt.$$

By changing the variable of the inner integral $y = \frac{e^{v+\lambda} - e^{t+\lambda}}{e^{u+\lambda} - e^{t+\lambda}}$

$$= \frac{1}{k^2\Gamma_k(\gamma)\Gamma_k(\beta)} \int\limits_a^u e^{t+\lambda} z(t) \left(e^{u+\lambda} - e^{t+\lambda}\right)^{\frac{\gamma+\beta}{k}-1} \left[\int\limits_0^1 (1-y)^{\frac{\gamma}{k}-1} y^{\frac{\beta}{k}-1} dy \right] dt.$$

According to the k-Beta function, we get

$${}_kI^{\gamma,\lambda}\left({}_kI^{\beta,\lambda} z(u)\right) = \frac{1}{k\Gamma_k(\gamma+\beta)} \int\limits_a^u e^{t+\lambda} \left(e^{u+\lambda} - e^{t+\lambda}\right)^{\frac{\gamma+\beta}{k}-1} z(t) dt$$

$${}_kI^{\gamma,\lambda}\left({}_kI^{\beta,\lambda} z(u)\right) = {}_kI^{(\gamma+\beta),\lambda} z(u) = {}_kI^{\beta,\lambda}\left({}_kI^{\gamma,\lambda} z(u)\right).$$

Theorem 2.3. *Let $\gamma, \beta > 0, a > 0$. Then*

$${}_kI_a^{\gamma,\lambda}\left(\left(e^{u+\lambda} - e^{a+\lambda}\right)^{\frac{\beta}{k}-1}\right) = \frac{\Gamma_k(\beta)}{\Gamma_k(\gamma+\beta)} \left(e^{u+\lambda} - e^{a+\lambda}\right)^{\frac{\gamma+\beta}{k}-1}, k > 0 \tag{2. 9}$$

where Γ_k represent the k-Gamma function.

Proof: According to the definition of new k-fractional integral and changing the variable $y = \frac{e^{v+\lambda} - e^{a+\lambda}}{e^{u+\lambda} - e^{a+\lambda}}$, that turns into

$${}_kI_a^{\gamma,\lambda}\left(\left(e^{u+\lambda} - e^{a+\lambda}\right)^{\frac{\beta}{k}-1}\right) = \frac{1}{k\Gamma_k(\gamma)} \int\limits_a^u \left(e^{u+\lambda} - e^{v+\lambda}\right)^{\frac{\gamma}{k}-1} \left(e^{v+\lambda} - e^{a+\lambda}\right)^{\frac{\beta}{k}-1} e^{v+\lambda} dv$$

$${}_kI_a^{\gamma,\lambda}\left(\left(e^{u+\lambda} - e^{a+\lambda}\right)^{\frac{\beta}{k}-1}\right) = \frac{\left(e^{u+\lambda} - e^{a+\lambda}\right)^{\frac{\gamma+\beta}{k}-1}}{k\Gamma_k(\gamma)} \int_0^1 (1-y)^{\frac{\gamma}{k}-1} y^{\frac{\beta}{k}-1} dy.$$

$${}_kI_a^{\gamma,\lambda}\left(\left(e^{u+\lambda} - e^{a+\lambda}\right)^{\frac{\beta}{k}-1}\right) = \frac{\Gamma_k(\beta)}{\Gamma_k(\gamma+\beta)} \left(e^{u+\lambda} - e^{a+\lambda}\right)^{\frac{\gamma+\beta}{k}-1},$$

which completes the proof of this theorem.

Remark 2.4. *For $k = 1$ in (2. 9), we gain the formula*

$$I_a^{\gamma,\lambda}\left(\left(e^{u+\lambda} - e^{a+\lambda}\right)^{\frac{\beta}{k}-1}\right) = \frac{\Gamma(\beta)}{\Gamma(\gamma+\beta)} \left(e^{u+\lambda} - e^{a+\lambda}\right)^{\gamma+\beta-1}.$$

Corollary 2.5. *Let $\gamma, \beta > 0, a > 0$. Then*

$${}_kI_a^{\gamma,\lambda}(1) = \frac{\left(e^{u+\lambda} - e^{a+\lambda}\right)^{\frac{\gamma}{k}}}{\Gamma_k(\gamma+k)}. \tag{2. 10}$$

Remark 2.6. *For $k=1$, the above relation (2. 10) becomes*

$$I_a^{\gamma,\lambda}(1)=\frac{\left(e^{u+\lambda}-e^{a+\lambda}\right)^{\gamma}}{\Gamma\left(\gamma+1\right)}.$$

3. Some Fractional Integral Inequalities Analogous New k-Fractional Integral operators

Chebyshev Type Inequalities corresponding to the new fractional integral operators can be represented as fellows:

Theorem 3.1. *Let two synchronous functions g,h on $[0,1)$, then for all $u>0,\gamma,\beta>0$, then for new k-fractional integral the following inequalities hold:*

$${}_kI_a^{\gamma,\lambda}gh(u)\geq\frac{1}{{}_kI_a^{\gamma,\lambda}(1)}{}_kI_a^{\gamma,\lambda}g(u)I_a^{\gamma,\lambda}h(u),$$

and

$${}_kI_a^{\gamma,\lambda}gh(u){}_kI_a^{\beta,\lambda}(1)+{}_kI_a^{\beta,\lambda}gh(u){}_kI_a^{\gamma,\lambda}(1)\geq{}_kI_a^{\gamma,\lambda}g(u){}_kI_a^{\beta,\lambda}h(u)+{}_kI_a^{\gamma,\lambda}h(u){}_kI_a^{\beta,\lambda}g(u).$$

Proof: Since g and h both are synchronous functions on the interval $[0,1)$, then for all $\tau,t\geq0$, we have

$$\left(g(\tau)-g(t)\right)\left(h(\tau)-h(t)\right)\geq0.$$

Therefor

$$g(\tau)h(\tau)+g(t)h(t)\geq g(\tau)h(t)+g(t)h(\tau). \tag{3. 11}$$

Multiplying both sides of (3. 11) by $\frac{1}{k\Gamma_k(\gamma)}\left(e^{u+\lambda}-e^{\tau+\lambda}\right)^{\frac{\gamma}{k}-1}e^{\tau+\lambda}$ and then, integrating the resulting inequality w.r.t. τ up to (a,u), we obtain

$$\frac{1}{k\Gamma_k\left(\gamma\right)}\int\limits_a^u\left(e^{u+\lambda}-e^{\tau+\lambda}\right)^{\frac{\gamma}{k}-1}e^{\tau+\lambda}g(\tau)h(\tau)d\tau$$

$$+\,g(t)h(t)\frac{1}{k\Gamma_k\left(\gamma\right)}\int\limits_a^u\left(e^{u+\lambda}-e^{\tau+\lambda}\right)^{\frac{\gamma}{k}-1}e^{\tau+\lambda}d\tau$$

i.e.

$${}_kI_a^{\gamma,\lambda}gh(u)+g(t)h(t){}_kI_a^{\gamma,\lambda}(1)\geq h(t){}_kI_a^{\gamma,\lambda}g(u)+g(t){}_kI_a^{\gamma,\lambda}h(u). \tag{3. 12}$$

Now, multiplying both sides of (3. 12) by $\frac{1}{k\Gamma_k(\beta)}\left(e^{u+\lambda}-e^{t+\lambda}\right)^{\frac{\beta}{k}-1}e^{t+\lambda}$ and then integrating the resulting inequality w.r.t. t over (a,u), we obtain

$${}_kI_a^{\gamma,\lambda}\quad gh(u)\frac{1}{k\Gamma_k\left(\beta\right)}\int\limits_a^u\left(e^{u+\lambda}-e^{t+\lambda}\right)^{\frac{\beta}{k}-1}e^{t+\lambda}dt$$

$$+\quad{}_kI_a^{\gamma,\lambda}(1)\frac{1}{k\Gamma_k\left(\beta\right)}\int\limits_a^u\left(e^{u+\lambda}-e^{t+\lambda}\right)^{\frac{\beta}{k}-1}e^{t+\lambda}g(t)h(t)dt$$

that is

$$ {}_kI_a^{\gamma,\lambda}gh(u){}_kI_a^{\beta,\lambda}(1)+{}_kI_a^{\gamma,\lambda}(1){}_kI_a^{\beta,\lambda}gh(u)\geq{}_kI_a^{\gamma,\lambda}g(u){}_kI_a^{\beta,\lambda}h(u)+{}_kI_a^{\gamma,\lambda}h(u){}_kI_a^{\beta,\lambda}g(u). $$

Now the second inequality is proved.

Theorem 3.2. *Let g,h both are synchronous on the interval $[0,\infty)$, and $J\geq 0$, then $\forall u>a\geq 0,\gamma,\beta>0$, the following k-fractional integral inequality hold:*

$$ \begin{aligned} &\frac{1}{\Gamma_k(\beta+k)}\left(e^{u+\lambda}-e^{a+\lambda}\right)^{\frac{\beta}{k}}{}_kI_a^{\gamma,\lambda}ghJ(u)+\frac{1}{\Gamma_k(\gamma+k)}\left(e^{u+\lambda}-e^{a+\lambda}\right)^{\frac{\gamma}{k}}{}_kI_a^{\beta,\lambda}ghJ(u)\\ &\geq{}_kI_a^{\gamma,\lambda}gJ(u){}_kI_a^{\beta,\lambda}h(u)+{}_kI_a^{\gamma,\lambda}hJ(u){}_kI_a^{\beta,\lambda}g(u)-{}_kI_a^{\gamma,\lambda}J(u){}_kI_a^{\beta,\lambda}gh(u)\\ &-{}_kI_a^{\gamma,\lambda}gh(u){}_kI_a^{\beta,\lambda}J(u)+{}_kI_a^{\gamma,\lambda}g(u){}_kI_a^{\beta,\lambda}hJ(u)+{}_kI_a^{\gamma,\lambda}h(u){}_kI_a^{\beta,\lambda}gJ(u). \end{aligned} $$

Proof: Since the functions g and h are synchronous on the interval $[0,1)$, then $\forall r,s\geq 0 and J\geq 0$, we get

$$ (g(r)-g(s))\,(h(r)-h(s))\,(J(r)+J(s))\geq 0. $$

Therefor

$$ \begin{aligned} g(r)h(r)J(r)+g(s)h(s)J(s)\geq& g(r)h(s)J((s)+g(s)h(r)J(r)+g(r)h(s)J(r)\\ &+g(s)h(r)J(s)-g(r)h(r)J(s)-g(s)h(s)J(r). \end{aligned} \tag{3. 13} $$

Both sides of (3. 13) multiplying by $\frac{1}{k\Gamma_k(\gamma)}\left(e^{u+\lambda}-e^{r+\lambda}\right)^{\frac{\gamma}{k}-1}e^{r+\lambda}$ and then integrating the resulting inequality w.r.t. r up to (a,u), we obtain

$$ \begin{aligned} &\frac{1}{k\Gamma_k(\gamma)}\int_a^u\left(e^{u+\lambda}-e^{r+\lambda}\right)^{\frac{\gamma}{k}-1}e^{r+\lambda}g(r)h(r)J(r)dr\\ &\qquad+g(s)h(s)J(s)\frac{1}{k\Gamma_k(\gamma)}\int_a^u\left(e^{u+\lambda}-e^{r+\lambda}\right)^{\frac{\gamma}{k}-1}e^{r+\lambda}dr \end{aligned} $$

$$\geq \quad h(s)J(s)\frac{1}{k\Gamma_k(\gamma)}\int_a^u \left(e^{u+\lambda}-e^{r+\lambda}\right)^{\frac{\gamma}{k}-1}e^{r+\lambda}g(r)dr$$

$$+ \quad g(s)\frac{1}{k\Gamma_k(\gamma)}\int_a^u \left(e^{u+\lambda}-e^{r+\lambda}\right)^{\frac{\gamma}{k}-1}e^{r+\lambda}h(r)J(r)dr$$

$$+ \quad g(s)J(s)\frac{1}{k\Gamma_k(\gamma)}\int_a^u \left(e^{u+\lambda}-e^{r+\lambda}\right)^{\frac{\gamma}{k}-1}e^{r+\lambda}h(r)dr$$

$$+ \quad h(s)\frac{1}{k\Gamma_k(\gamma)}\int_a^u \left(e^{u+\lambda}-e^{r+\lambda}\right)^{\frac{\gamma}{k}-1}e^{r+\lambda}g(r)J(r)dr$$

$$- \quad J(s)\frac{1}{k\Gamma_k(\gamma)}\int_a^u \left(e^{u+\lambda}-e^{r+\lambda}\right)^{\frac{\gamma}{k}-1}e^{r+\lambda}g(r)h(r)dr$$

$$- \quad g(s)h(s)\frac{1}{k\Gamma_k(\gamma)}\int_a^u \left(e^{u+\lambda}-e^{r+\lambda}\right)^{\frac{\gamma}{k}-1}e^{r+\lambda}J(r)dr$$

i.e.

$$\begin{aligned}{}_kI_a^{\gamma,\lambda}ghJ(u)+g(s)h(s)J(s)_kI_a^{\gamma,\lambda}(1) \geq & h(s)J(s)_kI_a^{\gamma,\lambda}g(u)+g(s)_kI_a^{\gamma,\lambda}hJ(u)+g(s)J(s)_kI_a^{\gamma,\lambda}h(u)\\ & +h(s)_kI_a^{\gamma,\lambda}gJ(u)-J(s)_kI_a^{\gamma,\lambda}gh(u)-g(s)h(s)_kI_a^{\gamma,\lambda}J(u).\end{aligned} \tag{3. 14}$$

Both sides of above equation multiplying by $\frac{1}{k\Gamma_k(\beta)}\left(e^{u+\lambda}-e^{s+\lambda}\right)^{\frac{\beta}{k}-1}e^{s+\lambda}$ and then integrating the resulting inequality w.r.t. s up to (a,u), we obtain

$$\begin{aligned}{}_kI_a^{\gamma,\lambda}ghJ(u)_kI_a^{\beta,\lambda}(1)+{}_kI_a^{\beta,\lambda}ghJ(u)_kI_a^{\gamma,\lambda}(1) \geq & {}_kI_a^{\beta,\lambda}hJ(u)_kI_a^{\gamma,\lambda}g(u)+{}_kI_a^{\beta,\lambda}g(u)_kI_a^{\gamma,\lambda}hJ(u)\\ & +{}_kI_a^{\beta,\lambda}gJ(u)_kI_a^{\gamma,\lambda}h(u)+{}_kI_a^{\beta,\lambda}h(u)_kI_a^{\gamma,\lambda}gJ(u)\\ & -{}_kI_a^{\beta,\lambda}J(u)_kI_a^{\gamma,\lambda}gh(u)-{}_kI_a^{\beta,\lambda}gh(u)_kI_a^{\gamma,\lambda}(u).\end{aligned} \tag{3. 15}$$

which is the required proof.

Corollary 3.3. *Let g,h both synchronous functions on the interval $[0,1)$, , and $J \geq 0$, then $\forall u > a \geq 0, \gamma > 0$, the following k-fractional integral inequalities hold:*

$$\frac{\left(e^{u+\lambda}-e^{a+\lambda}\right)^{\frac{\gamma}{k}}}{\Gamma_k(\gamma+k)}{}_kI_a^{\gamma,\lambda}ghJ(u)$$

$$\geq {}_kI_a^{\gamma,\lambda}g(u){}_kI_a^{\gamma,\lambda}hJ(u)+{}_kI_a^{\gamma,\lambda}gJ(u){}_kI_a^{\gamma,\lambda}h(u)-{}_kI_a^{\gamma,\lambda}J(u){}_kI_a^{\gamma,\lambda}gh(u). \quad (3.\ 16)$$

Theorem 3.4. *Let g, h and J be the three define monotonic functions on the interval $[0,\infty)$, satisfactory the following*

$$(g(r)-g(s)\left(h(r)-h(s)\right)\left(J(r)-J(s)\geq 0\right.$$

$[0,\infty)$, and $J \geq 0$, then $\forall u > a \geq 0, \gamma, \beta > 0$, following k-fraction integral inequality hold:

$$\frac{1}{\Gamma_k(\beta+k)}\left(e^{u+\lambda}-e^{a+\lambda}\right)^{\frac{\beta}{k}}{}_kI_a^{\gamma,\lambda}ghJ(u)-\frac{1}{\Gamma_k(\gamma+k)}\left(e^{u+\lambda}-e^{a+\lambda}\right)^{\frac{\gamma}{k}}{}_kI_a^{\beta,\lambda}ghJ(u)$$

$$\geq {}_kI_a^{\gamma,\lambda}gJ(u){}_kI_a^{\beta,\lambda}h(u)+{}_kI_a^{\gamma,\lambda}hJ(u){}_kI_a^{\beta,\lambda}g(u)-{}_kI_a^{\gamma,\lambda}J(u){}_kI_a^{\beta,\lambda}gh(u)$$

$$-{}_kI_a^{\gamma,\lambda}gh(u){}_kI_a^{\beta,\lambda}J(u)+{}_kI_a^{\gamma,\lambda}g(u){}_kI_a^{\beta,\lambda}hJ(u)+{}_kI_a^{\gamma,\lambda}h(u){}_kI_a^{\beta,\lambda}gJ(u).$$

Proof: We proof this theorem to use the same arguments as in theorem 3.2.

Theorem 3.5. *Suppose that $g : R \longrightarrow R$ be a function, that is represented as*

$$g'(u)=\int_a^u g(\tau)d\tau \ u > a \geq 0.$$

Then for $\gamma \geq k > 0$,

$${}_kI_a^{\gamma,\lambda}g'(u)=\frac{k}{\left(e^{u+\lambda}-e^{a+\lambda}\right)}{}_kI_a^{(\gamma+k),\lambda}g(u).$$

Proof: From the definition of the new k-fractional integraland and using the Dirichlets formula , we obtain

$${}_kI_a^{\gamma,\lambda}g'(u)=\frac{1}{k\Gamma_k(\gamma)}\int_a^u\left(e^{u+\lambda}-e^{v+\lambda}\right)^{\frac{\gamma}{k}-1}e^{v+\lambda}\int_a^v g(t)dtdv,$$

$${}_kI_a^{\gamma,\lambda}g'(u)=\frac{1}{k\Gamma_k(\gamma)}\int_a^u g(t)dt\int_t^u\left(e^{u+\lambda}-e^{v+\lambda}\right)^{\frac{\gamma}{k}-1}e^{v+\lambda}dv.$$

Also,

$${}_kI_a^{\gamma,\lambda}g'(u) \quad = \quad \frac{k}{\left(e^{u+\lambda}-e^{a+\lambda}\right)k\Gamma_k(\gamma+k)}\int_a^u\left(e^{u+\lambda}-e^{t+\lambda}\right)^{\frac{\gamma}{k}}e^{t+\lambda}g(t)dt,$$

$${}_kI_a^{\gamma,\lambda}g'(u)=\frac{k}{\left(e^{u+\lambda}-e^{a+\lambda}\right)}{}_kI_a^{(\gamma+k),\lambda}g(u).$$

Thus the required result is obtained.
Now, we present a hypothesized Cauchy Schwarz-Bunyakovsky inequality as take place:

Lemma 3.6. *Let* $g, h, l : [c, d] \to [0, \infty)$ *be the functions which are continuous and* $0 \leq c \leq d$*. Then*

$$\left(\int_c^d g^p(w)h^x(w)l(w)dw\right)\left(\int_c^d g^q(w)h^y(w)l(w)dw\right) \geq \left(\int_c^d g^{\frac{p+q}{2}}(w)h^{\frac{x+y}{2}}(w)l(w)dw\right)^2. \tag{3. 17}$$

Where x, y, p, q *are the arbitrary positive real numbers.*

Proof: It is clear that

$$\int_c^d \left[\sqrt{g^p(w)h^x(w)l(w)}\sqrt{\int_c^d g^q(w)h^y(w)l(w)dw} - \sqrt{g^q(w)h^y(w)l(w)}\sqrt{\int_c^d g^p(w)h^x(w)l(w)dw}\right]^2 \geq 0$$

then, it take place of

$$\int_c^d g^p(w)h^x(w)l(w)dw \int_c^d g^q(w)h^y(w)l(w)dw + \int_c^d g^q(w)h^y(w)l(w)dw \int_c^d g^p(w)h^x(w)l(w)dw$$

$$- \quad 2\left(\int_c^d g^{\frac{p+q}{2}}(w)h^{\frac{x+y}{2}}(w)l(w)dw\right)\sqrt{\int_c^d g^q(w)h^y(w)l(w)dw}\sqrt{\int_c^d g^p(w)h^x(w)l(w)dw} \geq 0$$

and also,

$$2\left(\int_c^d g^p(w)h^x(w)l(w)dw\right)\left(\int_c^d g^q(w)h^y(w)l(w)dw\right)$$

$$\geq 2\left(\int_c^d g^{\frac{p+q}{2}}(w)h^{\frac{x+y}{2}}(w)l(w)dw\right)\sqrt{\int_c^d g^q(w)h^y(w)l(w)dw}\sqrt{\int_c^d g^p(w)h^x(w)l(w)dw},$$

which provides the required inequality.

Theorem 3.7. *Let* $g \in L_1[c, d]$*. So, that for* $k, p, q, x, y > 0$ *also* $\gamma > 0$

$$\left({}_kI_a^{p\left(\frac{\gamma}{k}-1\right)+1} f^x(u)\right)\left({}_kI_a^{q\left(\frac{\gamma}{k}-1\right)+1} f^y(u)\right) \geq \left({}_kI_a^{\frac{p+q}{2}\left(\frac{\gamma}{k}-1\right)+1} f^{\frac{x+y}{2}}(u)\right)^2. \tag{3. 18}$$

Proof: By putting $g(v) = \left(e^{u+\lambda} - e^{v+\lambda}\right)^{\frac{\gamma}{k}-1}$, $l(v) = \frac{1}{k\Gamma_k(\gamma)} e^{v+\lambda}$ and $h(v) = f(v)$ in (3. 17), we get

$$\left(\frac{1}{k\Gamma_k(\gamma)} \int_c^u \left(e^{u+\lambda} - e^{v+\lambda}\right)^{p\left(\frac{\gamma}{k}-1\right)} f^x(v) e^{v+\lambda} dv\right)$$

$$\left(\frac{1}{k\Gamma_k(\gamma)} \int_c^u \left(e^{u+\lambda} - e^{v+\lambda}\right)^{q\left(\frac{\gamma}{k}-1\right)} f^y(v) e^{v+\lambda} dv\right)$$

$$\geq \left(\frac{1}{k\Gamma_k(\gamma)} \int_c^u \left(e^{u+\lambda} - e^{v+\lambda}\right)^{\frac{p+q}{2}\left(\frac{\gamma}{k}-1\right)} f^{\frac{x+y}{2}}(v) e^{v+\lambda} dv\right)^2,$$

which gives the required result (3. 18).

Remark 3.8. *For $k = 1$, the (3. 18) inequality becomes the following*

$$\left(I_a^{p(\gamma-1)+1} f^x(u)\right)\left(I_a^{q(\gamma-1)+1} f^y(u)\right) \geq \left(I_a^{\frac{p+q}{2}(\gamma-1)+1} f^{\frac{x+y}{2}}(u)\right)^2. \quad (3.\ 19)$$

4. Conclusion

The results of this research show how to examined the properties of fractional integral for the modified new k-fractional integral operators. This modification of the new k-fraction integral was depend on an exponential kernel of arbitrary exponent in the integral. Also corresponding to the new fractional integral operators we studied some k-fractional integral inequalities and Chebyshev type inequalities.

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