Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 53(3)(2021) pp. 29-45

Transmuted Inverted Kumaraswamy Distribution: Theory and Applications

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Received: 16 February, 2021 / Accepted:13 March, 2021 / Published online:22 March, 2021

Abstract.: In this research, a new probability distribution named Transmuted Inverted Kumaraswamy (TIK) distribution based on quadratic rank transmutation map has been proposed. The newly proposed distribution is an extension of the inverted Kumaraswamy distribution. Several statistical properties such as moments, probability weighted moments, moment generating function, incomplete moments and entropy measure are investigated. The parameters of the proposed distribution are estimated by using the maximum likelihood approach under type-II censoring, and the performance has been evaluated through mean squared error and bias by conducting Monte Carlo simulations. The proposed distribution showed, based on various goodness of fit indices, better suitability as compared to competitive distributions when applied on four real life datasets.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Transmuted distribution, Kumaraswamy distribution, Type-II censoring.

1. Introduction

Many probability distributions that are capable of reliability framing in life data analysis are called lifetime distributions. All distributions are explained by their density functions. Lifetime data comprises the life expectancy of any living organism mechanical system. Inverted distribution and its several formations are used in other areas like medical, engineering, and economics. This family of distributions contains the Lomax (Pareto type II) probability distribution, Inverted Kumaraswamy Distribution, Log-Logistic probability distribution, Beta type-II probability Distribution, and many more. Many generalization of the distribution used in life time testing are proposed. One such generalization is obtained by using the quadratic rank transmuted map proposed by [32]. Using this idea, many transmuted distributions have been proposed in the literature. For example,transmuted Weibull distribution [10], transmuted modified inverse Rayleigh distribution [15], transmuted Weibull-Rayleigh distribution [37], transmuted weighted exponential distribution [6], transmuted Weibull power function distribution [12], etc., and investigated their statistical properties. For more details on various transmuted distribution developed so far, we refer the reader [28] to [35], and references therein.

The well-know Kumarsawamy distribution proposed by Poondi Kumarswamy (1980) is applicable in hydrology where the processes are usually double bounded. A detailed investigation on Kumaraswamy distribution is done by [14]. Since the development of this distribution, a lot its variants have been proposed. For example, the Kumaraswamy modified Weibull distribution by [4], Kumaraswamy marshal-Olkin family of distributions by [2], Kumaraswamy exponentiated inverse Rayleigh distribution [11], Kumaraswamy transmuted exponentiated additive Weibull distribution [25], etc, and many other variants. For more variants of Kumaraswamy distribution, we refer [28], [18], [30],and references therein. In this study, we purpose transmuted inverted Kumarswamy (TIK) distribution. We further investigate its statistical properties including its special cases. Moreover, simulation study and real life applications is also given. The rest of the article is as follows: The statistical investigation of TIK distribution is given in Section 1. The estimation of proposed distribution under type-II censoring is given in Section 2. The performance is evaluated through simulation studies in Section 3 whereas the real life applications are given in Section 4. Finally, Section 5 covers the concluding remarks.

sectionTransmuted Inverted Kumaraswamy Distribution and Properties

In this section, the statistical properties of TIK distribution are derived together with the special cases of TIK distribution and entropy measures. The generalized transmuted family of distributions based on a cumulative distribution function (CDF), see [32], is defined by

$$F(t) := (1+\lambda)G_t(t) - \lambda(G_t(t))^2 \,, \tag{1. 1}$$

where $G_t(t)$ is the generalized CDF of any given continuous distribution and λ is the truncated parameter. The corresponding probability density function (PDF) can be written as

$$f(t) := g_t(t)[(1+\lambda) - 2\lambda G_t(t)]\,, \tag{1. 2}$$

where λ is the transmuted parameter and $g_t(t)$ is the generalized density function of any given continuous distribution. In order to develop the TIK model, we consider the inverted Kumaraswamy distribution as the baseline model. Therefore, the PDF and the CDF of inverted Kumaraswamy distribution are

$$G(t,\alpha,\beta) = ((1-(1+t)^{-}\alpha)^{\beta}\,, \tag{1. 3}$$

for $\alpha, \beta > 0$ and

$$g(t) = \alpha\beta(1+t)^{-(\alpha+1}[1-(1-t)^{-}\alpha]^{\beta-1}. \tag{1. 4}$$

By using (1. 1), the CDF of TIK distribution is

$$\begin{aligned} F(t,\alpha,\beta) &= [(1-(1+t)^{-\alpha})^{\beta}][(1+\lambda) - \lambda\{1-(1+t)^{-\alpha}\}^{\beta}]\,, \\ &= (1+\lambda)\{1-(1+t)^{\alpha}\}^{\beta} - \lambda\{1-(1+t)^{-\alpha}\}^{2\beta} \end{aligned} \tag{1. 5}$$

where $\alpha, \beta > 0$. Analogously, the PDF of TIK distribution by using (1. 2) is

$$\begin{aligned} f(t,\alpha,\beta,\lambda) &= \frac{\alpha\beta}{(1+t)^{\alpha+1}}\left(1-(1+t)^{-\alpha}\right)^{\beta-1}\left(1+\lambda-2\lambda\left(1-(1+t)^{-\alpha}\right)^{\beta}\right) \\ &= \frac{\alpha\beta(1+\lambda)}{(1+t)^{(\alpha+1)}}\left\{1-(1+t)^{-\alpha}\right\}^{\beta-1} - \frac{2\lambda\alpha\beta}{(1+t)^{(\alpha+1)}}\left\{1-(1+t)^{-\alpha}\right\}^{2\beta-1}, \end{aligned} \tag{1. 6}$$

where α and β are the scale and λ is the transmuted parameter.

1.1. **Survival Function.** Let $T \in \mathbb{R}$ with CDF $F(t)$on the interval $[0,\ \infty]$, then the survival function for TIK distribution is

$$\begin{aligned} R(t) &= 1 - F(t) \\ &= 1-(1+\ \lambda)\left\{1-(1+t)^{-\alpha}\right\}^{\beta} - \lambda\left\{1-(1+t)^{-\alpha}\right\}^{2\beta} \end{aligned} \tag{1. 7}$$

Hazzard Function. The hazard rate function of TIK distribution denoted by $h_{TIkum}(t)$ is obtained as

$$h(t) = \frac{\alpha\beta(1+t)^{-(\alpha+1)}\left\{1-(1+t)^{-\alpha}\right\}^{\beta-1}\left[1+\lambda-2\lambda\left\{1-(1+t)^{-\alpha}\right\}^{\beta}\right]}{1-(1+\ \lambda)\left\{1-(1+t)^{-\alpha}\right\}^{\beta} - \lambda\left\{1-(1+t)^{-\alpha}\right\}^{2\beta}} \tag{1. 8}$$

Colloray: Let t follows TIK distribution, then for t>0, the following holds

$$\lim_{t\to 0}\begin{cases} \infty & when \quad \beta < 1 \\ \alpha(1+\lambda) & when \quad \beta = 1 \\ 0 & when \quad \beta > 1 \end{cases} \tag{1. 9}$$

Proof:

We know that

$$h(t) = \Pr(T < t)\ = \frac{f(t)}{1-F(t)}$$

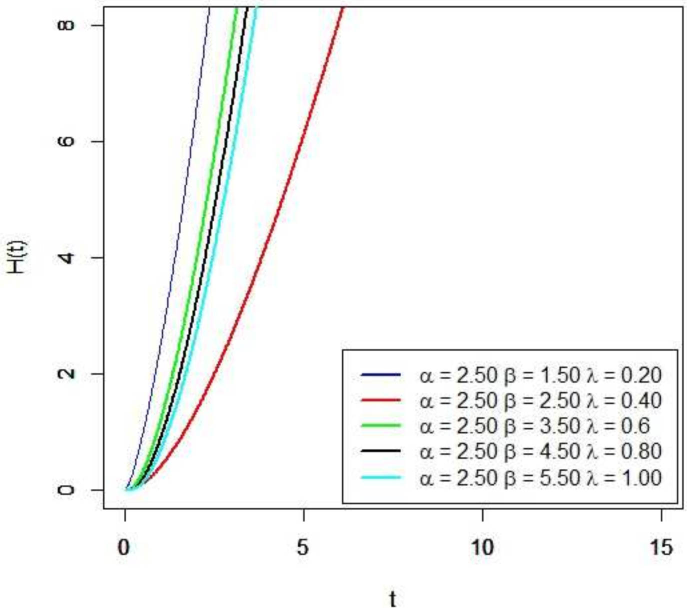

FIGURE 1. Cumulative hazard rate curves of TIK distribution for different values of parameters

$$\lim_{t\to 0} h\left(t\right) \ = \lim_{t\to 0} \frac{f\left(t\right)}{1-F\left(t\right)}$$
$$= \lim_{t\to 0} f(t) \lim_{t\to 0} \frac{1}{1-F\left(t\right)}$$

As we know that $\lim_{t\to 0} F(t) \ = F\left(0\right) = 0$
Therefore, $\lim_{t\to 0} h\left(t\right) \ = \lim_{t\to 0} \frac{f(t)}{1-F(t)} = \lim_{t\to 0} f(t)$

1.2. **Cummulative Hazzard Rate Function.** It is important to note that the the cumulative hazard rate can obtained by hazard rate function for the interval [0, *t*]. As compared to the hazard rate function, the cumulative hazard rate function is not probability density but it measures the risk which is infact is the risk of failure. Let H(t) denotes the hazar rate function at *t*, then H(.) for TIK distribution is

$$H\left(t\right) = \ -ln\left|1-\left(1+\ \lambda\right)\left\{1-\left(1+x\right)^{-\alpha}\right\}^{\beta} - \lambda\left\{1-\left(1+x\right)^{-\alpha}\right\}^{2\beta}\right) \quad (1.\ 10)$$

Figure 1 shows the cumulative hazard rate curves of TIK distribution for different parameters. Clearly, as t approaches to zero, $H(t)$ approach to zero too. Similarly, $H(t)$ approaches to infinity when t approaches to infinity.

1.3. **Probability Weighted Moments.** The probability weighted moments (PWMs) in the case of TIK distribution can be obtained by using

$$\beta_k = L\lambda(1+\lambda)^{k+1} B\left(k+1, \alpha\left(j+1+k+l\right)-k\right) - 2\lambda\alpha\beta M B\left(k+1, \alpha\left(i+1\right)-k\right) \tag{1. 11}$$

for all $k = 0, 1, 2, 3, 4, 5, \ \ldots$. Here we have

$$L = 2\lambda\alpha\beta \int_0^{\infty} t^k (1+t)^{-(\alpha+1)} \sum_{k=0}^{\infty} (-1)^k \left(\begin{array}{c} 2\beta-1 \\ i \end{array} \right) (1+t)^{-\alpha i}$$

and $M = \sum_{k=0}^{\infty} (-1)^k \left(\begin{array}{c} 2\beta-1 \\ i \end{array} \right) (1+t)^{-\alpha i}$

1.4. **Moment Generating Function.** Let *t* follows the TIK Distribution with parameters $\alpha, \beta > 0$ and $|\lambda| \leq 1$, then the moment genering function for TIK distribution is defined by

$$\begin{aligned} M_t\left(z\right) = {} & \alpha\beta\left(1+\lambda\right) \sum_{i=0}^{\infty} (-1)^i \left(\begin{array}{c} \beta-1 \\ i \end{array} \right) \sum_{k=0}^{\infty} \frac{(z)^k}{k!} B\left(1+k, \alpha\left(i+1\right)-k\right) \\ & - 2\alpha\beta\lambda \sum_{i=0}^{\infty} (-1)^i \left(\begin{array}{c} 2\beta-1 \\ i \end{array} \right) \sum_{k=0}^{\infty} \frac{(z)^k}{k!} B\left(1+k, \alpha\left(i+1\right)-k\right) \end{aligned} \tag{1. 12}$$

1.5. **Incomplete Moments.** Let *t* is randomly distributed variable follows the TIK Distribution with parameters $\alpha, \beta > 0$ and $|\lambda| \leq 1$, then the incomplete moment of TIK Distribution is can be obtained by.

$$\begin{aligned} E\left(t^r\right) = {} & \alpha\beta\left(1+\lambda\right) \sum_{i=0}^{\infty} (-1)^i \left(\begin{array}{c} \beta-1 \\ i \end{array} \right) B_{\frac{x}{1+x}}\left(r+1, \alpha\left(i+1\right)-r\right) \\ & - 2\lambda\alpha\beta \sum_{i=0}^{\infty} (-1)^i \left(\begin{array}{c} 2\beta-1 \\ i \end{array} \right) B_{\frac{x}{1+x}}\left(r+1, \alpha\left(i+1\right)-r\right) \end{aligned} \tag{1. 13}$$

where $r = 1, 2, 3, 4 \ldots$

1.6. **Renyi Entropy.**

$$\begin{aligned} I_T\left(\delta\right) = {} & \log\left(\alpha\right) + \frac{\delta}{1-\delta}\log\left(\theta\right) + \frac{\delta}{1-\delta}\log\left(1+\lambda\right)^{\delta} \beta\left(\delta\theta-\delta+1, \delta+\frac{\delta}{\alpha}-\frac{1}{\alpha}\right) \\ & - \log\left(2^{\delta}\lambda^{\delta}\alpha^{\delta-1}\theta^{\delta}\ \beta\left(2\delta\theta-\delta+1, \delta+\frac{\delta}{\alpha}-\frac{1}{\alpha}\right)\right) \end{aligned} \tag{1. 14}$$

1.7. **Special Cases of TIK Distribution.** Recall that the TIK distribution is

$$F(t,\ \alpha, \theta) = (1+\ \lambda)\left\{1-(1+t)^{-\alpha}\right\}^{\theta} - \lambda\left\{1-(1+t)^{-\alpha}\right\}^{2\theta},\ \alpha, \theta > 0$$

It can be written as

$$f\left(t,\alpha,\theta,\lambda\right)=\frac{\alpha\theta}{\left(1+t\right)^{(\alpha+1)}}\Big\{1-\left(1+t\right)^{-\alpha}\Big\}^{\theta-1}\left(1+\lambda-2\lambda\Big\{1-\left(1+t\right)^{-\alpha}\Big\}^{\theta}\right) \tag{1. 15}$$

for $\alpha,\theta>0,\ |\lambda|\ \leq 1\ and\ t>0$. Then following are the special cases of TIK distribution.

Case 1: Let us assume that $\theta=1$ and $\lambda=0$,then by substituting these in (1. 15), the TIK distribution leads to the density function of Lomax(Pareto type II) distribution, that is

$$f(t,\alpha,1,0)=\frac{\alpha(1)}{(1+t)^{(\alpha+1)}}\Big\{1-(1+t)^{-\alpha}\Big\}^{1-1}\left[1+0-2*0\Big\{1-(1+t)^{-\alpha}\Big\}^{1}\right], \tag{1. 16}$$

where $\alpha>0$ and $t>0$.

Case 2: Let us assume that $\lambda=0$and by putting it into equation (1. 15 we obtain the density function of Inverted Kumaraswamy Distribution probability, that is,

$$\begin{aligned} f(t,\alpha,\theta,0) &= \frac{\alpha\theta}{(1+t)^{(\alpha+1)}}\Big\{1-(1+t)^{-\alpha}\Big\}^{\theta-1}\left[1+0-2(0)\Big\{1-(1+t)^{-\alpha}\Big\}^{\theta}\right] \\ &= \frac{\alpha\theta}{(1+t)^{(\alpha+1)}}\Big\{1-(1+t)^{-\alpha}\Big\}^{\theta-1},\ \ \alpha,\theta>0,t>0 \end{aligned} \tag{1. 17}$$

Case 3: Let us assume that $\alpha=\theta=1$ and $\lambda=0$, then equation (1. 15 , after plugging these into it lead to the Log-Logistic probability distribution, that is,

$$\begin{aligned} f\left(t,1,1,0\right) &= \frac{1\left(1\right)}{\left(1+t\right)^{(1+1)}}\Big\{1-\left(1+t\right)^{-1}\Big\}^{1-1}\left[1+0-2\left(0\right)\Big\{1-\left(1+t\right)^{-1}\Big\}^{1}\right] \\ &= \frac{1}{\left(1+t\right)^{2}},t>0 \end{aligned} \tag{1. 18}$$

Case 4: Let us assume that $\alpha = 1$ and $\lambda = 0$ and plug them into equation (1. 15 that as a result produce the density function of Beta type-II Distribution, that is

$$\begin{aligned} f(t,\theta) &= \frac{\theta}{(1+t)^{(1+1)}}\left\{1-(1+t)^{-1}\right\}^{\theta-1}\left[1+0-2(0)\left\{1-(1+t)^{-1}\right\}^{\theta}\right] \\ &= \frac{\theta}{(1+t)^{2}}\left\{1-(1+t)^{-1}\right\}^{\theta-1} \\ &= \frac{\theta}{(1+t)^{2}}\left\{1-\frac{1}{(1+t)}\right\}^{\theta-1} \\ &= \frac{\theta}{(1+t)^{2}}\left\{\frac{t}{1+t}\right\}^{\theta-1} \\ &= \theta(1+t)^{-2}t^{\theta-1}(1+t)^{1-\theta} \\ &= \theta(1+t)^{1-2-\theta}t^{\theta-1} \\ &= \frac{1}{\theta(1,\theta)}t^{\theta-1}(1+t)^{-(\theta+1)} \end{aligned} \tag{1. 19}$$

2. Estimation Based on Type II Censored Data

It is well-kown that several estimation methods have been developed so far to estimate the parameters of an unknown distribution. One well-known approach is the method of maximum likelihood estimation (MLE) that we consider here to estimate the parameters for TIK distribution under type-II censoring. Recall that in censored data, the observed values of a random variable is partially known. There are different types of censored data, such as type-I and type-II censored data. In the case of type-I censored data, the experiment is stoped at a predetermined time which lead many objects unobserved. On the otherhand, in type-II censoring, the experimented is terminated when predetermined number of objects or units fails. To this end, we recall the PDF of TIK distribution.

$$f(t,\alpha,\theta,\lambda) = \frac{\alpha\theta}{(1+t)^{(\alpha+1)}}\left\{1-(1+t)^{-\alpha}\right\}^{\theta-1}\left[1+\lambda-2\lambda\left\{1-(1+t)^{-\alpha}\right\}^{\theta}\right], \tag{2. 20}$$

where $\alpha, \theta > 0,\ |\lambda| \leq 1$ and $t > 0$ For sample size n, we formulate (log) likelihood function, that is,

$$\begin{aligned} \ln(L) &= nln(\alpha) + nln(\theta) - (\alpha+1)\sum_{i=1}^{n} ln(1+t) \\ &+ \theta\sum_{i=1}^{n} ln\left\{1-((1+t))^{-\alpha}\right\} + \sum_{i=1}^{n} ln\left[((1+\lambda)) - 2\lambda\left\{1-(1+t)^{-\alpha}\right\}^{\theta}\right] \end{aligned} \tag{2. 21}$$

By differentiating w.r.t α we get

$$\frac{\partial ln(L)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\left(1+t\right) \; + \theta \sum_{i=1}^{n} \left[\frac{\left\{(1+t)^{-\alpha}\right\}}{1-(1+t)^{-\alpha}} \left\{-\ln(1+t)\right\} \right] \\ + -2\lambda\theta \sum_{i=1}^{n} \left\{ \frac{\left\{1-(1+t)^{-\alpha}\right\}^{\theta-1} \{1+t\}^{-\alpha} \left\{-\ln(1+t)\right\}}{(1+\lambda) - 2\lambda \left\{1-(1+t)^{-\alpha}\right\}^{\theta}} \right\} \tag{2. 22}$$

Now by differentiating w.r.t θ we get

$$\frac{\partial ln(L)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} ln\left(1-(1+t)^{-\alpha}\right) + 2\lambda \sum_{i=1}^{n} \left(\frac{\{1-(1+t)^{-\alpha}\}^{\theta-1} ln\{1-(1+t)^{-\alpha}\}}{(1+\lambda) - 2\lambda\{1-(1+t)^{-\alpha}\}^{\theta}} \right) \tag{2. 23}$$

Finally, be differentiating w.r.t. λ, we obtain

$$\frac{\partial ln(L)}{\partial\theta} = 2\lambda \sum_{i=1}^{n} \left\{ \frac{1 = 2\left\{1-(1+t)^{-\alpha}\right\}^{\theta}}{(1+\lambda) - 2\lambda \left\{1-(1+t)^{-\alpha}\right\}^{\theta}} \right\} \tag{2. 24}$$

By setting partial derivates w.r.t α, θ and λ equal to zero we obtain a set of simultaneous equation. One can obtain the estimators by solving these simultaneous equations.

3. SIMULATION STUDY

In this section we perform simulation study to evaluate the performance of TIK distribution. For simulatons, Monte Carlo approach is used with 10,000 repetitions and the comparasion is done on the basis of Mean Square Error (MSE) and absolute bias. The parameters are estimated using MLE technique. Following true parametric values are considered

$$\alpha = 0.5, \;\; \theta = 0.1, \;\; \lambda = 0.5$$
$$\alpha = 5.5, \;\; \theta = 1.0, \;\; \lambda = 0.9$$

Simulation is executed for sample sizes n=50,100,200,300,and 500. The results of ML estimated parametes in terms of mean, standard deviation, bias and mean squared error are presented in Table 1 and Table 2.

Sample size	Parameters1	Mean2	Standard2Error	Bias4	MSE5
25	α	4.788	2.3	4.288	23.679
	θ	37.597	94.241	37.497	10287.42
	λ	-799.255	10003.604	-798.755	1645231
50	α	4.285	1.739	3.785	17.351
	θ	18.466	44.065	18.366	2279.04
	λ	-594.221	558.347	-593.721	664255.9
100	α	4.05	1.221	3.55	14.093
	θ	8.705	17.446	8.605	378.414
	λ	-318.93	340.98	-318.43	217665.02
150	α	3.968	1.047	3.468	13.122
	θ	7.124	9.636	7.024	142.195
	λ	-194.47	216.54	-193.97	84513.93
200	α	3.904	0.893	3.404	12.384
	θ	6.142	6.041	6.042	72.995
	λ	-87.93	198.03	-87.43	46859.88
300	α	3.744	0.696	3.244	11.008
	θ	4.861	3.247	4.761	33.208
	λ	-54.31	141.02	-53.81	22782.15
400	α	3.726	0.673	3.226	10.86
	θ	4.467	2.097	4.367	23.468
	λ	-3.244	109.79	-2.744	12061.37
500	α	3.677	0.432	3.177	10.28
	θ	2.341	1.698	2.241	7.9052
	λ	-2.082	86.34	-1.582	7457.09
1000	α	1.987	0.319	1.487	2.3129
	θ	1.291	1.383	1.191	3.332
	λ	-1.643	71.94	-1.143	5176.67
1500	α	1.045	0.223	0.545	0.3467
	θ	0.653	1.127	0.553	1.575
	λ	-0.981	53.28	-0.481	2838.98
2000	α	0.641	0.182	0.141	0.053
	θ	0.201	0.732	0.101	0.546
	λ	-0.604	19.31	-0.104	372.886

TABLE 1. ML estimates of parameters on the basis of simulation and comparison with true values $\alpha = 0.5, \theta = 0.1$ and $\lambda = 0.5$.

Sample Size	Parameters1	Mean2	Standard Error3	Bias4	MSE5
25	α	11.602	1.095	6.102	38.434
	θ	598.907	683.766	597.907	825028.5
	λ	-1658.817	2079.904	-1657.917	7074690
50	α	10.992	0.302	5.492	30.253
	θ	305.312	203.422	304.312	133968.3
	λ	-801.3	616.019	-800.4	1020120
100	α	10.526	0.276	5.0262	25.337
	θ	253.258	81.177	252.258	70223.74
	λ	-648.602	236.916	-647.702	475647
150	α	7.452	0.198	1.952	3.849
	θ	218.421	69.821	217.421	52146.86
	λ	-501.43	209.321	-500.53	294345.56
200	α	6.873	0.134	1.373	1.90308
	θ	198.231	63.742	197.231	42962.98
	λ	-347.21	187.73	-346.31	155173.169
300	α	6.321	0.109	0.821	0.6859
	θ	109.37	57.291	108.37	15026.31
	λ	-191.94	109.65	-191.04	48519.404
400	α	6.098	0.103	0.598	0.3682
	θ	92.37	44.432	91.37	10322.67
	λ	-99.324	98.743	-98.424	19437.46
500	α	5.932	0.089	0.432	0.1945
	θ	61.327	39.778	60.327	5221.63
	λ	-57.84	59.345	-56.94	6763.99
1000	α	5.834	0.081	0.334	0.1181
	θ	39.231	26.234	38.231	2149.83
	λ	-31.54	47.021	-30.64	3149.78
1500	α	5.706	0.073	0.206	0.0477
	θ	8.765	19.456	7.765	438.831
	λ	-6.89	36.934	-5.99	1400
2000	α	5.6	0.065	0.1	0.0142
	θ	1.329	7.986	0.329	63.884
	λ	-0.999	17.405	-0.099	302.94

TABLE 2. ML estimates of parameters on the basis of simulation and comparison with true values $\alpha = 5.5, \theta = 1.0$ and $\lambda = 0.9$.

4. APPLICATIONS OF TRANSMUTED INVERTED KUMARASWAMY DISTRIBUTION TO REAL LIFE DATASETS

In this section, four real life applications of TIK distribution have been presented. The developed model is applied to data sets and the estimates of required parameters are obtained by using the MLE approach. Furthermore, the performance of TIK distribution is compared to Inverted Kumaraswamy Distribution, Lomax (Pareto type 2) distribution, Beta Type II(Inverted Beta) distribution, and Frechet distribution based on likelihood measures

4.1. **Application 1.** The first set of data comes from Lee and Wang (2003) comprises of the remission times (in months) of a random sample of 128 patients with bladder cancer.

5.62,32.15,2.26,6.76,14.24,9.47,11.98,5.71,26.31,5.06,7.26,0.20,2.23,20.28,5.41,5.34,25.74,7.32,3.52,12.02,7.62,8.37,0.40,0.51,4.34,5.32,11.64,14.76,6.97,5.85,4.40,4.50,46.12,8.65,9.22,7.28,3.70,12.03,4.87,4.26,7.63,11.79,6.93,12.63,2.62,2.83,36.66,5.17,13.29,1.46,8.66,21.73,4.98,2.07,13.80,10.06,2.75,7.09,2.69,14.77,2.02,2.87,17.14,6.94,4.51,9.02,5.09,3.02,18.10,5.49,16.62,2.54,3.31,23.63,17.36,1.40,1.26,1.19,10.66,11.25,3.36,5.41,9.74,6.54,10.34,3.64,0.81,7.66,13.11,3.82,7.39,4.33,10.75,0.90,0.08,19.13,12.07,7.93,25.82,2.02,2.69,2.09,5.32,2.46,0.50,14.83,1.35,1.76,4.18,3.48,34.26,4.23,22.69,3.57,3.36,2.64,8.26,3.25,1.05,79.05,43.01,8.53,17.12,15.96,7.87,7.59,6.25,3.88
The descriptive measures from the dataset are presented in Table 3 and the ML estimates alongwith their standard error for different distribution are presented in Table 4.

Mean	**Median**	**Mode**	**Variance**	**Minimum**	**Maximum**	**N**
9.36562	6.395	5	110.425	0.08	79.05	128

TABLE 3. Descriptive measures for the remission times (in month) of bladder cancer patients

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	**SE($\hat{\alpha}$)**	**S.E($\hat{\beta}$)**	**S.E($\hat{\lambda}$)**
TIK	1.217731	3.658443	-0.763852	0.09350652	0.6923991	0.13047099
Inverted Kumaraswamy	1.087677	4.657458		0.08580569	0.68545273	
Lomax (Pareto type-II)	0.5009143			0.04427481		
Beta Type II (Inverted Beta)		4.116951			0.3638905	
Frechet	0.7520845	3.2580877		0.04242387	0.40741298	

TABLE 4. MLEs for the remission times (in month) of bladder cancer patients data

In the following Table 5 and 6, different goodness of fit indices for different distributions are presented in order to compare the fitness of the proposed distribution with other considered distribution

Distribution	**lnL**	**AIC1**	**CAIC***	**BIC3**	**HQIC+**
TIK	420.7823	847.5645	847.7581	856.1206	851.0409
Inverted Kumaraswamy	426.3051	856.6102	856.7062	862.3142	858.9278
Lomax (Pareto Type-II)	472.0216	946.0433	946.075	948.8953	947.2021
Beta Type II (Inverted Beta)	426.8402	855.6804	855.7121	858.5324	856.8392
Frechet	444.0008	892.0015	892.0975	897.7056	894.3191

TABLE 5. Goodness of fit measures of proposed and other considered distributions for bladder cancer dataset

Distribution	Cramer Test	Anderson Darling Test	Kolmogorove Smirnov	p-value
TIK	0.26437	1.707203	0.083118	0.3395
Inverted Kumaraswamy	0.3859421	2.456708	0.10637	0.1104
Lomax (Pereto Type=II)	0.2738571	1.775642	0.31577	1.641x
Beta Type II (Inverted Beta)	0.3692528	2.356569	0.097025	0.1795
Frechet	0.7443207	4.546423	0.14079	0.01251

TABLE 6. Goodness of fit measures of proposed and other considered distributions for bladder cancer dataset.

From Table 5 and 6, it is observed that the TIK distribution reaches the minimum values of all the goodness of fit criteria described, indicating that the TIK distribution provides better performance for the remission times (in months) of patients with bladder cancer.

4.2. **Application 2.** The second set of data is taken from Murthy et al. (2006) where windshield maintenance times that were not defective at the time of the observations is given. The data is as follows:
0.046,1.436,2.592,0.140,1.492,2.600,0.150,1.580,2.670,0.248,1.719,2.717,0.280,1.794,2.819,0.313,1.915,2.820,0.389,1.920,2.878,0.487,1.963,2.950,0.622,1.978,3.003,0.900,2.053,3.102,0.952,2.065,3.304,0.996,2.117,3.483,1.003,2.137,3.500,1.010,2.141,3.622,1.085,2.163,3.665,1.092,2.183,3.695,1.152,2.240,4.015,1.183,2.341,4.628,1.244,2.435,4.806,1.249,2.464,4.881,1.262,2.543,5.140.
The descriptive statistics are presented in Table 7. The estimates of parameters using ML approach together with their standard error are given in Table 8. The goodness of fit measures considering different fit indices are presented in Table 9 and 10.

Mean	Median	Mode	Variance	Minimum	Maximum	N
2.08527	2.065	2.5	1.55059	0.046	5.14	63

TABLE 7. Descriptive measures for Service times of 63 Aircraft Windshield data.

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	**SE($\hat{\alpha}$)**	**S.E($\hat{\beta}$)**	**S.E($\hat{\lambda}$)**
TIK	2.0247256	2.7448902	-0.670917	0.22806	0.6692	0.18768
Inverted Kumaraswamy	1.789253	3.258129		0.21171	0.6299	
Lomax (Pareto type-II)	0.9625152			0.1212654		
Beta Type II (Inverted Beta)		1.699635			0.2141338	
Frechet	0.810356	0.930597		0.065629	0.154297	

TABLE 8. MLEs of TIK and other distribution on aircraft windshield data

Distribution	lnL	AIC	CAIC	BIC	HQIC
TIK	112.1855	230.371	230.7778	236.8004	232.8997
Inverted Kumaraswamy	115.1654	234.3308	234.5308	238.6171	236.0167
Lomax (Pareto Type-II)	130.8613	263.7226	263.7882	265.8657	264.5655
Beta Type II (Inverted Beta)	123.425	248.8501	248.9157	250.9932	249.693
Frechet	131.3029	266.6058	266.8058	270.8921	268.2916

TABLE 9. Gooddness of fit indices for TIK and other considered distributions using aircraft1 windshield data

Distribution	Crammer Test	Anderson Darling Test	Kolmogrov Smirnov Test	p-value
TIK	0.4531756	2.687421	0.16682	0.05315
Inverted Kumaraswamy	0.5463664	3.188067	0.18197	0.02684
Lomax (Pareto type 2)	0.4839982	2.846576	0.31801	3.642x
Beta Type II (Inverted Beta)	0.5123395	3.001137	0.26077	0.0002872
Frechet	0.9952383	5.418904	0.22147	0.003401

TABLE 10. Gooddness of fit indices for TIK and other considered distributions using aircraft1 windshield data

Clearly from Table 9 and Table 10, the TIK distribution shows better fit on the considered data as compared to all other distributions because it gain minimum values of most of th goodness of fit measures

4.3. **Application 3.** The third set of data, by Andrews and Herzberg (2012), represents the fatigue duration (fracture) of fatigue failure in Kevlar 373 epoxy under constant1pressure at a stress level of 90 percent until they all failed. The data is provided as:

0.0251,0.0886,0.0891,0.2501,0.3113,0.3451,0.4763,0.5650,0.5671,0.6566,0.6748,0.6751,0.6753,0.7696,0.8375,0.8391,0.8425,0.8645,0.8851,0.9113,0.9120,0.9836,1.0483,1.0596,1.0773,1.1733,1.2570,1.2766,1.2985,1.3211,1.3503,1.3551,1.4595,1.4880,1.5728,1.5733,1.7083,1.7263,1.7460,1.7630,1.7746,1.8275,1.8375,1.8503,1.8808,1.8878,1.8881,1.9316,1.9558,2.0048,2.0408,2.0903,2.1093,2.1330,2.2100,2.2460,2.2878,2.3203,2.3470,2.3513,2.4951,2.5260,2.9911,3.0256,3.2678,3.4045,3.4846,3.7433,3.7455,3.9143,4.8073,5.4005,5.4435,5.5295,6.5541,9.0960.

The descriptive statistics are presented in Table 11. The estimates of parameters using ML approach together with their standard error are given in Table 12. The goodness of fit measures considering different fit indices are presented in Table 13 and 14.

Mean	Median	Mode	Variance	Minimum	Maximum	N
1.95924	1.73615	1.5	2.47741	0.0251	9.096	76

TABLE 11. Descriptive measures for the fatigue fracture of Kevlar 373 epoxy data.

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	**SE($\hat{\alpha}$)**	**S.E($\hat{\beta}$)**	**S.E($\hat{\lambda}$)**
TIK	2.134827	2.469498	-0.752759	0.225614	0.546215	0.15845
Inverted Kumaraswamy	1.88626	3.143558		0.207126	0.556548	
Lomax (Pareto type-II)	1.026563			0.117755		
Beta Type II (Inverted Beta)		1.571857			0.180304	
Frechet	0.758853	0.820589		0.054088	0.132202	

TABLE 12. Gooddness of fit indices for TIK and other considered distributions using of Kevlar 373/epoxy data.

Distribution	**lnL**	**AIC1**	**CAIC***	**BIC3**	**HQIC+**
TIK	126.6522	259.3045	259.6378	266.2967	262.0989
Inverted Kumaraswamy	130.2268	264.4537	264.6181	269.1152	266.3166
Lomax (Pareto type 2)	148.0409	298.0817	298.1358	300.4124	299.0132
Beta Type II (Inverted Beta)	141.3447	284.6895	284.7435	287.0202	285.621
Frechet	153.5392	311.0784	311.2428	315.7399	312.9414

TABLE 13. Gooddness of fit indices for TIK and other considered distributions using of Kevlar 373/epoxy data.

Distribution	**Crammer Test**	**Anderson Darling Test**	**Kolmogrov Smirnov Test**	**p-value**
TIK	0.2233292	1.349718	0.1092	0.3026
Inverted Kumaraswamy	0.3085488	1.886975	0.12445	0.1744
Lomax (Pareto type 2)	0.2606781	1.593812	0.28597	5.39 10^{-6}
Beta Type II (Inverted Beta)	0.2833641	1.733491	0.22379	0.000796
Frechet	0.9168293	5.339595	0.18936	0.007352

TABLE 14. Gooddness of fit indices for TIK and other considered distributions using of Kevlar 373/epoxy data.

The above Table 13 and 14 addresse the goodness of fit criterion and ML estimates for examined models. We see that TIK distribution achieves the least estimations of all depicted goodness of fit standards. Therefore, TIK distribution can be considered better fit for fatigue fracture of Kevlar 373 epoxy data.

4.4. **Application 4.** The fourth data set is by Murty, Xie and Jiang (2004) present the failure time (in weeks) of 50 components commissioned at a time. The data is provided as follows
0.013,0.065,0.111,0.111,0.163,0.309,0.426,0.535,0.684,0.747,0.997,1.284,1.304,1.647,1.829,2.336,2.838,3.269,3.977,3.981,4.520,4.789,4.849,5.202,5.291,5.349,5.911,6.018,6.427,6.456,6.572,7.023,7.087,7.291,7.787,8.596,9.388,10.261,10.713,11.658,13.006,13.388,13.842,17.152,17.283,19.418,23.471,24.777,32.795,48.105.

The descriptive statistics are presented in Table 15. The estimates of parameters using ML approach together with their standard error are given in Table 16. The goodness of fit measures considering different fit indices are presented in Table 17 and 18.

Mean	Median	Mode	Variance	Minimum	Maximum	N
7.82102	5.32	2.5	84.75597	0.013	48.105	50

TABLE 15. Descriptive measures for the failure time (in weeks) of 500components data

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	S.E($\hat{\alpha}$)	S.E ($\hat{\beta}$)	S.E($\hat{\lambda}$)
TIK	0.8163129	1.1920156	-0.5980451	0.1243014	0.304033	0.2421881
Inverted Kumaraswamy	0.7211705	1.4436815		0.1162764	0.2734134	
Lomax (Pareto type 2)	0.581541			0.08224207		
Beta Type II (Inverted Beta)		1.876653			0.2653987	
Frechet	0.4790776	1.2802198		0.04541148	0.40276074	

TABLE 16. MLEs of TIK and other distributions for the failure time8(in weeks) of 500 components data.

Distribution	lnL	AIC1	CAIC*	BIC3	HQIC+
TIK	159.4382	324.8764	325.3981	330.6124	327.0607
Inverted Kumaraswamy	161.3341	326.6682	326.9235	330.4922	328.1244
Lomax (Pareto type 2)	163.0822	328.1644	328.2477	330.0764	328.8925
Beta Type II (Inverted Beta)	163.8393	329.6786	329.7619	331.5906	330.4067
Frechet	168.6388	341.2777	341.533	345.1017	342.7339

TABLE 17. Gooddness of fit indices for TIK and other considered distributions using for the failure time (in weeks) of 50 components data

Distribution	Crammer Test	Anderson Darling Test	Kolmogrov Smirnov Test	p-value
TIK	0.3745741	2.032786	0.1989	0.03827
Inverted Kumaraswamy	0.4320544	2.354821	0.21998	0.01583
Lomax (Pareto type 2)	0.4224282	2.301301	0.24674	0.004541
Beta Type II (Inverted Beta)	0.4369886	2.382256	0.29643	0.0003053
Frechet	0.6096729	3.313685	0.19935	0.0376

TABLE 18. Gooddness of fit indices for TIK and other considered distributions using for9the failure time (in weeks) of 50 components data.

For this dataset, we see from Table 17 and 18 that the TIK achieves the estimates with best results on goodness of fit indices. This indicates that the TIK distribution the best suitable distribution for the said dataset among the considered distributions.

5. Conclusion

In this study, a new TIK distribution is proposed to model the lifetime experiments and its statistical properties are investigated. The parameters of the TIK distribution were estimated through the MLE criteria and numerical results were estimated via Monte-Carlo simulation. The effects of the sample size on the shape and scale of the proposed distribution are also explored. The proposed distribution is compared, empirically, with some competitive distributions existing in the literature. The numerical results suggested that the proposed TIK distribution is suitable to the compared distributions when studying the life of an experiment. The proposed distribution can be extended to study the weighted dispersions, beta summed up conveyance, Zografos-Balakrishnan-G (ZB-G) appropriation, and Marshall Olkin circulation for some future work.

Note: This research is part of the thesis with turnitin similarity report ID: 1349085284 dated June 24, 2020 and submitted to Punjab University Library, Lahore [36].

Conflicts of Interest: The authors declare no conflict of interest.

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