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Approximation and Eventual periodicity of Generalized Kawahara equation using RBF-FD method.

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Abstract.: In engineering and mathematical physics nonlinear evolutionary equations play an important role. Kawahara equation is one of the famous nonlinear evolution equation appeared in the theories of shallow water waves possessing surface tension, capillary-gravity waves and also magneto-acoustic waves in a plasma. Another specific subjective parts of arrangements for some of evolution equations demonstrated by investigations, which connect alongwith their large-time behavior named as eventual time periodicity uncovered across solutions to IBVPs (initial-boundary-value problems). In this study eventual periodicity of solutions for the generalized fifth order Kawahara equation (IBVP) on bounded domain coupled with periodic boundary condition will explored numerically utilizing meshless technique called as Radial basis function generated finite difference (RBF-FD) method.

AMS (MOS) Subject Classification Codes: 65D12; 65J08; 65L06; 65M20
Key Words: RBFs Methods, RBF-FD Method, Kawahara equation, Eventual periodicity.

1. Introduction

In engineering and mathematical sciences such as solid state physics, plasma physics, chemical physics, fluid dynamics, chemical kinematics and geochemistry nonlinear evolutionary equations play an important role [1, 14, 24, 25, 26, 27, 17, 18, 19]. As an example, Kawahara equation is one of the famous nonlinear equation of evolution appeared in theories of shallow water waves possessing surface tension [7, 28, 9]. Various physical phenomena, suchlike plasma magneto-acoustic waves [22] and capillary gravity water waves [15] are described and represented by Kawahara and modified Kawahara equation respectively. KdV-Kawahara equation which is a particular form of Benney-Lin equation that accustomed to clarify the one-dimensional development in diverse media of small but finite amplitude long waves fluid dynamics problems [5, 23, 20, 21, 6]. Although the most general solution of the Kawahara equation is not available, the analytical solution for a special case in the form of solitary waves is given in [33]. Very few methods have been applied for the numerical solution of Kawahara equation [35, 10, 34]. It is worth nothing that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory signal, image processing and groundwater problems. In the past several decades, the investigation of travelling-wave solutions for nonlinear equations has played an important role in the study of nonlinear physical phenomena. An excellent literature of this can be found in fractional differentiation and integration operators were also used for extensions of the diffusion and wave equations. The HDM was recently applied to solve fractional modified Kawahara equation, fractional complex transform approximate is used for time fractional Kawahara and modified Kawahara equations, method based on the Jacobi elliptic functions for the fractional modified Kawahara equation has been found in [45, 16]. Another specific qualitative characteristic disclosed on solutions to IBVPs of some evolutionary equations that have been established through investigations and are linked by their large-time action named as eventual time periodicity. This enticing and appealing event take place by a piston or flap or paddle-type wave maker put on one of the channel's ends in research tests. As the wave generator oscillates at a predictable period $\mathrm{T}_0 > 0$, it appears that the amplitude of the waves becomes periodic at each place along the channel when a particular period of time has elapsed [36, 37]. Various studies have previously addressed this important and fascinating eventual periodic phenomena such as Burger-type equations, generalised equations for KdV, BBM, and its dissipating counterparts [38, 29, 31, 30, 32, 44, 2]. The goal of this study is to see whether the corresponding solution u of the following model problem for generalized fifth order Kawahara equation alongwith specified initial and boundary condition on bounded domain is eventually periodic by using a numerical scheme known as RBF-FD meshless method.

$$\begin{cases} u_t + \alpha u_x + (\beta + \delta u) u u_x + \gamma u_{xxx} - \mu u_{xxxxx} = 0, \ x \in [a, b], \ t \in (0, T], \\ u(a, t) = h_a(t), \ t \in (0, T], \\ u(b, t) = u_x(a, t) = u_x(b, t) = u_{xx}(b, t) = 0, \ t \in (0, T], \\ u(x, 0) = u_0(x), \ x \in [a, b]. \end{cases} \tag{1. 1}$$

Where α, β, δ, γ and μ are known and the boundary data $h_a(t)$ presumed to be periodic of period $\mathrm{T}_0 > 0$ such that $h_a(t) = h_a(t + \mathrm{T}_0)$ has asymptotic cycle of periodic behavior at any fixed point in space, supposing amplitude of the boundary forcing term $h_a(t)$ is minimal. So the wave-maker transfers energy from the left boundary ($x = a$, Place that mounts the wave-maker) into a finite channel while the channel at the right end ($x = b$) is free and open. In model (1. 1):

If $\alpha = \delta = 0$, then it is called Kawahara equation.
If $\alpha = \beta = 0$, then it is called Modified Kawahara equation.
If $\delta = 0$, then it is called KdV-Kawahara equation.

Meshfree methods are becoming more popular, emerging, interesting and fascinating numerical techniques due to the ability to solve those physical and engineering problems with no meshing or minimum of meshing for which the traditionally used mesh-based methods are not suited like finite volumes, finite differences, finite elements, Moving least square, Element free galerkin, Point interpolation method, Reproducing kernel particle method and Boundary element free method. RBFs methods appears to be really consists and most prominent meshless methods among the family of meshless methods while looking at the interpolation of multi dimensional scattered data and have received recently a tremendous and considerable attention in scientific community because of its capacity to achieve spectral accuracy, efficiency and high flexibility in solving complex PDEs, integral equations and fractional equations opposed to other advanced methods [4, 8, 11]. The most commonly used kernel in meshless techniques is the multi-quadric (MQ) kernel suggested by Hardy [13] to solve collocation scheme for PDEs employing radial basis function.

2. Description of RBF-Finite Differences method

We deal with general time dependent PDE for mathematical formulation and define the RBF-FD process in a gradual way. Take the problem of frame

$$u_t(x,t) = \mathcal{L}u(x,t), \text{ such that } x \in \Xi \subseteq \mathbb{R}^s,\ s \geq 1, t > 0, \tag{2. 2}$$

associated with initial and boundary conditions

$$u(x,0) = u_0(x),\ \mathcal{B}u(x,t) = h(x,t),\ x \in \partial\Xi, \tag{2. 3}$$

where u_0 and h are certain provided functions, while the spatial operators $\mathcal{L}$, $\mathcal{B}$ representing the differential operators. Assume $\{x_i\}_{i=1}^N$ denotes N number of nodes used for approximation in the domain Ξ for the given problem. RBF-FD is a mesh-free method and essentially a generalization of conventional finite difference (FD) method. In classical FD approach the derivative of a function u is defined as a linear combination of the values of u at some closest surrounding values (stencil) nodes. The difference is that RBF-FD methods use radial basis function instead of polynomials use in classical FD method [12].

2.1. **Global RBF differentiation matrix.** Discretization of equations (2. 2)-(2. 3) via global RBF method can be followed by approximating the unknown function u by the linear combination of radial kernel ϕ at the node x specified by

$$\hat{u}(x) = \sum_{j=1}^{N} c_j \phi(\|x - x_j\|) = \mathbf{\Phi}(\mathbf{x})^T \mathbf{c},\ x \in \Xi, \tag{2. 4}$$

such that $\mathbf{\Phi}(\mathbf{x})^T = (\phi\|x - x_1\|, \phi\|x - x_2\|,\ \dots, \phi\|x - x_N\|)$, and $\mathbf{c}$ is the expansion coefficients vector. Equation (2. 4) in Lagrange form is stated as

$$\hat{u}(x) = \mathbf{\Phi}(\mathbf{x})^T \mathbf{K}^{-1} u, \tag{2. 5}$$

here $\mathbf{K}$ representing system interpolation matrix for the global RBF. Now the interpolant (kernel-based) $\hat{u}$ in equation (2. 5) gives good approximation of u. Consequently any operator used on $\hat{u}$ also would be an excellent estimation of relevant operator employed on u (see [12, 11]). Applying linear differential operator $\mathcal{L}$ on above equation (2. 5) gives

$$\mathcal{L}\hat{u}(x) = \mathcal{L}\mathbf{\Phi}(\mathbf{x})^T\mathbf{K}^{-1}u. \tag{2. 6}$$

From equation (2. 6) we used the notation below for values

$$\mathbf{K}_{\mathcal{L}} = \begin{bmatrix} \mathcal{L}\mathbf{\Phi}(\mathbf{x_1})^T \\ . \\ . \\ . \\ \mathcal{L}\mathbf{\Phi}(\mathbf{x_N})^T \end{bmatrix}. \tag{2. 7}$$

The global discretization (differentiation) matrix $\mathbf{L}$ of size $N \times N$ may thus be considered as

$$\mathbf{L} = \mathbf{K}_{\mathcal{L}}\mathbf{K}^{-1}. \tag{2. 8}$$

Since from equation (2. 7), we see that the i^{th} row of $\mathbf{K}_{\mathcal{L}}$ corresponds to $\mathcal{L}\mathbf{\Phi}(\mathbf{x_i})^T$, therefore we observe from equation (2. 8) that the i^{th} row of $\mathbf{L}$,

$$\mathbf{L}_i = \mathcal{L}\mathbf{\Phi}(\mathbf{x_i})^T\mathbf{K}^{-1}, \tag{2. 9}$$

serve as global differentiation matrix $\mathbf{L}$ one single row.

2.2. **Local RBF differentiation matrix.** We now report derivation of local differentiation matrix and describe how to compute the local finite differences associated weights which give rise to local interpolant in a locally small neighborhood regarding point x_i exactly. Consider the set of points $\Xi = \{x_1, ..., x_N\}$ where we want the derivative approximation, these points can be regarded as stencil centers. For a given i^{th} evaluation node say x_i, the size of nearest neighboring nodes in stencil N_{x_i} of x_i is n. Specifying also the set of points $Z = \{z_1, ..., z_N\}$ at which we want to analyze (sample) data. The points inside the stencil having size n are collected at $Z_i \subset Z$. Now estimation of differential operator $\mathcal{L}$ on stencil with center node x_i and collected at Z_i is given by

$$\mathbf{L}_i = \mathbf{K}_{\mathcal{L}}^{x_i}\mathbf{K}_{Z_i}^{-1}. \tag{2. 10}$$

Actually it assemble a stencil having center node x_i hence we declare it as local differentiation matrix however it behaves globally since it operate whole entire data of that small stencil. All those $\mathbf{L}_i$ matrices contains non-zeros entries in sparse(global) matrix $\mathbf{L}^{FD}$, however their position must still be determined further in that sparse matrix $\mathbf{L}^{FD}$. Now $\mathbf{L}_i^{FD}$ which representing the i^{th} row of $\mathbf{L}^{FD}$ and holds non-zero values from matrix $\mathbf{L}_i$ (since it has one test node x_i so it is row vector). As the points in $Z_i \subset Z$ are used in constructing $\mathbf{L}_i$. Hence columns of $\mathbf{L}^{FD}$ connected alongwith those points which are non-zero columns of row i. Determining the position in the sparse row $\mathbf{L}_i^{FD}$ of those points correctly, define an incidence matrix having entries below

$$[\mathbf{P}_i]_{k,\ell} = \begin{cases} 1, & \text{if } k = \ell, \text{ i.e., } k^{th} \text{ entry in } Z_i \text{ meet the } \ell^{th} \text{ entry in Z,} \\ 0, & \text{else.} \end{cases}$$

Use this to describe the complete sparse matrix as

$$\mathbf{L}^{FD} = \begin{bmatrix} \mathbf{K}_{\mathcal{L}}^{x_1}\mathbf{K}_{Z_1}^{-1}\mathbf{P}_1 \\ . \\ . \\ . \\ \mathbf{K}_{\mathcal{L}}^{x_N}\mathbf{K}_{Z_N}^{-1}\mathbf{P}_N \end{bmatrix}. \tag{2. 11}$$

Ultimately the discretization for problem (2. 2)-(2. 3) can be written as

$$\acute{u} = \mathbf{M}u, \tag{2. 12}$$

where $\mathbf{M} = \begin{bmatrix} \mathbf{L}^{FD} \\ \mathbf{B}^{FD} \end{bmatrix}$, where $\mathbf{B}^{FD}$ stand for the discretization of operator applied at the boundary and can accordingly be found as $\mathbf{L}^{FD}$. Evolving in time the ODE system (2. 12), some solver ODE such as, ode113 ode23, ode45,and several others can be used from Matlab.

3. Stability Analysis

Applying θ-weighted scheme to equation (1. 1) in the form

$$\frac{u^{n+1} - u^n}{\tau} + \theta\mathcal{L}u^{n+1} + (1-\theta)u^n = g(x, t^{n+1}) \tag{3. 13}$$

where $0 \leq \theta \leq 1$, τ denotes time step and u^n (n is non-negative integer) indicates solution at time $t^n = n\tau$. We have

$$\begin{aligned} &\frac{u^{n+1} - u^n}{\tau} + \theta\left[\alpha u_x + (\beta + \delta u)uu_x + \gamma u_{xxx} - \mu u_{xxxxx}\right]^{n+1} \\ &\quad +(1-\theta)\left[\alpha u_x + (\beta + \delta u)uu_x + \gamma u_{xxx} - \mu u_{xxxxx}\right]^n = 0. \end{aligned} \tag{3. 14}$$

The nonlinear terms $(uu_x)^{n+1}$ and $(u^2u_x)^{n+1}$ in equation (3. 14) can be approximated by linear term [41] as

$$(u^m u_x)^{n+1} \approx (u^m)^n u_x^{n+1} + m(u^{m-1})^n u_x^n u^{n+1} - m(u^m)^n u_x^n, \ m = 1, 2, ... \tag{3. 15}$$

substituting equation (3. 15) into equation (3. 14) and rearranging it, equation (3. 14) will be written as

$$\begin{aligned} &\left[1 + \theta\tau\beta u_x^n + 2\theta\tau\delta u^n u_x^n\right] u^{n+1} + \left[\theta\tau\alpha + \theta\tau\beta u^n + \theta\tau\delta(u^2)^n\right] u_x^{n+1} + \theta\tau\gamma u_{3x}^{n+1} \\ &\quad -\theta\tau\mu u_{5x}^{n+1} = \left[1 + \theta\tau\beta u_x^n - (1-\theta)\tau\beta u_x^n - (1-\theta)\tau u^n + 2\theta\tau\delta u^n u_x^n\right] u^n \\ &\quad\quad -(1-\theta)\tau\alpha u_x^n - (1-\theta)\tau\gamma u_{3x}^n + (1-\theta)\tau\mu u_{5x}^n. \end{aligned} \tag{3. 16}$$

Through applying the Von Neumann stability analysis, the stability of the proposed system will be analysed. Despite the fact that the application of Von Neumann stability refers to linear difference equations, but it can provide the requisite condition and in practice can be useful for the nonlinear (linearized) difference equation (see [42] and references therein). For this reason, first, one variable should be local freezes in the nonlinear terms in equation (3. 16), i.e., $u^n = v$ and $u_x^n = v_x$, where v is local constant value of u^n and v_x is local constant value of u_x^n, then Von Neumann analysis used to determine the required necessary stability condition. If the nonlinear terms are freezed locally $u^n = v$ and $u^n_x = v_x$ in equation (3. 16), it could be written as

$$\begin{aligned}[1+\theta\tau\beta v_x+2\theta\tau\delta v v_x]\,u^{n+1}+\left[\theta\tau\alpha+\theta\tau\beta v+\theta\tau\delta(v^2)\right]u_x^{n+1}+\theta\tau\gamma u_{3x}^{n+1}\\ -\theta\tau\mu u_{5x}^{n+1}=[1+\theta\tau\beta v_x-(1-\theta)\tau\beta v_x-(1-\theta)\tau v+2\theta\tau\delta v v_x]\,u^n\\ -(1-\theta)\tau\alpha u_x^n-(1-\theta)\tau\gamma u_{3x}^n+(1-\theta)\tau\mu u_{5x}^n.\end{aligned} \quad (3.\ 17)$$

The Von Neumann method is applied for every j by using $u^n_j = \xi^n e^{i\eta x_j}$ and replacing it in equation (3. 17),

$$\begin{aligned}[1+\theta\tau\beta v_x+2\theta\tau\delta v v_x]\,\xi^{n+1}e^{i\eta x_j}+i\eta\left[\theta\tau\alpha+\theta\tau\beta v+\theta\tau\delta(v^2)\right]\xi^{n+1}e^{i\eta x_j}\\ -i\eta^3\theta\tau\gamma\xi^{n+1}e^{i\eta x_j}-i\eta^5\theta\tau\mu\xi^{n+1}e^{i\eta x_j}=\\ [1+\theta\tau\beta v_x-(1-\theta)\tau\beta v_x-(1-\theta)\tau v+2\theta\tau\delta v v_x]\,\xi^n e^{i\eta x_j}\\ -i\eta(1-\theta)\tau\alpha\xi^n e^{i\eta x_j}+i\eta^3(1-\theta)\tau\gamma\xi^n e^{i\eta x_j}+i\eta^5(1-\theta)\tau\mu\xi^n e^{i\eta x_j}.\end{aligned} \quad (3.\ 18)$$

After simplification of equation (3. 18) we have

$$\xi = \frac{P_1 - iQ_1}{P_2 + iQ_2} \quad (3.\ 19)$$

where
$P_1 = [1+\theta\tau\beta v_x-(1-\theta)\tau\beta v_x-(1-\theta)\tau v+2\theta\tau\delta v v_x]$
$Q_1 = \left[\eta(1-\theta)\tau\alpha-\eta^3(1-\theta)\tau\gamma-\eta^5(1-\theta)\tau\mu\right]$
$P_2 = [1+\theta\tau\beta v_x+2\theta\tau\delta v v_x]$
$Q_2 = \left[\eta\theta\tau\alpha+\eta\theta\tau\beta v+\eta\theta\tau\delta(v^2)-\eta^3\theta\tau\gamma-\eta^5\theta\tau\mu\right]$

$$|\xi|^2 = \frac{P_1^2+Q_1^2}{P_2^2+Q_2^2} = \frac{N}{D}. \quad (3.\ 20)$$

Where $N = P_1^2 + Q_1^2$ and $D = P_2^2 + Q_2^2$, we know that if $D - N \geq 0$ then $|\xi| \leq 1$ and the method is stable. Upon simplification, we get

$$D - N = \tau(v + \beta v_x). \quad (3.\ 21)$$

Now it is clear that $\tau > 0$, so for enough small value of τ that is $\lim_{\tau\longrightarrow 0}$, we neglect those terms containing product of τ. Thus equation (3. 21) is non-negative for $\theta \geq \frac{1}{2}$ and $\tau(v + \beta v_x) \geq 0$. Therefore $|\xi| \leq 1$. Hence the necessary condition is established for stability and it can be concluded that our method is convergent. In addition, the convergence analysis of the above mentioned RBF-FD method has been analytically proved by Bayona, and Moscoso et al., [3].

4. NUMERICAL RESULTS

4.1. **Usage and application of the numerical scheme suggested.** Within this section, the proposed method is implemented for finding the numerical solution of generalized Kawahara equation. The accuracy, efficiency and the success of this scheme is tested in terms of

L_∞ and L_2 (error norms) and the two invariants I_1 and I_2 which are defined by

$$L_\infty = \|u^* - u\|_\infty = \max|u^* - u|$$
$$L_2 = \|u^* - u\|_2 = \sqrt{h\sum_{i=1}^{N}(u^* - u)^2} \tag{4. 22}$$
$$I_j = \frac{1}{j}\int_{-\infty}^{\infty} u^j dx \simeq \frac{1}{j}h\sum_{i=1}^{N} u_i^j,\ j = 1, 2.$$

Now consider equation (1. 1) with parameter $\alpha = \beta = 0$ and $\delta = \gamma = \mu = 1$, alongwith analytical solitary wave solution [39] given by

$$u(x,t) = DSech^2(k(x - Bt)), \tag{4. 23}$$

where $D = \frac{-3}{\sqrt{10}}$, $B = \frac{4}{25}$ and $k = \frac{1}{2}\sqrt{\frac{1}{5}}$, the initial and boundary conditions are extracted from the exact solution (4. 23). The calculation are carried out by taking $[a, b] = [-30, 30]$, with $N = 61$. We use MQ-RBF $\Phi(r) = \sqrt{c^2 + r^2}$ with shape parameter $c = 5$. The L_∞ and L_2 norms at $t = 0, 5, 15, 25$ are seen in Table 1, and also the solitary wave profile in comparison with the exact solution is shown in Figure 1.

Method	time	L$_\infty$	L$_2$	I$_1$	I$_2$	CPU time/sec
[43]	0	0	0	-8.48525	2.68328	
	5	6.1995e-05	1.7896e-04	-8.48524	2.68317	0.187
	15	1.0717e-04	2.7337e-04	-8.48487	2.68296	0.313
	25	1.2130e-04	3.4855e-04	-8.48464	2.68275	0.453
RBF-FD	0	0	0	-8.34616	2.63929	0.013
	5	4.9580e-04	9.4940e-04	-8.34743	2.63929	0.114
	15	1.0017e-03	2.6235e-03	-8.34357	2.63930	0.268
	25	2.8298e-03	5.9075e-03	-8.33515	2.63931	0.415

TABLE 1. Comparison table for problem-(4. 23).

Similarly if we consider equation (1. 1) with parameter $\alpha = \delta = 0$ and $\beta = \gamma = \mu = 1$, having the following exact solution [40],

$$u(x,t) = \frac{105}{169}Sech^4(\frac{1}{2\sqrt{13}}(x - \frac{205}{169}t - x_0)). \tag{4. 24}$$

The initial and boundary conditions are extracted from the exact solution (4. 24). The simulation are performed by taking $[a, b] = [0, 200]$, with $\Delta x = 1$. The L_∞ and L_2 error norms are calculated with MQ-RBF at $t = 5$. From results shown in Table 2 and Figure 2, we can see that our method (RBF-FD) showing good agreement with the exact solution.

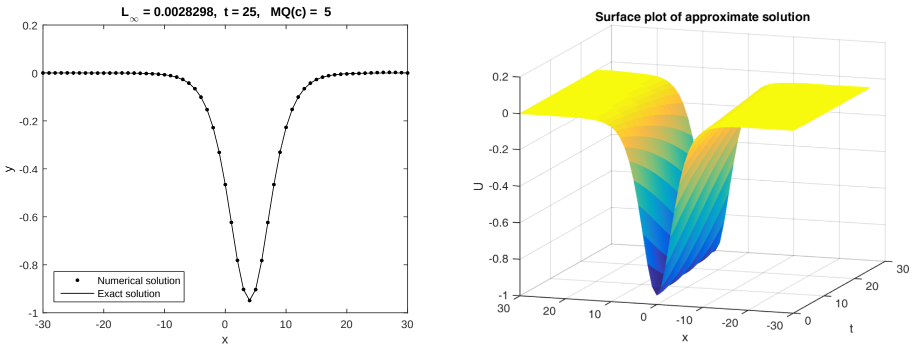

FIGURE 1. Solitary wave solution (showing the amplitude and trough position of the solitary wave) for problem (4. 23) in comparison with exact solution (solid lines show exact solution and "." showing numerical solution).

Method	time	L_∞	L_2	I_1	I_2	CPU
[43]	5	1.0977e-04	3.7679e-04	5.97559	1.27250	3.266
RBF-FD	5	1.1860e-04	4.0509e-04	5.94404	1.26614	0.399

TABLE 2. Comparison table for problem-(4. 24).

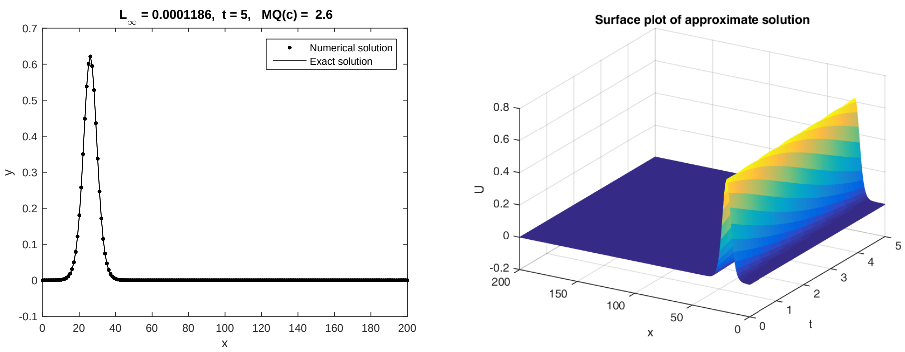

FIGURE 2. Solitary wave solution (showing the amplitude and peak position of the solitary wave) for problem (4. 24) in comparison with exact solution (solid lines show exact solution and "." showing numerical solution).

5. EVENTUAL PERIODICITY.

Now we present the results of our method investigating the eventual periodicity of generalized fifth order Kawahara equation (1. 1) in graphical form along with appropriate boundary data $h_a(t)$. The initial data u_0 is not necessarily necessary in eventual periodicity so take it zero. For each problem the amplitudes $u(x,t)$ produced in six graphs at particular points. N indicates complete domain points, while N_x denotes points in respective sub-domains. The X and Y axes are representative in these graphs of time t and amplitude u respectively. The last graph shows the amplitude remains zero in every problem.

5.1. **Eventual periodicity of Kawahara equation.** We compute the solutions of model equation (1. 1), for Kawahara equation with parameters $\alpha = 0$, $\beta = 1$, $\delta = 0$, $\gamma = 0.027$ and $\mu = 10^{-3}$. The amplitudes $u(x,t)$ for this model is shown in six plots in Figure 3 at given specific points viz $x = -19.5$, -17.5, -7.5, 5.0, 17.5 and 30.0 in the domain $[-20, 30]$ and in a time domain $[0, 5]$. The plots below clearly confirm the subsequent periodic activity of the solution in the specified domain at these particular positions.

5.2. **Eventual periodicity of Modified-Kawahara equation.** We compute the solutions of model equation (1. 1), for Modified-Kawahara equation using parameters $\alpha = 0$, $\beta = 0$, $\delta = 1$, $\gamma = 0.08$ and $\mu = 10^{-3}$. The amplitudes $u(x,t)$ for this model is shown in six plots in Figure 4 at given specific points viz $x = -29.4$, -27.0, -15.0, 0.0, 15.0 and 30.0 in the domain $[-30, 30]$ and in a time domain $[0, 5]$. The plots below clearly confirm the subsequent periodic activity of the solution in the specified domain at these particular positions.

5.3. **Eventual periodicity of KdV-Kawahara equation.** Finally we compute the solutions of model equation (1. 1), for KdV-Kawahara equation with parameters $\alpha = 0.4$, $\beta = 1.5$, $\delta = 0$, $\gamma = 4$ and $\mu = 10^{-3}$. The amplitudes $u(x,t)$ for this model is shown in six plots in Figure 5 at given particular points viz $x = 2$, 10, 50, 100, 150 and 200 in the domain $[0, 200]$ and in a time domain $[0, 5]$. The plots below clearly confirm the subsequent periodic activity of the solution in the specified domain at these particular positions.

6. CONCLUSION

In this study we have discussed RBF-FD method in detail and also implemented on the solution of IBVPs for generalized fifth order Kawahara equation and to examine the eventual periodicity in graphical form. The amplitudes recorded in different graphs at particular points in domain. In each problem the last graph shows the amplitude remain zero. We integrate our method with the RK-4 approach for time integration. The spatial operators in multi-dimensions are approximated by RBF in the finite difference (FD) setting which generates small size differentiation matrices in local sub-domains and these are assembled as a single sparse matrix in the global domain. So large amount of data can be manipulated very easily and accurately. The construction of our approach is simpler and easier to solve any nonlinear higher order PDEs as compared to other numerical methods. The efficiency, capacity, and high order accuracy of our suggested approaches are demonstrated using examples and results.

7. ACKNOWLEDGMENTS

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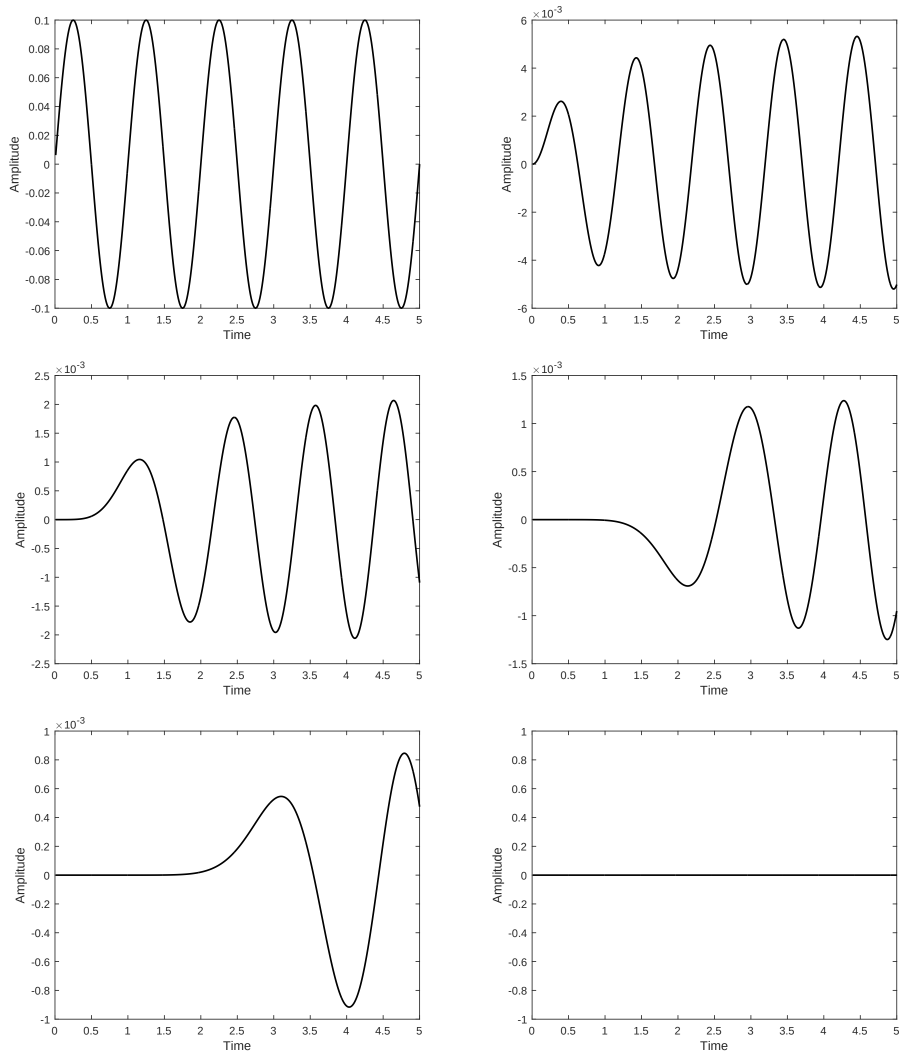

FIGURE 3. Kawahara equation eventual periodicity for $x = -19.5$, -17.5, -7.5, 5.0, 17.5 and $30.0 \in [-20, 30]$, $N = 100$, $N_x = 25$, $\delta t = 0.01$, $tmax = 5$, $h_a(t) = 0.1\sin(2\pi t)$.

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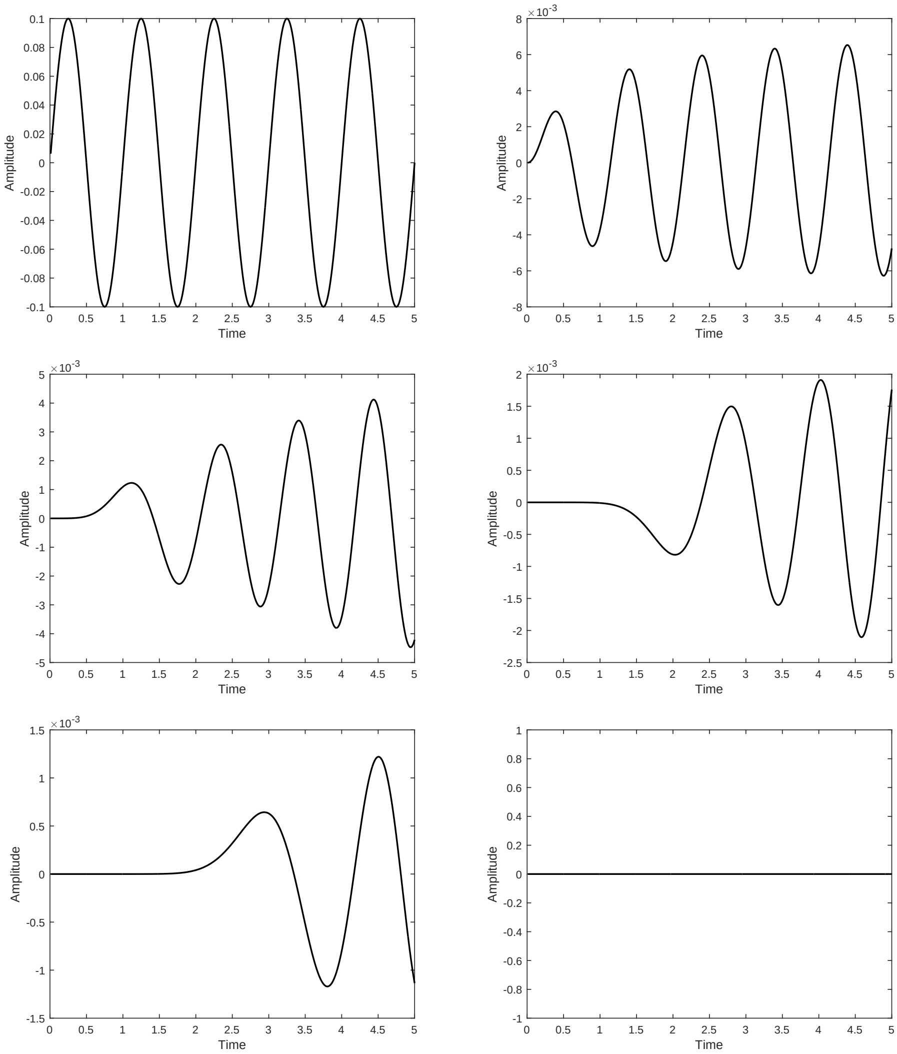

FIGURE 4. Modified-Kawahara equation eventual periodicity for $x = -29.4,\ -27.0,\ -15.0,\ 0.0,\ 15.0$ and $30.0 \in [-30, 30]$, $N = 100$, $N_x = 25$, $\delta t = 0.01$, $tmax = 5$, $h_a(t) = 0.1\sin(2\pi t)$.

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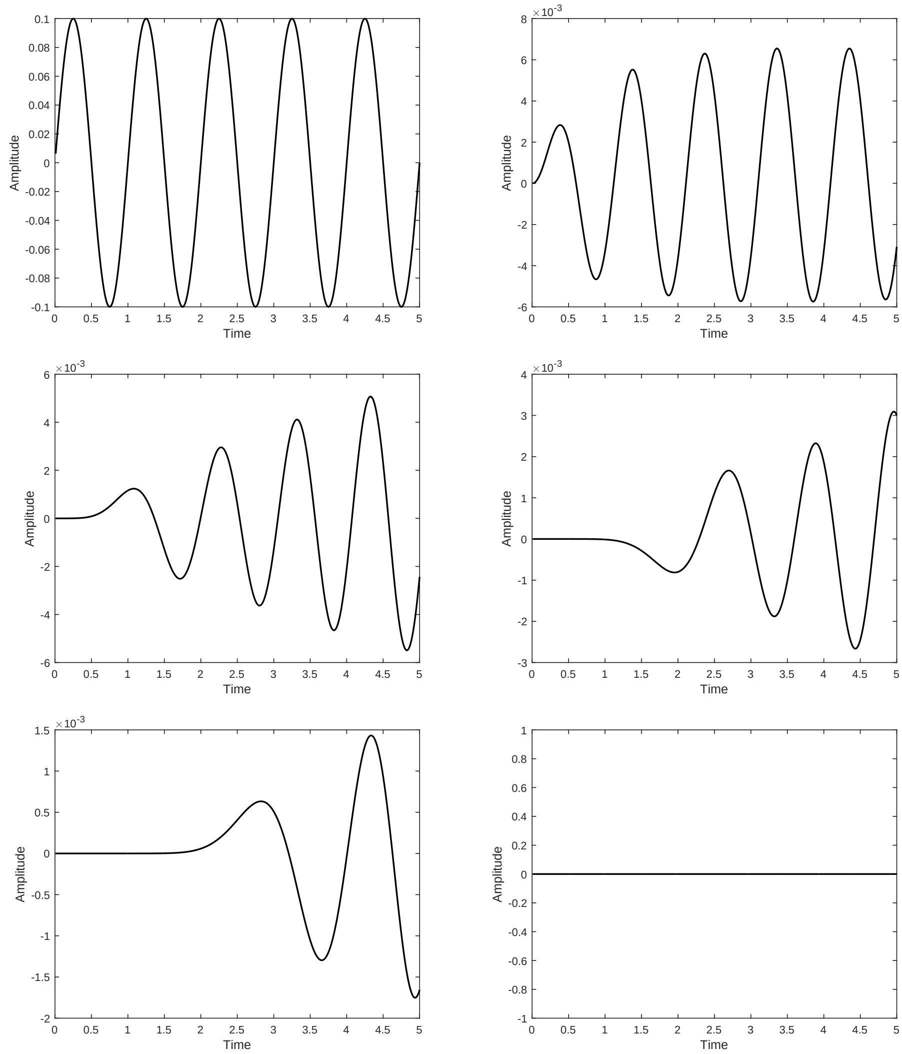

FIGURE 5. KdV-Kawahara equation eventual periodicity for x = 2, 10, 50, 100, 150 and 200 $\in [0, 200]$, $N = 100$, $N_x = 25$, $\delta t = 0.01$, $tmax = 5$, $h_a(t) = 0.1\sin(20\pi t)$.

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