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Numerical Solution of Ninth and Tenth Order Ordinary Differential Equations via Hermite Wavelet Method

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Abstract.: In this paper, we study Hermite wavelet method (HWM) for numerical solutions of higher-order ordinary differential equations. The Hermite wavelet used Hermite polynomial which is the basis for this method. This technique uses collocation points that transform the differential equation into an algebraic system of equations which reduces difficult computations to easier form as compared with other numerical techniques. We consider and evaluate two test problems, one of them is of order nine and the other is of order ten in order to show the applicability of the method. The outcomes that we get from the considered approach are approximately similar to the exact solution and easily acquired. The absolute errors for different number of collocation points are calculated and compared with the results obtained by other methods present in the available literature, and the graphical results obtained showed comparison between present numerical results and analytical solutions. The proposed methodology is also computationally efficient relative to other numerical approaches and the results obtained via the proposed scheme are precise and correct.

AMS (MOS) Subject Classification Codes: 34-XX; 34Bxx; 65-XX; 65Lxx
Key Words: Wavelet, Boundary value problems, Hermite wavelet method, Hermite polynomial.

1. Introduction

Differential equations play a vital role in modeling many real world problems [2]. Many physical phenomena can be modeled and can further be investigated for more interesting results via differential equations [29]. Due to their important applications, many researchers have given attention to the study of differential equations in various aspects. These aspects include qualitative theory of existence, analytical solutions and numerical results. For this purpose, fixed point theory has been used to study existence of solutions to differential equations. For analytical results, various techniques can be used such as Adomian decomposition method (ADM) [13], homotopy perturbation method (HPM) [14], new homotopy perturbation method (NHPM) [11], Chebyshev wavelet method [7] etc. However, in most situations finding exact and analytical solution is quite difficult job for many differential equations problems. Therefore, various mathematicians developed different numerical schemes to find the approximate solutions of various differential equations, for example, see [12, 4, 1, 5, 6]. Wavelet method is also one of the interesting numerical method. Wavelet family or wavelet was first introduced by a Hungarian mathematician Haar in his Ph.D. thesis in 1909 [23]. But Morlet and the team under Grossmann in France first proposed the present theoretical form of wavelet. Meyer and his colleagues developed the wavelet analysis method and the main algorithm date back to Mallat's work in 1988. The Wavelet family is broadly divided into some categories i.e., discrete wavelet, continuous wavelet, etc. Some wavelets techniques are discrete wavelets which constitute Haar wavelet [20], Legendre wavelet [22], Villasenor wavelet (VW)[19], Cohen-Daubechies-Feauveau wavelet [17], Daubechies wavelet [33], etc. While continuous wavelets contain Beta wavelet [8], Poisson wavelet [18], Hermite wavelet [26], Hermitian hat wavelet [30], Mexican hat wavelet [21], Spline wavelet [15].

Wavelet analysis is a new technique though its mathematical support date backs to the work of Joseph Fourier in the nineteenth century. Fourier laid the foundation with his theories of frequency analysis which is more considerable and impressive. However, the word wavelet was firstly used by Haar but the main algorithm of wavelet was developed by Mallat. After that, the researches on wavelets become international. The wavelet is a wave like oscillation having an amplitude that begins at zero, increases and then decreases back to zero. Wavelets are an interesting and useful mathematical tool for solving many differential equations, integral equations as well as integro-differential equations in an easy way and in less amount of time. These are functions that satisfy different mathematical concepts and are easily used for the representation of data or functions. In wavelet analysis, there is a scaling function that plays an important role. This analysis procedure is to develop a wavelet function, known as analysing wavelet or mother wavelet. There is a type of wavelets namely Hermite wavelets, which are a family of continuous functions and introduced by a French mathematician Charles Hermite. It is also constructed from dilation and translation of a single function of mother wavelet or analysing wavelet. Applications of wavelet exists in many applied areas such as in music [27], optics [24], signal and image processing [9], astronomy [28], radar [34], nuclear engineering [10], earthquake-prediction [3], magnetic resonance [16], fingerprint compression [25], etc. The following family of continuous wavelets is forming, when dilation parameter α and translation parameter β are changing continuously as

$$\phi_{\alpha,\beta}(y) = |\alpha|^{\frac{-1}{2}}\phi(\frac{y-b}{\alpha}), \text{ for all } \alpha \in \mathbb{R}, \beta \in \mathbb{R} - \{0\}. \tag{1. 1}$$

Now, if we apply restrictions on the parameters α and β to discrete values

$$\alpha = \alpha_0^{-r}, \beta = s\beta_0\alpha_0^{-r}, \ \alpha_0 > 1, \beta_0 > 0, \tag{1. 2}$$

then the family of discrete wavelets are as follow

$$\phi_{r,s}(y) = |\alpha|^{\frac{1}{2}}\phi(\alpha_0^r y - s\beta_0), \text{ for all } \alpha \in \mathbb{R}, \beta \in \mathbb{R} - \{0\}. \tag{1. 3}$$

where $\phi_{r,s}(y)$ form a wavelet basis for $L^2(\mathbb{R})$.

In the present article, we study Hermite wavelet method to obtain approximate solutions of higher-order differential equations. The Hermite wavelet method uses Hermite polynomial as a basis function. The unknown function is to be approximated through the Hermite wavelet in which collocation points will be used. With the help of collocation points, we will obtain a system of algebraic equations for the given differential equation. This system of equations will then be solved to get the desired solution of the problem. Test problems of ninth and tenth order ordinary differential equations will be considered to study the effectiveness and validity of the method.

The organization of the paper is as follows. In Section 2 we describe basic concepts related to our work. In Section 3, we discuss Hermite wavelet method and function approximation for solution of ordinary differential equations. Test problems and its solution are given in Section 4. While in Section 5, we provided conclusion of our work.

2. Preliminaries

In this section we introduce some basic concepts which are used in our work.

Boundary Value Problems (BVPs): The problems of finding the solution of differential equations such that the associated conditions relate at two or more than two different values of the independent variable. For example, for second order ordinary differential equation, we have

$$y(x_0) = y_0,\ y'(x_1) = y_1,$$

then these are called boundary conditions and the problem with given boundary conditions is called boundary value problem (BVP).

Hermite Wavelet: Hermite wavelets are defined as

$$\phi_{s,j}(y) = \begin{cases} \frac{2^{\frac{r+1}{2}}}{\sqrt{\pi}} N_j(2^r y - 2s + 1), & \frac{s-1}{2^{r-1}} \leq y < \frac{s}{2^{r-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2. 4}$$

where $j = 0, 1, 2, \ldots,$. Here $N_j(y)$ is Hermite polynomial of degree j on the real line $\mathbb{R}$.

Hermite Polynomial: Hermite polynomial $N_j(y)$ of degree j can be defined on the real line $\mathbb{R}$ and satisfies the following recurrence formula.

$$\begin{aligned} N_0(y) &= 1, \\ N_1(y) &= 2y, \\ &\vdots \\ N_{j+2}(y) &= 2yN_{j+1}(y) - 2(j+1)N_j(y), \end{aligned}$$

where $j = 0, 1, 2, \ldots,$.

There are some properties of Hermite polynomial. It is the beginner step for the generation of algorithm of Hermite wavelet method. The whole algorithm is based on the Hermite polynomial, it works like a backbone. Another property is that, the first two steps are already defined and the remaining steps are generating by the generalized formulae.

3. SOLUTION OF DIFFERENTIAL EQUATION BY APPLYING HERMITE WAVELET METHOD

Let we have a general ordinary differential equation of order s. To obtain the solution for this ordinary differential equation by Hermite wavelet method (HWM), we establish an algorithm for solution of generalized ordinary differential equation (ODEs) as follow. Considering the following equation

$$z^{(s)}(y) = z^{(j)}(y) + z(y) + g(y),\ j < s,\ s, j \in \mathbb{Z}^+, \tag{3. 5}$$

with the conditions

$$z(0) = w_0,\ z(r) = w_r,\ z^{(s)}(0) = z_0,\ z^{(s)}(r) = z_r. \tag{3. 6}$$

To obtain a better solution function $z(y)$ by Hermite wavelet method, we apply the following steps in our proposed HWM algorithm.
Step 1. Consider

$$z(y) = \sum_{s=1}^{\infty}\sum_{j=0}^{\infty} \mho_{s,j}\phi(y), \tag{3. 7}$$

where $\phi_{s,j}(y)$ is given as

$$\phi_{s,j}(y) = \begin{cases} \frac{2^{\frac{r+1}{2}}}{\sqrt{\pi}} N_j(2^r y - 2s + 1), & \frac{s-1}{2^{r-1}} \le y < \frac{s}{2^{r-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{3. 8}$$

where $j = 0, 1, 2, \ldots, .$
Step 2. Here $N_j(y)$ is Hermite polynomial of degree j on the real line $\mathbb{R}$ and satisfies the following recurrence formula

$$\begin{aligned} N_0(y) &= 1, \\ N_1(y) &= 2y, \\ &\vdots \\ N_{j+2}(y) &= 2yN_{j+1}(y) - 2(j+1)N_j(y), \end{aligned}$$

where $j = 0, 1, 2, \ldots, .$
Step 3. In this step we approximate $z(y)$ by truncating the series represented in Eq. (3. 7) as

$$z(y) \approx \sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1} \mho_{s,j}\phi(y) = \mho^T\phi(y), \tag{3. 9}$$

where $\mho$ and $\phi(y)$ are $2^{r-1}\times\ J-1$ matrices and $r, J \in \mathbb{Z}^+$. These are represented as

$$\mho^T = [\mho_{1,0}, \ldots, \mho_{1,J-1}, \mho_{2,0}, \ldots, \mho_{2,J-1}, \ldots, \mho_{2^{r-1},0}, \ldots, \mho_{2^{r-1},J-1}], \tag{3. 10}$$

$$\phi(y) = [\phi_{1,0}, \ldots, \phi_{1,J-1}, \phi_{2,0}, \ldots, \phi_{2,J-1}, \ldots, \phi_{2^{r-1},0}, \ldots, \phi_{2^{r-1},J-1}]. \tag{3. 11}$$

Step 4. The Eq. (3. 5) is approximated by using Eq. (3. 9) as below

$$\frac{d^s}{dy^s}\sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1} \mho_{s,j}\phi(y) = \frac{d^j}{dy^j}\sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1} \mho_{s,j}\phi(y) + \sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1} \mho_{s,j}\phi(y) + g(y). \tag{3. 12}$$

Now, using the subjected conditions of Eq. (3. 6) and Eq. (3. 9) we have

$$\sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1}\mho_{s,j}\phi(0)=w_0,\ \frac{d^s}{dy^s}\sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1}\mho_{s,j}\phi(0)=z_0,$$

$$\sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1}\mho_{s,j}\phi(r)=z_0,\ \frac{d^s}{dy^s}\sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1}\mho_{s,j}\phi(r)=w_0, \tag{3. 13}$$

Then, solving Eq. (3. 12) and Eq. (3. 13) with the help of Maple software, we obtain the values of the following coefficient constants as

$$\mho_{1,0},\ldots,\mho_{1,J-1},\mho_{2,0},\ldots,\mho_{2,J-1},\ldots,\mho_{2^{r-1},0},\ldots,\mho_{2^{r-1},J-1}, \tag{3. 14}$$

$$\phi_{1,0},\ldots,\phi_{1,J-1},\phi_{2,0},\ldots,\phi_{2,J-1},\ldots,\phi_{2^{r-1},0},\ldots,\phi_{2^{r-1},J-1}. \tag{3. 15}$$

Step 5. By putting the unknown constants of (3. 14) and (3. 15) in Eq. (3. 12) and Eq. (3. 13), we get the approximation. Hence the equations (3. 7)-(3. 15) constitute the numerical procedure for problem described in Eq. (3. 5) and Eq. (3. 6).

Hermite wavelet have many advantages in solving many problems in different fields. Some of them are: It can easily be applied to find the solution of different types of differential equation. It can also generate many other conditions (initial conditions or boundary conditions) which help in the solution of the given differential equations. The error occurring in the solution of this method is small as compared to other wavelets method. The results obtained by this method (HWM) is very coincided with the exact solution. With the help of this method we get approximation in a short interval of time and in an easy way. Hermite wavelet method is also used to solve problems or equations arising in the field of physics, engineering and in many other areas.

4. Numerical Results and Applications

In this section, we apply the constructed algorithm procedure on test problems.

Example 1. Let us consider the following ordinary differential equation of order nine with some given boundary conditions [34].

$$\begin{aligned} z^{(9)}+z^{(7)}+yz^{(4)}+z^{(3)}+z\sin(y)+z&=5y\sin(y)-\cos(y)+y^2\cos(y)-y\sin^2(y)\\ &\quad+\sin(y)\cos(y)+y\cos(y),\ 0<y<1, \end{aligned} \tag{4. 16}$$

and

$$\begin{aligned} &z(0)=0,\ z(1)=\cos(1),\ z^{'}(0)=1,\ z^{'}(1)=\cos(1)-\sin(1),\ z^{''}(0)=0,\\ &z^{''}(1)=-2\sin(1)-\cos(1),z^{'''}(0)=-3,\ z^{'''}(1)=-3\cos(1)+\sin(1),\ z^{iv}(1)=0. \end{aligned} \tag{4. 17}$$

Exact solution is

$$z=y\cos(y). \tag{4. 18}$$

Consider

$$z(y)=\sum_{s=1}^{\infty}\sum_{j=0}^{\infty}\mho_{s,j}\phi(y), \tag{4. 19}$$

where $\phi_{s,j}(y)$ is given as

$$\phi_{s,j}(y) = \begin{cases} \frac{2^{\frac{r+1}{2}}}{\sqrt{\pi}} N_j(2^r y - 2s + 1), & \frac{s-1}{2^{r-1}} \leq y < \frac{s}{2^{r-1}}, \\ 0, & \text{otherwise}, \end{cases} \quad (4.\ 20)$$

where $j = 0, 1, 2, \ldots, .$

We solve this problem (4. 16) by the proposed method (Hermite wavelet method) for r=1 and J=10. We apply the proposed method algorithm to approximate $z(y)$ by truncating (4. 19) as

$$z(y) \approx \sum_{j=0}^{10-1} \mho_{1,j}\phi(y) = \mho^T \phi(y), \quad (4.\ 21)$$

where $\mho$ and $\phi(y)$ are matrices of appropriate dimensions.

$$\mho^T = [\mho_{1,0}, \mho_{1,1}, \mho_{1,2}, \mho_{1,3}, \mho_{1,4}, \mho_{1,5}, \mho_{1,6}, \mho_{1,7}, \mho_{1,8}, \mho_{1,9}, \mho_{1,10}], \quad (4.\ 22)$$

$$\phi(y) = [\phi_{1,0}, \phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \phi_{1,4}, \phi_{1,5}, \phi_{1,6}, \phi_{1,7}, \phi_{1,8}, \phi_{1,9}, \phi_{1,10}]^T. \quad (4.\ 23)$$

To find the values of eleven unknown constants namely

$$\mho_{1,0}, \mho_{1,1}, \mho_{1,2}, \mho_{1,3}, \mho_{1,4}, \mho_{1,5}, \mho_{1,6}, \mho_{1,7}, \mho_{1,8}, \mho_{1,9}, \mho_{1,10}, \quad (4.\ 24)$$

we need eleven algebraic equations for the solution values of the constants of (4. 24), but there are only eight boundary conditions given, which are not sufficient for finding the eleven unknown constants. Therefore we have to find out three more conditions from which we generate three more equations which help us in finding the above (4. 24) unknown constants $\mho_{s,j}$. The remaining conditions are obtained by substituting Eq. (4. 16) in Eq. (4. 21), we obtain

$$z^{(9)} \sum_{j=0}^{10-1} \mho_{1,j}\phi(y)) + z^{(7)} \sum_{j=0}^{10-1} \mho_{1,j}\phi(y) + yz^{(4)} \sum_{j=0}^{10-1} \mho_{1,j}\phi(y) + z^{(3)} \sum_{j=0}^{10-1} \mho_{1,j}\phi(y)$$
$$+ z^{'} \sin(y) + z = 5y\sin(y) - \cos(y) + y^2\cos(y) - y\sin^2(y). \quad (4.\ 25)$$

Then, we have to collocate the Eq. (4. 25) by limit points of the following sequence, to obtain the remaining three equations,

$$\{y_i\} = \{\frac{1}{2}(1 + cos\frac{(i-1)\pi}{9})\},$$

where $i = 2, 3, \ldots, .$ Hence, we obtain

$$z^{(9)} \sum_{j=0}^{10-1} \mho_{1,j}\phi(y_i)) + z^{(7)} \sum_{j=0}^{10-1} \mho_{1,j}\phi(y_i) + y_i z^{(4)} \sum_{j=0}^{10-1} \mho_{1,j}\phi(y_i) + z^{(3)} \sum_{j=0}^{10-1} \mho_{1,j}\phi(y_i)$$
$$+ z^{'} \sin(y_i) + z = 5y_i\sin(y_i) - \cos(yi) + y_i^2\cos(y_i) - y_i\sin^2(y_i). \quad (4.\ 26)$$

By using the boundary conditions of Eq. (4. 17) and Eq. (4. 26), we get the following system of equations

$$
\begin{aligned}
&2.678819088 \times 10^{8}\mho_{1,10} + 6.988223640 \times 10^{6}\mho_{1,9} - 9.705866244 \times 10^{6}\mho_{1,8}\\
&+ 9.705866232 \times 10^{5}\mho_{1,7} + 2.079828480 \times 10^{5}\mho_{1,6}\\
&- 69327.61602\mho_{1,5} + 6932.761602\mho_{1,4} = 0,\\
&- 2.412601044 \times 10^{7}\mho_{1,10} + 6.697047720 \times 10^{6}\mho_{1,9} + 1.941173250 \times 10^{5}\mho_{1,8}\\
&- 3.033083198 \times 10^{5}\mho_{1,7} + 34663.80804\mho_{1,6} + 8665.951998\mho_{1,5}\\
&- 3466.380800\mho_{1,4} + 433.2976002\mho_{1,3} = -3,\\
&2.41260104 \times 10^{7}\mho_{1,10} + 6.697048020 \times 10^{6}\mho_{1,9} - 1.94117314 \times 10^{5}\mho_{1,8}\\
&- 3.033083188 \times 10^{5}\mho_{1,7} - 34663.80796\mho_{1,6} + 8665.95200\mho_{1,5}\\
&+ 3466.380802\mho_{1,4} + 433.2976002\mho_{1,3} = -3\cos(1) + \sin(1),\\
&- 2.677779146 \times 10^{6}\mho_{1,10} - 6.03150260 \times 10^{5}\mho_{1,9} + 1.860291030 \times 10^{5}\mho_{1,8}\\
&+ 6066.16634\mho_{1,7} - 10832.44000\mho_{1,6} + 1444.325332\mho_{1,5}\\
&+ 433.2976000\mho_{1,4} - 216.6488000\mho_{1,3} + 36.10813334\mho_{1,2} = 0,\\
&- 2.6777790 \times 10^{6}\mho_{1,10} + 6.0315032 \times 10^{5}\mho_{1,9} + 1.86029101 \times 10^{5}\mho_{1,8}\\
&- 6066.1656\mho_{1,7} - 10832.43996\mho_{1,6} - 1444.32534\mho_{1,5} + 433.297601\mho_{1,4}\\
&+ 216.6488002\mho_{1,3} + 36.10813334\mho_{1,2} = -2\sin(1) - \cos(1),\\
&- 4.838489 \times 10^{5}\mho_{1,10} - 66944.46908\mho_{1,9} + 16754.172\mho_{1,8} + 5813.4095\mho_{1,7}\\
&- 216.64878\mho_{1,6} - 451.351671\mho_{1,5} - 72.216266\mho_{1,4}\\
&+ 27.0811001\mho_{1,3} + 18.054066667\mho_{1,2} + 4.5135166668\mho_{1,1} = \cos(1) - \sin(1),\\
&4.838489867 \times 10^{5}\mho_{1,10} + 66944.47908\mho_{1,9} - 16754.17386\mho_{1,8} + 5813.40948\mho_{1,7}\\
&+ 216.648801\mho_{1,6} - 451.3516666\mho_{1,5} + 72.2162667\mho_{1,4}\\
&+ 27.08110002\mho_{1,3} - 18.05406667\mho_{1,2} + 4.5135516668\mho_{1,1} = 1,\\
&9279.79038\mho_{1,10} + 12096.22465\mho_{1,9} - 1859.568869\mho_{1,8} - 523.567934\mho_{1,7}\\
&+ 207.6217667\mho_{1,6} + 9.0270334\mho_{1,5} - 22.56758335\mho_{1,4}\\
&+ 4.5135166664\mho_{1,3} + 2.2567583334\mho_{1,2} - 2.2567583334\mho_{1,1} + 1.1283791667\mho_{1,0} = 0,\\
&9279.801\mho_{1,10} - 12096.223\mho_{1,9} - 1859.568969\mho_{1,8} + 523.56795\mho_{1,7} + 207.62178\mho_{1,6}\\
&- 9.027035\mho_{1,5} - 22.56758335\mho_{1,4} - 4.51351659\mho_{1,3}\\
&+ 2.25675833\mho_{1,2} + 2.2567583334\mho_{1,1} + 1.1283791667\mho_{1,0} = \cos(1),\\
&- 4.204195981 + 1.876818634 \times 10^{12}\mho_{1,10} + 1.115135716 \times 10^{11}\mho_{1,9} + 1.334438606 \times 10^{9}\mho_{1,8}\\
&+ 9.178759093 \times 10^{7}\mho_{1,7} + 79881.04795\mho_{1,6} + 64358.84551\mho_{1,5} + 9614.423334\mho_{1,4}\\
&+ 441.3492882\mho_{1,3} + 14.61610214\mho_{1,2} + 5.702446940\mho_{1,1} + 1.1283791667\mho_{1,0} = 0,\\
&- 3.586855568 + 1.439649437 \times 10^{12}\mho_{1,10} + 1.072350470 \times 10^{11}\mho_{1,9} + 1.041235660 \times 10^{9}\mho_{1,8}\\
&+ 9.189602127 \times 10^{7}\mho_{1,7} - 51707.30948\mho_{1,6} + 40985.35543\mho_{1,5} + 8233.933679\mho_{1,4}\\
&+ 426.5085778\mho_{1,3} + 9.44943140\mho_{1,2} + 4.9706474665\mho_{1,1} + 1.1283791667\mho_{1,0} = 0.
\end{aligned}
$$

Then solving the above equations, we obtain the unknown constants as below

$$\mho_{1,0} = 0.3154160820,\ \mho_{1,1} = 0.1099302198,\ \mho_{1,2} = -0.348028622e^{-1},$$
$$\mho_{1,3} = -0.4950358146e^{-2},\ \mho_{1,4} = 0.31118836e^{-3},\ \mho_{1,5} = 0.27369780e^{-4},$$
$$\mho_{1,6} = -9.197627 \times 10^{-7},\ \mho_{1,7} = -5.788142 \times 10^{-8}, \mho_{1,8} = 1.2813069 \times 10^{-9},$$
$$\mho_{1,9} = 7.75299 \times 10^{-11},\ \mho_{1,10} = -2.034180 \times 10^{-12}. \tag{4. 27}$$

By putting the values of unknown constants of Eq. (4. 27) in Eq. (4. 26), we obtain the numerical result of Eq. (4. 16),

$$z(y) = -0.2406824105e^{-5}y^{10} + 0.3496729659e^{-4}y^{9} - 0.2198572828e^{-4}y^{8}$$
$$- 0.1363934584e^{-2}y^{7} - 0.14216927e^{-4}y^{6} + 0.4166988271e^{-1}y^{5} + 1 \times 10^{-12}y^{4}$$
$$- 0.5000000001y^{3} + 1.39 \times 10^{-10}y^{2} + 1.000000001y - 1 \times 10^{-10}. \tag{4. 28}$$

TABLE 1. HWM and analytical solutions, and absolute error (AE) comparison with Petrov-Galerkin method [31] for the Example 1.

y	Exact solution	Approximate solution by HWM	AE in solution [31]	AE in HWM
0.00	0.000000000000000	-0.000000000100000	1.457692×10^{-07}	$1.0000000000 \times 10^{-10}$
0.01	0.009999500004000	0.009999499914000	2.458692×10^{-07}	$9.0000000000 \times 10^{-11}$
0.02	0.019996000130000	0.019996000050000	7.003546×10^{-07}	$8.0000000000 \times 10^{-11}$
0.03	0.029986501010000	0.029986500940000	1.430511×10^{-06}	$7.0000000000 \times 10^{-11}$
0.04	0.039968004270000	0.039968004210000	2.324581×10^{-06}	$6.0000000000 \times 10^{-11}$
0.05	0.049937513020000	0.049937512970000	1.668930×10^{-06}	$5.0000000000 \times 10^{-11}$
0.06	0.059892032390000	0.059892032360000	2.086163×10^{-07}	$3.0000000000 \times 10^{-11}$
0.07	0.069828570020000	0.069828569990000	1.430511×10^{-06}	$3.0000000000 \times 10^{-11}$
0.08	0.079744136500000	0.079744136490000	1.609325×10^{-06}	$1.0000000000 \times 10^{-11}$
0.09	0.089635745970000	0.089635745970000	1.072884×10^{-06}	$0.0000000000 \times 10^{+00}$

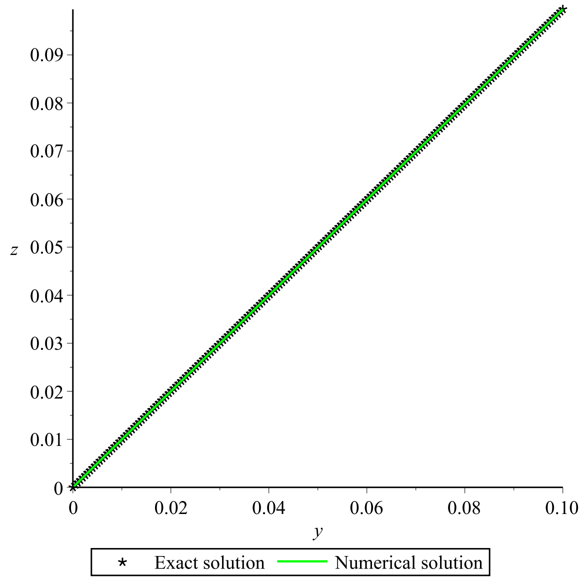

FIGURE 1. Comparison graph between exact and approximate solution for example 1.

Example 2. Let us consider the following ordinary differential equation of order ten [10].

$$z^{(10)} - z^{(3)} = 2e^{y}z^{2},\ 0 < y < 1, \tag{4. 29}$$

with boundary conditions

$$\begin{aligned} &z(0) = 1,\ z(1) = e^{-1},\ \ z^{'}(0) = -1,\ z^{'}(1) = -e^{-1},\ z^{''}(0) = 1, \\ &z^{''}(1) = e^{-1},\ \ z^{'''}(0) = -1,\ z^{'''}(1) = -e^{-1},\ z^{iv}(0) = 1,\ z^{iv}(1) = e^{-1}. \end{aligned} \tag{4. 30}$$

Exact result is given by

$$z = e^{-y}. \tag{4. 31}$$

We solve this problem by the proposed method for r=1 and J=12. We truncate the series to approximate $z(y)$ by using Eq. (3. 7) as

$$z(y) \approx \sum_{j=0}^{12-1} \mho_{1,j}\phi(y) = \mho^{T}\phi(y), \tag{4. 32}$$

where $\mho$ and $\phi(y)$ are matrices of appropriate dimensions.

$$\mho^{T} = [\mho_{1,0}, \mho_{1,1}, \mho_{1,2}, \mho_{1,3}, \mho_{1,4}, \mho_{1,5}, \mho_{1,6}, \mho_{1,7}, \mho_{1,8}, \mho_{1,9}, \mho_{1,10}, \mho_{1,11}, \mho_{1,12}], \tag{4. 33}$$

$$\phi(y) = [\phi_{1,0}, \phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \phi_{1,4}, \phi_{1,5}, \phi_{1,6}, \phi_{1,7}, \phi_{1,8}, \phi_{1,9}, \phi_{1,10}, \phi_{1,11}, \phi_{1,12}]^{T}. \tag{4. 34}$$

Further we need thirteen equations to find the values of eleven unknown constants

$$\mho_{1,0}, \mho_{1,1}, \mho_{1,2}, \mho_{1,3}, \mho_{1,4}, \mho_{1,5}, \mho_{1,6}, \mho_{1,7}, \mho_{1,8}, \mho_{1,9}, \mho_{1,10}, \mho_{1,11}, \mho_{1,12}, \tag{4. 35}$$

but there are only ten boundary conditions given, therefore, we have to find out other three conditions which help in finding the above unknown constants $\mho_{s,j}$. The remaining conditions are obtained by substituting Eq. (4. 29) in Eq. (4. 32), and we get

$$z^{(10)}\sum_{j=0}^{12-1}\mho_{1,j}\phi(y)-z^{(3)}\sum_{j=0}^{12-1}\mho_{1,j}\phi(y)=2e^{y}z^{2}. \tag{4. 36}$$

Then, we have to collocate the Eq. (4. 36) by limit points of the following sequence, to obtain the remaining three equations

$$\{y_i\}=\{\frac{1}{2}(1+cos\frac{(i-1)\pi}{9})\},$$

where $i=2,3,\ldots,$.. From which we get the following

$$z^{(10)}\sum_{j=0}^{12-1}\mho_{1,j}\phi(y_i)-z^{(3)}\sum_{j=0}^{12-1}\mho_{1,j}\phi(y_i)=2e^{y}z^{2}, \tag{4. 37}$$

From the boundary conditions in Eq. (4. 30) and Eq. (4. 37), the system of equations obtained by computation in Maple software will become sufficient to find the unknown constants. Then solving the system of equations with the help of Maple software, we obtain the unknown constants as follow

$$\begin{aligned}
&\mho_{1,0}=0.8074957166,\ \mho_{1,1}=-0.2061596808,\ \mho_{1,2}=0.2603949965e^{-1},\\
&\mho_{1,3}=-0.2183499505e^{-2},\ \mho_{1,4}=0.1370350229e^{-3},\ \mho_{1,5}=-0.6872029673e^{-5},\\
&\mho_{1,6}=2.869676903\times10^{-7},\ \mho_{1,7}=-1.026574756\times10^{-8},\ \mho_{1,8}=3.210828766\times10^{-10},\\
&\mho_{1,9}=-8.897106673\times10^{-12},\ \mho_{1,10}=2.184740907\times10^{-13},\\
&\mho_{1,11}=-4.562480041\times10^{-15},\ \mho_{1,12}=6.909943807\times10^{-17}.
\end{aligned} \tag{4. 38}$$

By putting the values of unknown constants from Eq. (4. 38) in Eq.(2.9), we obtain the numerical solution of Eq. (4. 29)

$$\begin{aligned}
&z(y)=8.707755555\times10^{-10}y^{12}-1.959850734\times10^{-8}y^{11}+2.595102538\times10^{-7}y^{10}\\
&-0.2721387724e^{-5}y^{9}+0.2474878748e^{-4}y^{8}-0.1983595486e^{-3}y^{7}+0.1388858384e^{-2}y^{6}\\
&-0.8333325823e^{-2}y^{5}+0.4166666665e^{-1}y^{4}-0.1666666666y^{3}+0.5000000000y^{2}\\
&-0.9999999997y+0.9999999999.
\end{aligned} \tag{4. 39}$$

TABLE 2. Absolute error comparison (AE) of HWM with Galerkin method with septic B-splines [32], HWM and analytical results for Example 2.

y	Exact solution	Approximate solution by HWM	AE in solution [32]	AE in HWM
0.1	0.904837418000000	0.904837418000000	6.735325E-06	0.0000000000E+00
0.2	0.818730753100000	0.818730753100000	4.410744E-06	0.0000000000E+00
0.3	0.740818220700000	0.740818221100000	3.629923E-05	4.0000000000E-10
0.4	0.670320046000000	0.670320046100000	4.839897E-05	1.0000000000E-10
0.5	0.606530659700000	0.606530660200000	4.929304E-05	5.0000000000E-10
0.6	0.548811636100000	0.548811636200000	3.945827E-05	1.0000000000E-10
0.7	0.496585303800000	0.496585304200000	9.834766E-06	4.0000000000E-10
0.8	0.449328964100000	0.449328964200000	1.996756E-06	1.0000000000E-10
0.9	0.406569659700000	0.406569660300000	5.066395E-06	6.0000000000E-10

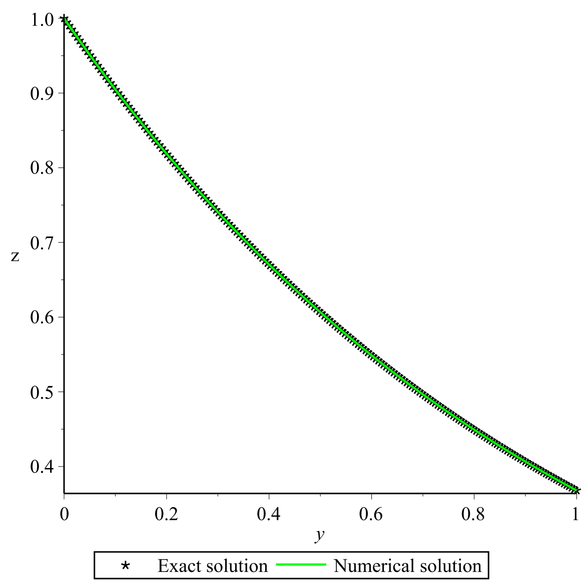

FIGURE 2. Comparison graph between exact and approximate solution for Example 2.

5. CONCLUSION

In this paper, a numerical scheme based on Hermite wavelet method has been developed to solve higher order ordinary differential equations. We used Maple software for compilation of numerical algorithm. The proposed method has been tested on two boundary value problems of order nine and ten. The solutions obtained through proposed method has also been compared with exact solutions and with other numerical results. From this, it is observed that the numerical results obtained by the proposed method are better as compared to other numerical methods and also approximately coincides with the exact solutions.

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References

[1] Z. E. Abo-Hammour, O. Alsmadi, S. Momani, O. A. Arqub, *A genetic algorithm approach for prediction of linear dynamical systems*, Math. Probl. Eng., **2013**, (2013)831657.

[2] F. Ahmad, E. Tohidi, M. Z. Ullah, J. A. Carrasco, *Higher order multi-step Jarratt-like method for solving systems of nonlinear equations: Application to PDEs and ODEs*, Comput. Math. Appl., **70**, No. 4 (2015) 624-636.

[3] A. Ali, R. Ghazali, M. M. Deris, *The wavelet multilayer perceptron for the prediction of earthquake time series data*, In Proceedings of the 13^{th} International Conference on Information Integration and Web-based applications and services (2011) 138-143.

[4] O. A. Arqub, *Numerical algorithm for the solutions of fractional order systems of Dirichlet function types with comparative analysis*, Fundam. Inform., **166**, No. 21 (2019)111-137.

[5] O. A. Arqub, H. Rashaideh, *The RKHS method for numerical treatment for integrodifferential algebraic systems of temporal two-point BVPs*, Neural. Comput. Appl., **30**, No.8 (2018) 2595-2606.

[6] O. A. Arqub, *Numerical solutions of systems of first-order, two-point BVPs based on the reproducing kernel algorithm*, Calcolo, **55**, No. 3 (2018) 1-28.

[7] E. Babolian, F. Fattahzadeh, *Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration*, Appl. Math. Comput., **188**, No. 1 (2007) 417-426.

[8] W. Bellil, C. B. Amar, A. M. Alimi, *Beta wavelet networks for function approximation*, In Adaptive and Natural Comput. Alg., (2005) 18-21.

[9] A. Bultheel, D. Huybrechs, *Wavelets with applications in signal and image processing*, Course material University of Leuven, Belgium (2014).

[10] T. Cong, G. Su, S. Qiu, W. Tian, *Applications of ANNs in flow and heat transfer problems in nuclear engineering: a review work*, Prog. Nucl. Energy, **62**, (2013) 54-71.

[11] A. Demir, S. Erman, B. Ozgur, E. Korkmaz, *Analysis of the new homotopy perturbation method for linear and nonlinear problems*, Bound. Value Probl., **2013** No. 1 (2013) 1-11.

[12] S. Djennadi, N. Shawagfeh, O. A. Arqub,*A fractional Tikhonov regularization method for an inverse backward and source problems in the time-space fractional diffusion equations*, Chaos Solit. Fractals, **150**,(2021) 111127.

[13] D. J. Evans, K. R. Raslan, *The Adomian decomposition method for solving delay differential equation*, Int. J. Comput. Math., **82(1)**, No. 1 (2005) 49-54.

[14] J. He, *Homotopy perturbation method: a new nonlinear analytical technique*, Appl. Math. Comput., **135**, No. 1 (2003) 73-79.

[15] J. G. Han, W. X. Ren, Y. Huang, *A spline wavelet finiteelement method in structural mechanics*, Int. J. Numer. Methods Eng., **66**, No. 1 (2006) 166-190.

[16] D. M. Healy, J. B. Weaver, *Two applications of wavelet transforms in magnetic resonance imaging*, IEEE Trans. Inf. Theory, **38**, No. 2 (1992) 840-860.

[17] S. Khalid, U. Jamil, K. Saleem, M. U. Akram, W. Manzoor, W. Ahmed, A. Sohail, *Segmentation of skin lesion using Cohen-Daubechies-Feauveau biorthogonal wavelet*, Springerplus, **5**, No. 1 (2016) 1-17.

[18] K. A. Kosanovich, A. R. Moser, M. J. Piovoso, *A new family of wavelets: the Poisson wavelet transform*, Comput. Chem. Eng., **21**, No. 6 (1997) 601-620.

[19] D. U. Lee, L. W. Kim, J. D. Villasenor, *Precision-aware self-quantizing hardware architectures for the discrete wavelet transform*, IEEE Trans. Image Process., **21**, No. 2 (2011) 768-777.

[20] U. Lepik, *Numerical solution of differential equations using Haar wavelets*, Math. Comput. Simul., **68**, No. 2 (2005) 127-143.

[21] Z. Masood, K. Majeed, R. Samar, M. A. Z. Raja, *Design of Mexican Hat Wavelet neutral networks for solving Bratu type nonlinear systems*, Neurocomputing, **221** (2017) 1-14.

[22] F. Mohammadi, M. M. Hosseini, *A new Legendre wavelet operational matrix of derivative and its applications in solving the singular ordinary differential equations*, J. Franklin Inst., **348**, No. 8 (2011) 1787-1796.

[23] R.Polikar, *The story of wavelets*, Phy. Modern topics in Mech. Elec. Eng., (1999) 192-197.

[24] P. Sandoz, *Wavelet transform as a processing tool in white-light interferometry*, OPt. Lett., **22**, No. 14 (1997) 1065-1067.

[25] B. G. Sherlock, D. M. Monro, *Optimized wavelets for fingerprint compression*, Proc.-ICASSP IEEE Int. Conf. Acoust. Speech Signal Process., **3** (1996) 1447-1450.

[26] S. C. Shiralashetti, B. S. Hoogar, S. Kumbinarasaiah, *Hermite wavelet based method for the numerical solution of linear and nonlinear delay differential equations*, Int. J. Eng. Sc. Math., **6**, No. 8 (2017) 71-79.

[27] W. C. Shih, *Time Frequency Analysis and Wavelet Transform Tutorial*, Wavelet for Music Signals Analysis (2006).

[28] J. L. Starck, *Multiscale methods in astronomy: Beyond wavelets*, In Astronomical Data Analysis Software and Systems XI, **(281)** (2002) 391-399.

[29] H. Sun, W. Chen, Y. Chen, *Variable-order fractional differential operators in anomalous diffusion modeling*, Phys. A: Stat. Mech. Appl., **388**, No. 21 (2009) 4586-4592.

[30] H. H. Szu, C. C. Hsu, L. D. Sa, W. Li, *Hermitian hat wavelet design for singularity detection in the Paraguay river-level data analyses*, In Wavelet Appl. Int. Society Opt. Photonics, **3078** (1997) 96-115.

[31] K. K. Viswanadham, S. M. Reddy, *Numerical solution of ninth order boundary value problems by Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions*, Procedia Eng., **127** (2015) 1227-1234. *Fuzzy Sets and Systems*, **309** (2017) 131-144.

[32] K. K. Viswanadham, S. Ballem, *Numerical solution of tenth order boundary value problems by Galerkin method with Septic B-splines*, Int. J. Appl. Sci. Eng., **13**, No. 3 (2015) 247-260.

[33] C. Vonesch, T. Blu, M. Unser, *Generalized Daubechies wavelet families*, IEEE Trans. Signal Process., **55**, No. 9 (2007) 4415-4429.

[34] V. V. Zhirnov, S. V. Solonskaya, I. I. Zima, *Application of wavelet transform for generation of radar virtual images*, Telecommun. Radio Eng., **73**, No. 17 (2014).