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ON THE NEW FAMILY OF KIES BURR XII DISTRIBUTION

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Abstract.: We propose a new lifetime model derived from the T-X generator called the new family of Kies Burr XII (NFKBXII) distribution. The NFKBXII density function is symmetrical, right-skewed, left-skewed, J, reverse-J, U and exponential shaped. The NFKBXII model failure rates can be monotone and non-monotone in shapes depending on the selection of the parameters. To show the importance of the NFKBXII distribution, we establish various mathematical properties such as random number generator, moments, density functions of record values, reliability and characterizations. We address the maximum likelihood estimates (MLE) for the NFKBXII parameters. We estimate the precision of the maximum likelihood estimators via a simulation study. We consider an application to serum-reversal times to clarify the potentiality and utility of the NFKBXII model along with NKBXII, KMBXII, KBXII, NKL, KL, MBXII, MBIII, Weibull and inverse Weibull distributions. We apply goodness of fit statistics (GOFs), various model selection criteria, and graphical tools to examine the adequacy of the NFKBXII distribution.

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1. INTRODUCTION

In the recent decade, many continuous distributions have been introduced in the statistical literature. These distributions do not pay vital role in survival analysis, life testing, reliability, finance, environmental sciences, biometry, hydrology, ecology and geology. So, applications of the generalized models to these sciences are clear requisites.
The generalization of the distribution is the only way to increase the applicability of the parent distribution. There are several ways to add one or more parameters in the parent

distribution. Using baseline distribution as generator attracts several researchers in recent past. These generalized distributions contain more flexibility as compare to its sub-models and have sometime inferiority as compare to the competitor models. Many authors studied various forms of BXII such Paranaba et al.; [16] (beta BXII), Paranaba et al.; [17] (Kumaraswamy BXII), Silva and Cordeiro; [19] (BXII power series), Gomes et al.; [6] (McDonald BXII), Muhammad; [14] (generalized BXII-Poisson), Korkmaz et al. [9](Burr XII-Geometric), Ramos et al. [18] (Burr XII negative binomial), Mdlongwa et al.; [13] (BXII modified Weibull), Nasir et al.; [15] (BXII-Uniform), Guerra et al.; [7] (gamma BXII), Cordeiro et al.; [4] (BXII system of densities) and Bhatti et al.;[2] (Modified Burr XII Power).

The study of the NFKBXII distribution focuses on the following motivations: (i) to derive the NFKBXII model; (ii) to generate distributions with symmetrical, right-skewed, left-skewed, J, reverse-J, U, exponential shaped as well as high kurtosis; (iii) to have monotone and non-monotone failure rate function; (iv) to study numerically descriptive measures for the NFKBXII distribution based on parameter values; (v) to derive mathematical properties such as random number generator, ordinary moments, conditional moments, density functions of record values, reliability measures and characterizations; (vi) to estimate the precision of the maximum likelihood estimators via a simulation study; (vii) to reveal the potentiality and utility of the NFKBXII model; (viii) to work as the preeminent substitute model to other existing models and to model the real data; (ix) to deliver better fits model than other models and (x) to infer empirically from goodness of fit statistics (GOFs) and graphical tools.

The contents of the article are structured as follows. Section 2 derives the NFKBXII model. We study basic structural properties, random number generator and sub-models for the NFKBXII model. Section 3 presents certain mathematical properties such as ordinary moments, conditional moments, density functions of record values and reliability measures. Section 4 characterizes the NFKBXII distribution. Section 5, we address the maximum likelihood estimation for the NFKBXII parameters. In Section 6, we evaluate the precision of the maximum likelihood estimators via a simulation study. In Section 7, we consider an application to elucidate the potentiality and utility of the NFKBXII model. In Section 8, we conclude the article.

2. THE NFKBXII DISTRIBUTION

In this section we derive the NFKBXII distribution from the T-X family technique. We also derive the NFKBXII model from a link concerning the exponential and gamma variables. We derive basic structural properties. We highlight the nature of density and failure rate functions of the proposed model.

2.1. **T-X Family Technique.**

The generalized uniform (GU) distribution has the following cumulative distribution function (cdf)

$$G(x;a,b,\kappa)=\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-a^{\kappa}},\qquad a\leq x<b, a>0, b>0, \kappa>0. \tag{2. 1}$$

The odds ratio $W\left(G(x,\xi)\right)=\frac{G(x;a,b,\kappa)}{\bar{G}(x;a,b,\kappa)}$ for the GU random variable (rv) X is

$$W\left(G(x,\xi)\right)=\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}. \tag{2. 2}$$

To derive the wider families, the cdf for the T-X family (Alzaatreh et al. [1]) of distributions is

$$F\left(x\right)=\int_{a}^{W[G(x;\xi)]} r(t)dt, -\infty<x<\infty, \tag{2. 3}$$

where $r\left(t\right)$ is the probability density function (pdf) of rv T, where $T\in[a,b]$ for $-\infty\leq a<b<\infty$ and $W\left[G\left(x;\xi\right)\right]$ is a function of the baseline cdf of a rv X, subject to the vector parameter ξ and justifies the situations

- $W\left[G\left(x;\xi\right)\right]\in[a,\mathrm{b}]$
- $W\left[G\left(x;\xi\right)\right]$ is differentiable and monotonically non-decreasing and
- $\lim\limits_{x\to-\infty}W\left[G\left(x;\xi\right)\right]\to a$ and $\lim\limits_{x\to\infty}\left[G\left(x;\xi\right)\right]\to b.$

For the T-X family of distributions, the pdf of X is

$$f\left(x\right)=\left\{\frac{\partial}{\partial x}W\left[G\left(x;\xi\right)\right]\right\}r\left\{W\left[G\left(x;\xi\right)\right]\right\}, x\in[-\infty,\infty]. \tag{2. 4}$$

We derive the cdf of the NFKBXII distribution from the T-X family technique by setting

$$r\left(t\right)=\alpha\beta t^{\beta-1}\left(1+\gamma t^{\beta}\right)^{-\frac{\alpha}{\gamma}-1}, t>0,\alpha>0,\beta>0,\gamma>0$$

and

$$W\left[G\left(x;\xi\right)\right]=\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}$$

as

$$F\left(x\right)=\int_{0}^{\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)}\alpha\beta t^{\beta-1}\left(1+\gamma t^{\beta}\right)^{-\frac{\alpha}{\gamma}-1}dt,$$

$$F\left(x\right)=1-\left[1+\gamma\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}}, x\geq a, \tag{2. 5}$$

where $\alpha>0$, $\beta>0$, $\gamma>0, \kappa>0, a\geq 0, b>0$, $a<b$ are parameters. The pdf corresponding to (2. 5) is given by

$$f\left(x\right)=\alpha\beta\kappa\left(b^{\kappa}-a^{\kappa}\right)x^{\kappa-1}\frac{\left(x^{\kappa}-a^{\kappa}\right)^{\beta-1}}{\left(b^{\kappa}-x^{\kappa}\right)^{\beta+1}}\left[1+\gamma\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}, x>a. \tag{2. 6}$$

Hereafter, the rv with pdf (2. 6) is indicated as X $\sim$ NFKBXII $\left(\alpha,\beta,\gamma,\kappa,a,b\right)$. The NFKBXII model is also well-known as modified Burr XII generalized uniform distribution. The parameters a and b are extreme values of rv.

2.2. **Nexus between Gamma and Exponential Variables.** We derive the NFKBXII distribution from the nexus concerning exponential and gamma variables.

2.2.1: Let W_1 and W_2 be independent continues random variable. If $W_1 \sim exp(1)$ and $W_2 \sim \text{gamma}\,(\alpha/\gamma, 1)$, then for $W_1 = \gamma\left(\frac{X^\kappa - a^\kappa}{b^\kappa - X^\kappa}\right)^\beta W_2$ we reach at

$$X = \left[\left(a^\kappa + b^\kappa\left[\frac{W_1}{\gamma W_2}\right]^{\frac{1}{\beta}}\right) / \left(1 + \left[\frac{W_1}{\gamma W_2}\right]^{\frac{1}{\beta}}\right)\right]^{\frac{1}{\kappa}} \sim \text{NFKBXII}\,(a, b, \alpha, \beta, \gamma, \kappa)\,.$$

proof

If $W_1 \sim \exp(1)$ i.e $f(w_1) = e^{-w_1}$, $w_1 > 0$ and $W_2 \sim \text{gamma}\,(\alpha/\gamma, 1)$ i.e $f(w_2) = \frac{w_2{}^{\alpha/\gamma-1} e^{-w_2}}{\Gamma(\alpha/\gamma)}$, $w_2 > 0$.

The joint density for W_1 and W_2 is $f(w_1, w_2) = \frac{w_2{}^{\alpha/\gamma-1} e^{-w_2} e^{-w_1}}{\Gamma(\alpha/\gamma)}$, $w_1 > 0,\ w_2 > 0$.

Letting $W_1 = \gamma\left(\frac{X^\kappa - a^\kappa}{b^\kappa - X^\kappa}\right)^\beta W_2$, the joint density of X and W_2 is

$$f(x, w_2) =$$

$$\gamma\beta\kappa\,(b^\kappa - a^\kappa)\,x^{\kappa-1}\frac{(x^\kappa - a^\kappa)^{\beta-1}}{(b^\kappa - x^\kappa)^{\beta+1}} w_2 \frac{w_2{}^{\alpha/\gamma-1} e^{-w_2} e^{-\gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta w_2}}{\Gamma(\alpha/\gamma)},\ \ x > 0,\ w_2 > 0.$$

The NFKBXII of the rv X is

$$f(x) = \gamma\beta\kappa\,(b^\kappa - a^\kappa)\,x^{\kappa-1}\frac{(x^\kappa - a^\kappa)^{\beta-1}}{(b^\kappa - x^\kappa)^{\beta+1}}\frac{1}{\Gamma(\alpha/\gamma)}\int_0^\infty w_2{}^{\alpha/\gamma} e^{-\left[1+\gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta\right]w_2}\,dw_2\,.$$

After simplifying, we attain at

$$f(x) = \alpha\beta\kappa\,(b^\kappa - a^\kappa)\,x^{\kappa-1}\frac{(x^\kappa - a^\kappa)^{\beta-1}}{(b^\kappa - x^\kappa)^{\beta+1}}\left[1 + \gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1},\ \ x > a.$$

2.3. **Basic Structural Properties.** For X$\sim$NFKBXII$(\alpha, \beta, \gamma, \kappa, a, b)$, the survival, failure rate, cumulative failure rate, reverse failure rate, elasticity functions and the Mills ratio are given, respectively, by

$$S(x) = \left[1 + \gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}}, x \geq a, \tag{2. 7}$$

$$h(x) = \frac{(x^\kappa - a^\kappa)^{\beta-1}}{(b^\kappa - x^\kappa)^{\beta+1}}\left\{\frac{\alpha\beta\kappa\,(b^\kappa - a^\kappa)\,x^{\kappa-1}}{1 + \gamma(x^\kappa - a^\kappa)^\beta (b^\kappa - x^\kappa)^{-\beta}}\right\}, x > a, \tag{2. 8}$$

$$H(x) = \frac{\alpha}{\gamma}\ln\left[1 + \gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta\right]\ \ x \geq a, \tag{2. 9}$$

$$\text{r(x)} = \frac{d\ln}{dx}\left\{1 - \left[1 + \gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}}\right\},\ \ x > a, \tag{2. 10}$$

$$e(x) = \frac{d\ln}{d\ln x}\left\{1 - \left[1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}}\right\} \tag{2. 11}$$

and

$$m(x) = \frac{(b^{\kappa} - x^{\kappa})^{\beta+1}}{(x^{\kappa} - a^{\kappa})^{\beta-1}}\left\{\frac{1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}}{\alpha\beta\kappa\,(b^{\kappa} - a^{\kappa})\,x^{\kappa-1}}\right\}. \tag{2. 12}$$

For $X \sim$NFKBXII$(\alpha, \beta, \gamma, \kappa, a, b)$, the quantile function with $0 < q < 1$ is

$$x = \left\{a^{\kappa} + b^{\kappa}\left[\frac{(1-q)^{-\frac{\gamma}{\alpha}} - 1}{\gamma}\right]^{\frac{1}{\beta}}\right\}^{\frac{1}{\kappa}}\left\{1 + \left[\frac{(1-q)^{-\frac{\gamma}{\alpha}} - 1}{\gamma}\right]^{\frac{1}{\beta}}\right\}^{-\frac{1}{\kappa}} \tag{2. 13}$$

and its random number generator with $Z \sim Uniform\,(0, 1)$ is

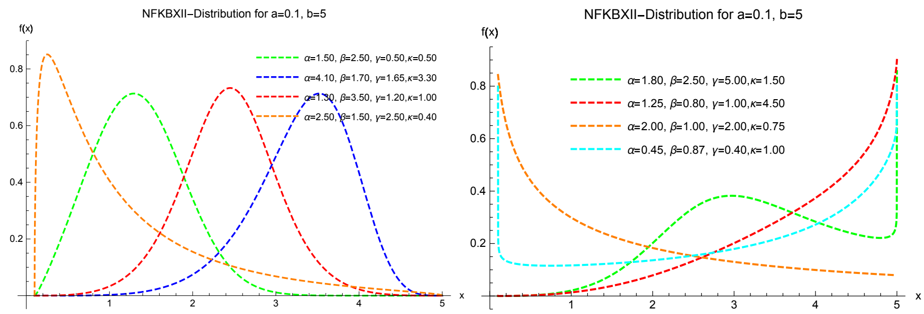
$$X = \left\{a^{\kappa} + b^{\kappa}\left[\frac{(1-Z)^{-\frac{\gamma}{\alpha}} - 1}{\gamma}\right]^{\frac{1}{\beta}}\right\}^{\frac{1}{\kappa}}\left\{1 + \left[\frac{(1-Z)^{-\frac{\gamma}{\alpha}} - 1}{\gamma}\right]^{\frac{1}{\beta}}\right\}^{-\frac{1}{\kappa}}. \tag{2. 14}$$

2.4. **Sub-Models.** The sub-models for the NFKBXII distribution are given in Table 1.

Table 1: Sub-models for the NFKBXII Distribution

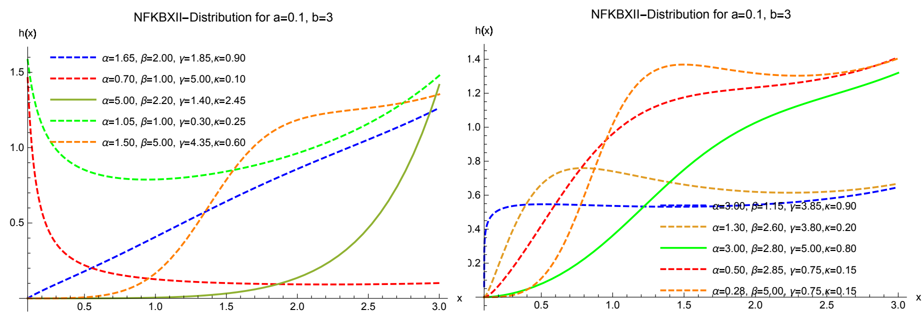
Sr. No.	Parameters						Model
1	a	b	α	β	γ	κ	New Family of Kies Burr XII (NFKBXII)
2	a	b	α	β	γ	1	Kies Modified Burr XII (KMBXII)
3	a	b	α	β	1	κ	New Kies Burr XII (NKBXII)
4	0	1	α	β	γ	κ	Reduced New Kies Burr XII (RNKBXII)
5	a	b	α	1	γ	κ	New Kies Modified Lomax (NKML)
6	a	b	α	1	γ	1	Kies Modified Lomax (KML)
7	a	b	α	1	1	1	Kies Lomax (KL)
8	0	1	α	1	1	1	Reduced Kies Lomax (RKL)
9	0	1	α	β	γ	κ	Reduced Kies Burr XII (RKBXII)
10	0	1	α	β	γ	κ	Reduced New Kies Burr XII (RNKBXII)
11	a	b	α	β	$\gamma \to 0$	κ	Generalized Kies (Gen. K)
12	0	1	α	β	$\gamma \to 0$	κ	Reduced Generalized Kies (RGK)
13	0	1	1	β	$\gamma \to 0$	1	Reduced New Kies (RNK)
14	0	1	α	β	$\gamma \to 0$	1	Reduced Kies (RK)
15	a	b	1	β	$\gamma \to 0$	1	Kies(Kies[8])and Kumar et al.;[11]
16	0	b	α	β	γ	κ	Modified Burr XII Power (MBXII-Power)
17	a	0	α	β	γ	κ	Modified Burr XII Pareto (MBXII-Pareto)
18	0	b	α	β	1	κ	Burr XII Power (MBXII-Power)
19	a	0	α	β	1	κ	Burr XII Pareto (MBXII-Pareto)
20	0	b	1	β	$\gamma \to 0$	κ	Weibull-Power
21	a	0	1	β	$\gamma \to 0$	κ	Weibull-Pareto

2.5. **Plots of the NFKBXII Density and Failure Rate Functions.** We plot the pdf and failure rate function of the NFKBXII distribution for selected values of the parameters. The NFKBXII density can display numerous natures such as symmetrical, right-skewed, U, left-skewed, J, reverse-J and exponential in Figure 1. The failure rate function can highlight shapes as modified bathtub, bathtub, decreasing, increasing and increasing-decreasing-increasing in Figure 2. Therefore, the NFKBXII distribution is quite flexible and can be applied excellently in evaluating numerous data sets.



(a) (b)

Figure 1: PDF Plots



(a) (b)

Figure 2: Hazard Function Plots

3. Mathematical Properties

Here, we present certain mathematical and statistical properties such as the ordinary moments, the Mellin transform, order statistics and their related moments, conditional moments, density functions of record values and reliability measures.

3.1. **Moments of the NFKBXII Distribution.** The moments are significant tools for statistical analysis in pragmatic sciences. The descriptive tools such as μ_1' (mean), σ(standard deviation), γ_1(skewness) and γ_2(kurtosis) can be obtained from the moments. For $X \sim NFKBXII\,(\alpha,\beta,\gamma,\kappa,a,b)$, the rth raw or ordinary moment is

$$\mu_r' = E(X^r) = \int_a^b x^r f(x)dx,$$

$$E(X^r) = \alpha\beta\kappa\left(b^\kappa - a^\kappa\right)\int_a^b x^r x^{\kappa-1}\frac{\left(x^\kappa - a^\kappa\right)^{\beta-1}}{\left(b^\kappa - x^\kappa\right)^{\beta+1}}\left[1+\gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1} dx.$$

letting $w = \gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta$, then we arrive at

$$E(X^r) = \frac{\alpha}{\gamma}\int_0^\infty\left[\frac{\left(a^\kappa - b^\kappa\right)}{\left[1+\left(\gamma^{-1}w\right)^{\frac{1}{\beta}}\right]} + b^\kappa\right]^{\frac{r}{\kappa}}[1+w]^{-\frac{\alpha}{\gamma}-1}dx,$$

$$E(X^r) = \frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left(a^\kappa - b^\kappa\right)^\ell b^{r-\kappa\ell}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu(\ell)_i\gamma^{-\frac{\nu}{\beta}}}{\nu!}\left[\int_0^\infty w^{\frac{\nu}{\beta}}[1+w]^{-\frac{\alpha}{\gamma}-1}dw\right].$$

Observe that

$$\mu_r' = \alpha\sum_{\ell=0}^{\frac{r}{\kappa}}\sum_{\nu=0}^{\infty}(-1)^\nu\binom{\frac{r}{\kappa}}{\ell}(\ell)_\nu\frac{b^{(r-\kappa\ell)}\;\left(a^\kappa - b^\kappa\right)^\ell}{\gamma^{\frac{\nu}{\beta}+1}\;\nu!}\,B\left(\frac{\nu}{\beta}+1,\frac{\alpha}{\gamma}-\frac{\nu}{\beta}\right),\quad r = 1,2,3,..., \tag{3. 15}$$

where $(\ell)_\nu = \frac{\Gamma(\ell+\nu)}{\Gamma(\ell)}$ is the Pochhammar symbol and $B\,[.,.]$ is a beta function. The Mellin transformation is applied to get the moments of a probability distribution.
For $X \sim NFKBXII$ model, the Mellin transform is

$$\begin{aligned}
&\mathrm{M}\left\{f\left(x\right);r\right\} = E\left(X^{r-1}\right)\\
&= \int_a^b x^{r-1}\alpha\beta\kappa\left(b^\kappa - a^\kappa\right)x^{\kappa-1}\frac{\left(x^\kappa - a^\kappa\right)^{\beta-1}}{\left(b^\kappa - x^\kappa\right)^{\beta+1}}\left[1+\gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1}dx,
\end{aligned}$$

$$\mathrm{M}\left\{f\left(x\right);r\right\} = \alpha\sum_{\ell=0}^{\frac{r-1}{\kappa}}\sum_{\nu=0}^{\infty}\binom{\frac{r-1}{\kappa}}{\ell}\frac{(-1)^\nu\left(a^\kappa - b^\kappa\right)^\ell b^{(r-1-\kappa\ell)}(\ell)_\nu}{\nu!\;\gamma^{\frac{\nu}{\beta}+1}}\;B\left(\frac{\nu}{\beta}+1,\frac{\alpha}{\gamma}-\frac{\nu}{\beta}\right). \tag{3. 16}$$

From Mellin transform, the rth ordinary moment is $\mu_r' = \mathrm{M}\left\{f\left(x\right);r+1\right\}$. The rth central moment μ_r, coefficient of skewnessγ_1 and kurtosis γ_2 for the NFKBXII model are attained from

$$\mu_r = \sum_{\ell=1}^{r}(-1)^\ell\binom{r}{\ell}\;\mu'_\ell\,\mu'_{r-\ell},\gamma_1 = \mu_3/\sqrt[3]{\mu_2}\;\;\text{and}\beta_2 = \mu_4\Big/(\mu_2)^2.$$

The numerical values for the mean (μ'_1), median $(\tilde{\mu})$, standard deviation (σ), skewness γ_1 and kurtosis γ_2 for the NFKBXII model for selected values of $\alpha,\beta,\gamma,\kappa$ are listed in Table

2. We also depict that the NFKBXII model can be effective to model data sets in term of the descriptive measures.

Table 2: $\mu_1', \tilde{\mu}, \sigma, \gamma_1$ and $\gamma_2 of the NFKBXII Distribution$

Parameters $\alpha,\beta,\gamma,\kappa, a = 0.1, b = 5$	μ_1'	$\tilde{\mu}$	σ	γ_1	γ_2
0.5,0.5,0.5,0.5	2.8938	3.4288	1.8814	-0.3102	1.4514
1,0.5,0.5,1	2.2868	2.0876	1.7337	0.1774	1.503
1,1,0.5,1	2.3263	2.3167	1.2699	0.5675	1.9293
0.5,1,1,1	3.3646	3.7709	1.4608	-0.6368	2.1404
1,1,1,1	2.5475	2.546	1.4138	0.0019	1.8006
1,1,1.5,1.5	3.1574	3.3585	1.3691	-0.4033	1.9986
1.5, 1.5, 1.5, 1.5	2.8327	2.8676	1.0272	-0.1547	2.3389
1.5,2,2,2	3.366	3.4114	0.7802	-0.3628	2.8385
2,2,2,2	3.1772	3.2173	0.7454	-0.3142	2.8922
2,0.5,0.5,1	1.2978	0.711	1.3393	1.0818	2.9803
3,0.5,0.5,1	0.8365	0.376	0.9977	1.7462	5.4416
5,5,5,1	2.169	2.1583	0.4167	0.2157	3.6534
5,2,0.5,1	1.4433	1.4452	0.5389	0.047	2.5273
5,5,5,1	2.169	2.1583	0.4167	0.2157	3.6534
1,1,5,5	4.4588	4.8515	0.7535	-1.6114	5.0274
1.5,1.5,5,5	4.3504	4.5111	0.6112	-1.0637	3.7569
2,2,5,5	4.2789	4.3371	0.4923	-0.7274	3.3926
2.5,2.5,5,5	4.2357	4.2603	0.3985	-0.5334	3.4162
3,3,5,5	4.2114	4.2253	0.3272	-0.4387	3.5929
3.5,3.5,5,5	4.1991	4.2099	0.2730	-0.4075	3.8003
4,4,5,5	4.1937	4.2035	0.2316	-0.4105	3.9827
4.5,4.5,5,5	4.1930	4.2026	0.1997	-0.4370	4.1191
5,5,5,5	4.1943	4.2037	0.1745	-0.4717	4.2674
6,6,6,1.35	2.6703	2.6704	0.3308	-0.0032	3.7595
6,6,6,1.3	2.6104	2.6096	0.3350	0.0180	3.7560

3.2. **Order Statistics.** Order statistics (OS) have extensive uses in climatology, reliability and life testing. The moments of OS are very useful tools to predict upcoming failure items. For $X \sim NFKBXII\,(\alpha,\beta,\gamma,\kappa,a,b)$; the pdf of $X_{i:n}$ is

$$f\left(x_{i:n}\right) = \frac{\alpha\beta\kappa\left(b^{\kappa}-a^{\kappa}\right)}{B\left(i,\,n-i+1\right)}\sum_{j=0}^{i-1}\left(-1\right)^{j}\binom{i-1}{j}x^{\kappa-1}\frac{\left(x^{\kappa}-a^{\kappa}\right)^{\beta-1}}{\left(b^{\kappa}-x^{\kappa}\right)^{\beta+1}}\times \left[1+\gamma\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}(n-i+j+1)-1}.$$

For $X \sim NFKBXII\left(\alpha,\beta,\gamma,\kappa,a,b\right)$, ordinary moments of $X_{i:n}$ are
$E\left(X^r{}_{i:n}\right) = \int\limits_a^b x^r f\left(x_{i:n}\right)dx,$

$$E\left(X^r{}_{i:n}\right) = \frac{\alpha\beta\kappa\left(b^\kappa - a^\kappa\right)}{B\left(i,\,n-i+1\right)}\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\int\limits_a^b x^r x^{\kappa-1}\frac{\left(x^\kappa - a^\kappa\right)^{\beta-1}}{\left(b^\kappa - x^\kappa\right)^{\beta+1}}\times \left[1+\gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}(n-i+j+1)-1} dx,$$

$$E\left(X^r{}_{i:n}\right) = \alpha\sum_{j=0}^{i-1}\sum_{\ell=0}^{\frac{r}{\kappa}}\sum_{\nu=0}^{\infty}(-1)^{j+\nu}\binom{i-1}{j}\binom{\frac{r}{\kappa}}{\ell}\frac{(\ell)_\nu}{\nu!\;\gamma^{1+\frac{\nu}{\beta}}}\frac{\left(b^\kappa - a^\kappa\right)^\ell}{b^{\kappa\ell-r}}\times \frac{B\left[\frac{\nu}{\beta}+1,\frac{\alpha}{\gamma}\left(n-i+j+1\right)-\frac{\nu}{\beta}\right]}{B\left(i,\,n-i+1\right)},\quad r=1,2,3...$$

3.3. **Conditional Moments.** Life expectancy, mean waiting time and inequality measures can be obtained from incomplete moments. The sth conditional moment $E\left(\left.X^s\right|X>z\right)$ for $X \sim NFKBXII\left(\alpha,\beta,\gamma,\kappa,a,b\right)$ is

$$E\left(\left.X^r\right|X>z\right) = \frac{1}{S\left(z\right)}\left[E\left(X^s\right) - E_{X\le z}\left(X^s\right)\right].$$

The sth lower incomplete $E_{X\le z}\left(X^s\right)$ for $X \sim NFKBXII\left(\alpha,\beta,\gamma,\kappa,a,b\right)$ is

$$E_{X\le z}\left(X^s\right) = \alpha\beta\kappa\left(b^\kappa - a^\kappa\right)\int\limits_a^z x^s x^{\kappa-1}\frac{\left(x^\kappa - a^\kappa\right)^{\beta-1}}{\left(b^\kappa - x^\kappa\right)^{\beta+1}}\left[1+\gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1} dx.$$

letting $\gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta = w$, we obtain

$$E_{X\le z}\left(X^s\right) = \alpha\sum_{\ell=0}^{\frac{s}{\kappa}}\sum_{\nu=0}^{\infty}\binom{\frac{s}{\kappa}}{\ell}\frac{(-1)^\nu(\ell)_\nu}{\nu!\;\gamma^{\frac{\nu}{\beta}+1}}\left(a^\kappa - b^\kappa\right)^\ell b^{(s-\kappa\ell)}B_w\left(\frac{\nu}{\beta}+1,\frac{\alpha}{\gamma}-\frac{\nu}{\beta}\right). \tag{3. 17}$$

s = 1,2,3,..
The sth conditional moment is

$$E\left(\left.X^s\right|X>z\right) = \frac{1}{S(z)}\alpha\sum_{\ell=0}^{\frac{s}{\kappa}}\sum_{\nu=0}^{\infty}\binom{\frac{s}{\kappa}}{\ell}\frac{(-1)^\nu(\ell)_\nu}{\nu!\;\gamma^{1+\frac{\nu}{\beta}}}\left(a^\kappa - b^\kappa\right)^\ell b^{(s-\kappa\ell)}\times \left[B\left(\frac{\nu}{\beta}+1,\frac{\alpha}{\gamma}-\frac{\nu}{\beta}\right) - B_w\left(\frac{\nu}{\beta}+1,\frac{\alpha}{\gamma}-\frac{\nu}{\beta}\right)\right]$$

where $B_w\left(.,.\right)$ is the incomplete beta function.
The sth reversed conditional moment $E\left(\left.X^r\right|X\le z\right)$ for $X \sim NFKBXII\left(\alpha,\beta,\gamma,\kappa,a,b\right)$

is

$$E\left(X^s \middle| X \leq z\right) = \frac{1}{F(z)} \sum_{\ell=0}^{\frac{s}{\kappa}} \sum_{\nu=0}^{\infty} \binom{\frac{s}{\kappa}}{\ell} \frac{(-1)^{\nu} (\ell)_{\nu}}{\nu!\ \gamma^{1+\frac{\nu}{\beta}}} (a^{\kappa} - b^{\kappa})^{\ell} b^{(s-\kappa\ell)} B_w \left(\frac{\nu}{\beta} + 1, \frac{\alpha}{\gamma} - \frac{\nu}{\beta}\right). \tag{3. 18}$$

3.4. **Record Values Distributions.** Record values have wide applications in the life testing, industry, hydrology and economics. Here we study the record values distributions using the NFKBXII distribution as baseline.

Definition of Record Value 3.4.1: Based on a sequence $X_i, \ldots,\ i = 1, 2...,$ of i.i.d random variables with a cdf F and record times $U(1) = 1$ and $U(n+1) = \min\left\{j > U(n); X_j > X_{U(n)}\right\},\ n \in N$, the random variables $X_{U(n)}$, $n \in N$ are called upper record values. The pdf for the ith upper record value $R_i = X_{U(i)}$, with $R_1 = X_1$, for the NFKBXII model is

$$f_{R_i}(x) = \frac{f(x)}{\Gamma(i)} \{-\log[S(x)]\}^{i-1}$$

$$f_{R_i}(x) = \tfrac{\alpha\beta\kappa(b^{\kappa}-a^{\kappa})\ x^{\kappa-1}}{\Gamma(i)} \tfrac{(x^{\kappa}-a^{\kappa})^{\beta-1}}{(b^{\kappa}-x^{\kappa})^{\beta+1}} \left\{\log\left[1+\gamma\left(\tfrac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right]^{\frac{\alpha}{\gamma}}\right\}^{i-1} \times \left[1+\gamma\left(\tfrac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}, x > a.$$

for $X \sim NFKBXII(\alpha, \beta, \gamma, \kappa, a, b)$, the joint pdf of $(R_1, ...R_n)$ is

$$f_{(R_1,...,R_n)}(x_1, ..., x_n) = f(x_n) \prod_{\ell=1}^{n-1} h(x_\ell)$$

$$f_{(R_1,...,R_n)}(x_1, ..., x_n) = \alpha\beta\kappa(b^{\kappa} - a^{\kappa})\, {x_n}^{\kappa-1} \tfrac{({x_n}^{\kappa}-a^{\kappa})^{\beta-1}}{(b^{\kappa}-{x_n}^{\kappa})^{\beta+1}} \left[1+\gamma\left(\tfrac{{x_n}^{\kappa}-a^{\kappa}}{b^{\kappa}-{x_n}^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1} \times \prod_{\ell=1}^{n-1} \alpha\beta\kappa(b^{\kappa} - a^{\kappa})\, {x_\ell}^{\kappa-1} \tfrac{({x_\ell}^{\kappa}-a^{\kappa})^{\beta-1}}{(b^{\kappa}-{x_\ell}^{\kappa})^{\beta+1}} \left[1+\gamma\left(\tfrac{{x_\ell}^{\kappa}-a^{\kappa}}{b^{\kappa}-{x_\ell}^{\kappa}}\right)^{\beta}\right]^{-1}.$$

3.5. **Stress-Strength Reliability.** Let X_1 be strength and X_2 be stress and X_1 follows the NFKBXII distribution $(\alpha_1, \beta, \gamma, \kappa, a, b)$ and X_2 follows the NFKBXII distribution $(\alpha_2, \beta, \gamma, \kappa, a, b)$, then the reliability of a component is

$$R = Pr(X_2 < X_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} f(x_1, x_2) dx_2 dx_1 = \int_{0}^{\infty} f_{x_1}(x) F_{x_2}(x)\, dx.$$

(Kotz et al. [10]),

$$R = \int_a^b \alpha_1\beta\kappa(b^{\kappa} - a^{\kappa})\, x^{\kappa-1} \tfrac{(x^{\kappa}-a^{\kappa})^{\beta-1}}{(b^{\kappa}-x^{\kappa})^{\beta+1}} \left[1+\gamma\left(\tfrac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha_1}{\gamma}-1} \times \left[1-\left[1+\gamma\left(\tfrac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha_2}{\gamma}}\right] dx,$$

$$R = 1 - \int_a^b \alpha_1 \beta\kappa \left(b^\kappa - a^\kappa\right) x^{\kappa-1} \frac{(x^\kappa - a^\kappa)^{\beta-1}}{(b^\kappa - x^\kappa)^{\beta+1}} \left[1 + \gamma\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta\right]^{-\frac{\alpha_1}{\gamma} - \frac{\alpha_2}{\gamma} - 1} dx.$$

$$R = \frac{\alpha_2}{(\alpha_1 + \alpha_2)}$$

Therefore the stress-strength reliability parameter R is independent of parameters β, γ, κ, a, b.

3.6. **Reliability Estimation of Multicomponent Stress-Strength model.** Consider a machine with n identical elements, out of which s elements are operative. Let $X_i, i = 1, 2...n$ represents strengths of n elements with cdf F while, the stress Y enforced on the elements have cdf G. The strengths X_i and stress Y are independently distributed. The probability that machine maneuvers well is reliability of the system i.e.

$R_{s,n} = P[strengths\,(X_i, i = 1, 2...n) > stress\,(Y)]$(Bhattacharyya and Johnson, [3]),

$$R_{s,n} = P[\text{at the minimum}\, s\, of\ (X_i, i = 1, 2...n)\, exceed\, Y],$$

$$R_{s,n} = \sum_{\ell=s}^{n} \binom{n}{\ell} \int_{-\infty}^{\infty} [1 - F(y)]^\ell [F(y)]^{n-\ell} dG(y). \quad (3.\ 19)$$

Let X_i follows the NFKBXII distribution $(\alpha_1, \beta, \gamma, \kappa, a, b)$ and Y follows the NFKBXII distribution $(\alpha_2, \beta, \gamma, \kappa, a, b)$ with unknown α_1, α_2 and common β, γ and κ. Then, the probability that machine maneuvers correctly in multicomponent stress- strength model is

$$R_{s,n} = \sum_{\ell=s}^{n} \binom{n}{\ell} \int_a^b \left(\begin{array}{c} \left\{\left[1 + \gamma\left(\frac{y^\kappa - a^\kappa}{b^\kappa - y^\kappa}\right)^\beta\right]^{-\frac{\alpha_1}{\gamma}}\right\}^\ell \left\{1 - \left[1 + \gamma\left(\frac{y^\kappa - a^\kappa}{b^\kappa - y^\kappa}\right)^\beta\right]^{-\frac{\alpha_1}{\gamma}}\right\}^{(n-\ell)} \times \\ \alpha_2 \beta\kappa \left(b^\kappa - a^\kappa\right) y^{\kappa-1} \frac{(y^\kappa - a^\kappa)^{\beta-1}}{(b^\kappa - y^\kappa)^{\beta+1}} \left[1 + \gamma\left(\frac{y^\kappa - a^\kappa}{b^\kappa - y^\kappa}\right)^\beta\right]^{-\frac{\alpha_2}{\gamma} - 1} dy \end{array} \right).$$

Letting $u = \left[1 + \gamma\left(\frac{y^\kappa - a^\kappa}{b^\kappa - y^\kappa}\right)^\beta\right]^{-\frac{\alpha_2}{\gamma}}$, we attain

$$R_{s,n} = \sum_{\ell=s}^{n} \binom{n}{\ell} \int_0^1 \left(u^\vartheta\right)^\ell \left(1 - u^\vartheta\right)^{n-\ell} du.$$

Let $w = u^\vartheta$ then

$$R_{s,n} = \sum_{\ell=s}^{n} \binom{n}{\ell} \int_0^1 (w)^\ell (1 - w)^{(n-\ell)} \frac{1}{\vartheta} w^{\frac{1}{\vartheta} - 1} dw,\ where\ \vartheta = \frac{\alpha_1}{\alpha_2},$$

$$R_{s,n} = \frac{1}{\vartheta} \sum_{\ell=s}^{n} \binom{n}{\ell} B\left(\ell + \frac{1}{\vartheta}, n - \ell + 1\right). \quad (3.\ 20)$$

The probability in (3. 20) is known as reliability of multicomponent stress-strength model.

4. CHARACTERIZATIONS

We characterize the NFKBXII distribution through: (i) Truncated moment and (ii) Mills ratio.

4.1. **Truncated Moment.** We characterize the NFKBXII distribution via truncated moment (Glänzel,[5]).
Proposition 4.1.1 Let $X : U \rightarrow (a, b)$ be a continuous rv and let

$$\xi(x) = \left[1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}\right]^{-1}, \; x > a.$$

The pdf of X is (2.6) iff the function $\omega(x)$ (clear in theorem G) is

$$\omega(x) = \frac{\alpha}{\alpha + \gamma}\left[1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}\right]^{-1}, \quad x \geq a,$$

Proof Let $X \sim NFKBXII(\alpha, \beta, \gamma, \kappa, a, b)$, then

$$[1 - F(x)]\, E[\xi(X)|\, X \geq x] = \frac{\alpha}{\alpha + \gamma}\left[1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma} - 1} \quad x > a,$$

or

$$E[\xi(X)|\, X \geq x] = \frac{\alpha}{\alpha + \gamma}\left[1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}\right]^{-1}, \; x > a.$$

$$\omega(x) = \frac{\alpha}{\alpha + \gamma}\left[1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}\right]^{-1}, \; x \geq a$$

$$\omega'(x) = -\beta\kappa\left(b^{\kappa} - a^{\kappa}\right) x^{\kappa - 1} \frac{\left(x^{\kappa} - a^{\kappa}\right)^{\beta - 1}}{\left(b^{\kappa} - x^{\kappa}\right)^{\beta + 1}} \frac{\alpha\gamma}{\alpha + \gamma}\left[1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}\right]^{-2} < 0, \quad x \geq a$$

and

$$\omega(x) - \xi(x) = -\frac{\gamma}{\alpha + \gamma}\left[1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}\right]^{-1} < 0, \quad x \geq a.$$

Conversely if $\omega(x)$ is given as above, then

$$s'(x) = \frac{\omega'(x)}{\omega(x) - \xi(x)} = \frac{\alpha\beta\kappa\left(b^{\kappa} - a^{\kappa}\right) x^{\kappa - 1} \frac{\left(x^{\kappa} - a^{\kappa}\right)^{\beta - 1}}{\left(b^{\kappa} - x^{\kappa}\right)^{\beta + 1}}}{\left[1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}\right]} \quad x > a,$$

and hence

$$s(x) = \ln\left\{\left[1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}\right]^{\frac{\alpha}{\gamma}}\right\} \quad x > a,$$

and

$$e^{-s(x)} = \left[1 + \gamma\left(\frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}} \quad x > a.$$

Now, according to Theorem G (Glänzel, [5]), $X \sim NFKBXII\left(\alpha,\beta,\gamma,\kappa,a,b\right)$.
Corollary 4.1.1: Let $X : U \to (a,b)$ be a continuous rv. The pdf of X is (2. 6) iff there exist functions $\omega\left(x\right)$ and $\xi\left(x\right)$ (clear in Theorem G) justifying the differential equation

$$\frac{\omega'\left(x\right)}{\omega\left(x\right)-\xi\left(x\right)}=\frac{\alpha\beta\kappa\left(b^{\kappa}-a^{\kappa}\right)x^{\kappa-1}\frac{\left(x^{\kappa}-a^{\kappa}\right)^{\beta-1}}{\left(b^{\kappa}-x^{\kappa}\right)^{\beta+1}}}{\left[1+\gamma\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right]},x>a. \tag{4. 21}$$

Remarks 4.1.1: The general solution of (4. 21) is

$$\omega\left(x\right)=\left[1+\gamma\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right]^{\frac{\alpha}{\gamma}}\left\{-\int\left[\frac{\alpha\beta\kappa\left(b^{\kappa}-a^{\kappa}\right)x^{\kappa-1}\frac{\left(x^{\kappa}-a^{\kappa}\right)^{\beta-1}}{\left(b^{\kappa}-x^{\kappa}\right)^{\beta+1}}}{\left[1+\gamma\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right]^{\left(\frac{\alpha}{\gamma}+1\right)}}\xi\left(x\right)\right]dx+D\right\}$$

where D is constant.

4.2. **Mills Ratio.** We characterize the NFKBXII distribution via Mills ratio.
Definition 4.2.1: Let X :$U \to (a,b)$ be a continuous rv with cdf $F(x)$ and pdf $f(x)$. The Mills ratio, $m\left(x\right)$ of a twice differentiable distribution function F, fulfills the differential equation

$$\frac{d}{dx}\left[\ln f\left(x\right)\right]+\frac{1}{m\left(x\right)}+\frac{m'\left(x\right)}{m\left(x\right)}=0. \tag{4. 22}$$

Definition 4.2.2: Let X :$U \to (a,b)$ be a continuous rv. The $X \sim NFKBXII\left(\alpha,\beta,\gamma,\kappa,a,b\right)$, iff the Mills ratio, $m\left(x\right)$ justifies the first order differential equation

$$m'\left(x\right)+\left[\left(\kappa-1\right)+\frac{\kappa\left(\beta-1\right)x^{\kappa}}{\left(x^{\kappa}-a^{\kappa}\right)}+\frac{\kappa\left(\beta+1\right)x^{\kappa}}{\left(b^{\kappa}-x^{\kappa}\right)}\right]m\left(x\right)=\frac{\gamma}{\alpha}\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta-1}. \tag{4. 23}$$

Proof If $X \sim NFKBXII\left(\alpha,\beta,\gamma,\kappa,a,b\right)$, iff the Mills ratio, $m\left(x\right)$, then (4.23) holds. Now if (4. 23) holds, then

$$\frac{d}{dx}\left[\left(\alpha\beta\kappa\left(b^{\kappa}-a^{\kappa}\right)x^{\kappa-1}\frac{\left(x^{\kappa}-a^{\kappa}\right)^{\beta-1}}{\left(b^{\kappa}-x^{\kappa}\right)^{\beta+1}}\right)m\left(x\right)\right]=\frac{d}{dx}\left[1+\gamma\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right]$$

or

$$m\left(x\right)=\left(\alpha\beta\kappa\left(b^{\kappa}-a^{\kappa}\right)x^{\kappa-1}\frac{\left(x^{\kappa}-a^{\kappa}\right)^{\beta-1}}{\left(b^{\kappa}-x^{\kappa}\right)^{\beta+1}}\right)^{-1}\left[1+\gamma\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta}\right],\ x\geq a,$$

that is the Mills ratio of $NFKBXII\left(\alpha,\beta,\gamma,\kappa,a,b\right)$.

5. ESTIMATION

Here, we adopt maximum likelihood estimation technique for the NFKBXII parameters. Let $\xi=\left(\alpha,\beta,\gamma,\kappa\right)^{T}$ be unknown parameter vector. The log likelihood function $\ell\left(\xi\right)$ for the NFKBXII distribution is

$$\ell = \ln L(x_i, \xi) = n\ln\alpha + n\ln\beta + n\ln\kappa + n\ln(b^\kappa - a^\kappa) + (\kappa-1)\sum_{i=1}^{n}\ln x_i - (\beta-1)\sum_{i=1}^{n}\ln({x_i}^\kappa - a^\kappa) - (\beta+1)\sum_{i=1}^{n}\ln(b^\kappa - {x_i}^\kappa) - \left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\ln\left[1+\gamma\left(\frac{{x_i}^\kappa - a^\kappa}{b^\kappa - {x_i}^\kappa}\right)^\beta\right], \tag{5. 24}$$

where the location parameters a and b are supposed to be well-known, as its smallest and supreme likelihood are equal to smallest and supreme OS. We can obtain the maximum likelihood estimators (MLEs) of α, β, γ and κ by solving (5.25)-(5.28) either directly or using the quasi-Newton procedure, computer packages/ software such as R, SAS, Ox, MATHEMATICA, MATLAB and MAPLE.

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - \frac{1}{\gamma}\sum_{i=1}^{n}\ln\left[1+\gamma\left(\frac{{x_i}^\kappa - a^\kappa}{b^\kappa - {x_i}^\kappa}\right)\right] = 0, \tag{5. 25}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\ln({x_i}^\kappa - a^\kappa) - \sum_{i=1}^{n}\ln(b^\kappa - {x_i}^\kappa) - (\alpha+\gamma)\sum_{i=1}^{n}\frac{\left(\frac{{x_i}^\kappa - a^\kappa}{b^\kappa - {x_i}^\kappa}\right)^\beta \ln\left(\frac{{x_i}^\kappa - a^\kappa}{b^\kappa - {x_i}^\kappa}\right)}{\left[1+\gamma\left(\frac{{x_i}^\kappa - a^\kappa}{b^\kappa - {x_i}^\kappa}\right)^\beta\right]} = 0, \tag{5. 26}$$

$$\frac{\partial\ell}{\partial\gamma} = \alpha\gamma^{-2}\sum_{i=1}^{n}\ln\left[1+\gamma\left(\frac{{x_i}^\kappa - a^\kappa}{b^\kappa - {x_i}^\kappa}\right)^\beta\right] - \left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\left[\left(\frac{{x_i}^\kappa - a^\kappa}{b^\kappa - {x_i}^\kappa}\right)^{-\beta} + \gamma\right] = 0, \tag{5. 27}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\kappa} &= \frac{n}{\kappa} + \frac{n(b^\kappa\ln b - a^\kappa\ln a)}{(b^\kappa - a^\kappa)} + \sum_{i=1}^{n}\ln x_i - (\beta-1)\sum_{i=1}^{n}\frac{({x_i}^\kappa\ln x_i - a^\kappa\ln a)}{({x_i}^\kappa - a^\kappa)} \\ &- (\beta+1)\sum_{i=1}^{n}\frac{(b^\kappa\ln b - {x_i}^\kappa\ln x_i)}{(b^\kappa - {x_i}^\kappa)} \\ &- (\alpha+\gamma)\beta\sum_{i=1}^{n}\frac{\left(\frac{{x_i}^\kappa - a^\kappa}{b^\kappa - {x_i}^\kappa}\right)^\beta\left[\frac{({x_i}^\kappa\ln x_i - a^\kappa\ln a)}{({x_i}^\kappa - a^\kappa)} + \frac{(b^\kappa\ln b - {x_i}^\kappa\ln x_i)}{(b^\kappa - {x_i}^\kappa)}\right]}{\left[1+\gamma\left(\frac{{x_i}^\kappa - a^\kappa}{b^\kappa - {x_i}^\kappa}\right)^\beta\right]} = 0.\end{aligned} \tag{5. 28}$$

6. MONTE CARLO SIMULATION STUDIES

We evaluate the precision of the MLEs of the proposed distribution via a simulation study regarding the sample size n. We generate N=10000 samples of sizes n=50,100,150 from the inverse cdf of the NFKBXII distribution with true parameter values $(\alpha, \beta, \gamma, \kappa, a, b)$ =(0.5,0.8,1.2,1.5, 0.10,4), (0.8,1.0,1.5,2.0, 0.5, 5) and (1.0,1.25,1.75, 2.25,0.75, 5.5). We estimate the MLEs $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\kappa}, \hat{a}, \hat{b})$ for 10000 samples from the non-linear optimization techniques. We also work out the means, biases and mean squared errors (MSE) of the MLEs. From the simulation results given in Table 3, we infer that as sample size (n) increases, the MSEs reduce and biases fall to zero. We observe that as the shape parameter increases, MSEs of estimated parameters increase. Finally, we infer that MLEs for the NFKBXII distribution are consistent.

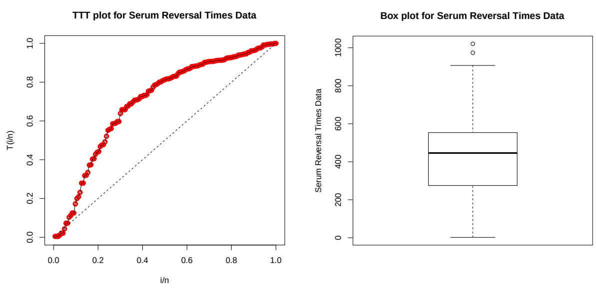
Table 3: Means, Bias and MSEs of NFKBXII Distribution

n	Statistics	$\alpha = 0.5$	$\beta = 0.8$	$\gamma = 1.2$	$\kappa = 1.5$	a=0.10	b=4
50	Means	0.5246	0.8081	1.2818	1.5747	0.162	3.9999
	Bias	0.0246	0.0081	0.0818	0.0747	0.062	-0.0001
	MSE	0.0036	0.005	0.0219	0.028	0.0302	0.
100	Means	0.5192	0.8045	1.2609	1.5849	0.1104	4.
	Bias	0.0192	0.0045	0.0609	0.0849	0.0104	0.
	MSE	0.0017	0.0024	0.0122	0.0238	0.0047	0.
150	Means	0.5174	0.8018	1.2526	1.5758	0.1012	4.
	Bias	0.0174	0.0018	0.0526	0.0758	0.0012	0.
	MSE	0.0012	0.0015	0.0088	0.0195	0.0005	0.
		$\alpha = 0.8$	$\beta = 1$	$\gamma = 1.5$	$\kappa = 2$	a=0.5	b=5
50.	Means	0.8192	1.0004	1.5721	2.0056	0.8865	4.997
	Bias	0.0192	0.0004	0.0721	0.0056	0.3865	-0.003
	MSE	0.0072	0.007	0.0306	0.0424	0.1954	0.0001
100.	Means	0.8109	0.9992	1.5239	2.0145	0.9281	4.9995
	Bias	0.0109	-0.0008	0.0239	0.0145	0.4281	-0.0005
	MSE	0.0019	0.0016	0.0083	0.0138	0.2013	0.
150.	Means	0.807	0.9998	1.5094	2.0077	0.9652	4.9999
	Bias	0.007	-0.0002	0.0094	0.0077	0.4652	-0.0001
	MSE	0.0007	0.0003	0.0027	0.0041	0.2238	0.
		$\alpha = 1$	$\beta = 1.25$	$\gamma = 1.75$	$\kappa = 2.25$	a=0.75	b=5.5
50.	Means	1.0271	1.2361	1.9304	2.2553	1.0474	5.4816
	Bias	0.0271	-0.0139	0.1804	0.0053	0.2974	-0.0184
	MSE	0.0133	0.0191	0.1004	0.0802	0.2106	0.0011
100.	Means	1.0303	1.2427	1.8936	2.2697	0.9237	5.4927
	Bias	0.0303	-0.0073	0.1436	0.0197	0.1737	-0.0073
	MSE	0.0066	0.0058	0.0595	0.036	0.1098	0.0002
150.	Means	1.0289	1.2495	1.877	2.2782	0.8503	5.4959
	Bias	0.0289	-0.0005	0.127	0.0282	0.1003	-0.0041
	MSE	0.0045	0.0031	0.0437	0.0238	0.0601	0.0001

7. Applications

We consider an application to serum-reversal time (in days) of children for authentication of the flexibility, utility and potentiality of the NFKBXII model. We compare the NFKBXII distribution with models such as NKBXII, KMBXII, KBXII, NKL, KL, MBXII, MBIII, Weibull and inverse Weibull. For selection of the optimum distribution, we compute the estimate of GOFs such as Cramer-von Mises (W*), Anderson Darling (A*) and Kolmogorov- Smirnov (K-S) statistic with p-values and various model selection criteria such as the estimate of likelihood ratio statistic ($-2\widehat{\ell}$), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC) and Bayesian information criterion (BIC) for all competing and sub distributions. We compute the MLEs for the parameters and their standard errors (in parentheses). We also compute model selection criteria ($-2\widehat{\ell}$, AIC, CAIC, BIC, HQIC) and GOFs (W*, A*, K-S) values for the NKBXII, KMBXII, KBXII, NKL, KL, MBXII, MBIII, Weibull and inverse Weibull models.

7.1. **Serum Reversal Time Data:** (Lee, [12]): The serum-reversal time (in days) of 143 children born from HIV-infected mothers are 2, 2, 2, 5,9, 9, 19, 32, 32, 46, 50, 56, 56, 78, 91, 95, 106, 129, 129, 148, 149, 156, 175, 176, 191, 192, 204, 209, 211, 225, 229, 230, 238, 254, 271, 274, 276, 290, 291, 292, 297, 297, 322, 334, 334, 334, 344, 346, 353, 353, 359, 365, 366, 367, 370, 378, 378, 382, 382, 385, 398, 400, 402, 414, 422, 424, 428, 434, 435, 440, 443, 446, 448, 448, 451, 454, 459, 460, 461, 473, 480, 481, 484, 487, 493, 497, 498, 502, 511, 511, 513, 514, 516, 521, 524, 526, 537, 538, 541, 543, 544, 544, 545, 549, 551, 553, 553, 554, 556, 559, 571, 576, 577, 578, 582, 588, 590, 596, 609, 610, 615, 619, 626, 627, 648, 653, 678, 680, 687, 696, 729, 744, 748, 777, 847, 848, 867, 874, 894, 901, 907, 974 and 1021. A descriptive summary for the serum-reversal time data set provides the following values: 143 (sample size), 2 (minimum), 1021 (maximum), 446(median), 424.3147 (mean), 227.6823 (standard deviation), 53.65884 (coefficient of variation), 0.07878 (coefficient of skewness) and 2.7737 (coefficient of kurtosis). The TTT (total time on test) plot for serum-reversal time data is concave [Figure 3(a)] which infers increasing shaped hazard rate. The boxplot for serum-reversal time data is positively skewed [Figure 3(b)]. So, the NFKBXII distribution is suitable to model serum-reversal time data.



(a) (b)

Figure 3: (a) TTT Plot, (b)Box Plot

Table 4 reports the MLEs (standard errors). Table 5 displays model selection criteria such as $-2\widehat{\ell}$, AIC, CAIC and BIC.

Table 4: MLEs(Standard errors) and $W*$, $A*$, K-S (p-values) for Serum-reversal times

Model	α	β	γ	κ	a	b
NFKBXII	0.0387(0.0571)	1.8572 (0.1972)	0.0296 (0.0434)	0.0370 (0.2566)	2	1021
NKBXII	1.3808(0.4113)	1.4219 (0.2389)	—–	0.9336 (0.2046)	2	1021
KMBXII	1.2920 (0.1988)	1.2498 (0.1305)	0.6133 (0.2793)	—–	2	1021
KBXII	1.5081 (0.1298)	1.3557 (0.0985)	—–	—–	2	1021
NKL	2.3322 (0.3162)	—–	—–	1.47086 (0.1396)	2	1021
KL	1.5760(0.1337)	—–	—–	–	2	1021
MBXII	82380 (2914.938)	0.1615 (0.0142)	76069.00(1507.748)		–	–
MBIII	82380.3854 (261.13678)	1.9115 (0.0225)	76069.2894 (1857.7229)	–	–	–
Weibull	0.0022 (0.0002)	1.02139 (0.0181)	—–	–	–	–
Inverse Weibull	16.7307 (2.3641)	0.5614 (0.0289)	—–	–	–	–

Table 5: $-2\hat{\ell}$,AIC, CAIC, BIC, W$*$,A$*$ and K-S(p-values) for Serum-reversal times

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	W^*	A^*	K-S(p-values)
NFKBXII	1878.55	1886.55	1886.84	1898.28	0.2346	1.3469	0.0769 (0.3843)
NKBXII	1884.77	1890.77	1890.94	1899.57	0.3513	1.9313	0.0935 (0.1761)
KMBXII	1883.62	1889.62	1889.8	1898.43	0.3137	1.7617	0.0843(0.277)
KBXII	1884.86	1888.86	1888.95	1894.73	0.3495	1.925	0.0923 (0.1869)
NKL	1887.77	1891.77	1891.85	1897.63	0.3983	2.1905	0.1061 (0.087)
KL	1900.73	1902.73	1902.76	1905.67	0.4445	2.4331	0.175 (0.0004)
MBXII	5900.84	5906.84	5907.01	5915.73	2.5844	14.019	1 ($<$ 2.2e-16)
MBIII	2040.3	2046.3	2046.47	2055.19	1.7553	9.7468	0.1438(0.0054)
Weibull	2014.2	2018.2	2018.28	2024.12	1.2003	6.7351	0.2662 (3.163e-09)
Inverse Weibull	2198.32	2202.32	2202.4	2208.24	4.0129	20.9785	0.2924 (4.835e-11)

From the tables 4 and 5, it is clear that our proposed model is best fitted, with the smallest values for all statistics and maximum p-value.

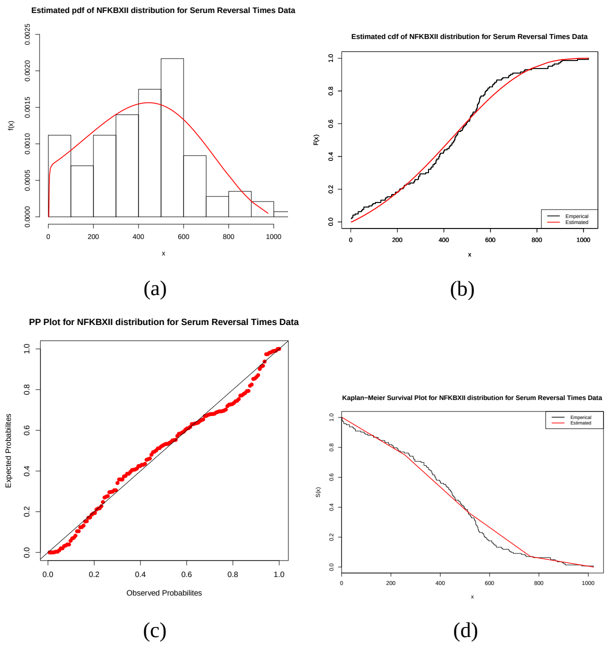


Figure 4: (a)PDf Plot, (b) CDF Plot, (c) PP Plot, (d) Survival Plot

Figure 4 infers that the proposed model is closely fitted to Serum-reversal time data.

8. Conclusion

We derive the NFKBXII distribution from (i) the T-X family procedure and (ii) the nexus concerning exponential and gamma variables. The NFKBXII density can be symmetrical, right-skewed, left-skewed, J, reverse-J, U and exponential shapes. The NFKBXII failure rate can take such as monotone and non-monotone shapes. We derive certain mathematical properties such as random number generator, sub-models, ordinary moments, conditional moments, density functions of record values and reliability measures. We characterize the NFKBXII distribution via innovative techniques. We address the maximum likelihood estimation for the NFKBXII parameters. We evaluate the precision of the maximum likelihood estimators via a simulation study. We consider an application to serum-reversal times of children to elucidate the flexibility, utility and potentiality of the NFKBXII model. We compute the estimate of GOFs and various model selection criteria for testing the acceptability and competency of the NFKBXII distribution. We ascertain empirically that the proposed model is suitable for serum-reversal times analysis.

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9. APPENDIX A

Theorem 1. Let (U, F, P) be given probability space and let $H = [a_1,\, a_2]$ an interval with $a_1 < a_2\,(a_1 = -\infty, a_2 = \infty)$. Let $X : U \to [a_1, a_2]$ be a continuous rv with distribution function F and Let $\xi(x)$ be real function defined on $H = [a_1, a_2]$ such that $E[\xi(x)\big|X \geq x] = \omega(x);\, x \in H$ is defined with some real function $h(x)$ should be in simple form. Assume that $\xi(x)\varepsilon C([a_1, a_2]),\ \omega(x)\varepsilon C^2([a_1, a_2])$ and F is twofold continuously differentiable and strictly monotone function on the set $[a_1, a_2]$: To conclude, assume that the equation $\omega(x) = \xi(x)$ has no real solution in the inside of $[a_1, a_2]$. Then F is achieved from the functions $\xi(x)$ and $\omega(x)$ as $F(x) = \int_a^x k\left|\frac{\acute{\omega}(t)}{\omega(t)-\xi(t)}\right| \exp(-s(t))\, dt$, where $s(t)$ is the solution of equation $\acute{s}(t) = \frac{\acute{\omega}(t)}{\omega(t)-\xi(t)}$ and k is a constant, chosen to make $\int_{a_1}^{a_2} dF = 1$.