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ROOTS OF TRIANGULAR MATRICES

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Abstract. This work is concerned with determining a general expression for roots of upper and lower triangular matrices with a constant diagonal. Numerical examples of the results are also presented.

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1. Introduction

Roots of matrices can be useful in many technical problems such as the resolution of matrix differential equations, nonlinear matrix equations, Markov models of finance and matrix sign functions and the computation of matrix logarithm. Recently, the computation of roots of matrices has been a popular problem. There are many papers on this topic [1, 4, 8, 13, 14, 16]. We recall that all diagonalizable and nonsingular matrices have a p^{th} root. Furthermore, a p^{th} root of matrix is not necessarily unique. Some singular matrices have p^{th} roots (see [16]). Various methods are used by researchers for finding roots of matrices. For example, diagonalization method for diagonalizable matrices [16], Shur decomposition method for nonsingular matrices [15, 16], in [8, 9] Guo and Lannazzo used the famous iterative method (Newton's iteration method) to compute a p^{th} root of matrices having a norm less than 1, in [8] Guo used Halley's method which is another iterative method converges faster than Newton's method to find a p^{th} root of matrix, an iterative algorithm to compute a p^{th} root of matrix was given by Arias et al in [1] by Newton's binominal application, in [3] Ben Taher et al developed two methods based on the primary matrix functions and minimal polynomial. For details about matrix functions and theory of matrices we refer the reader to the books [7, 6, 2, 5, 11, 12].

2. MAIN RESULTS

Our goal in this paper is to find a p^{th} root of a unipotent triangular matrix A based on positive integer power of its nilpotent part and the series obtained by Newton's binominal. Main results are illustrated by given examples.
Let $A = (a_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{R})$ be a triangular matrix with constant diagonal $(a_{ii} = 1, \forall i \in \{1, \dots, n\})$.

- If A is an upper triangular matrix, then it's given by

$$A = \begin{pmatrix} 1 & a_{1\,2} & \cdots & a_{1\,n} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1\,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{2. 1}$$

- If A is a lower triangular matrix, then it's given by

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_{2\,1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n\,1} & \cdots & a_{n\,n-1} & 1 \end{pmatrix} \tag{2. 2}$$

In both cases, A is a unipotent matrix, that is $A = I + T$ with T nilpotent matrix.
More precisely, $T = (t_{i\,j})$ where

$$t_{i\,j} = \begin{cases} a_{i\,j} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

Let p be a positive integer, any matrix R such that $R^p = A$ is called a p^{th} root of A.
The problem is to compute a p^{th} root (denoted by $A^{\frac{1}{p}}$) of A for all positive integers rather greater than 1 in both the cases where A is an upper triangular and a lower triangular unipotent matrix A given by (2. 1) and (2. 2).
The following theorem gives a method to calculate the entries of $A^{\frac{1}{p}}$.

Theorem 2.1. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a triangular unipotent matrix with constant diagonal $(a_{ii} = 1, \forall i \in \{1, \dots, n\})$ and let $A^{\frac{1}{p}} = (\alpha_{ij})_{1\leq i,j\leq n}$ for all positive integer $p > 1$, then*

- *if A is an upper triangular matrix, we have*

$$\alpha_{i\,j} = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \\ \frac{1}{p}t_{ij} + \frac{\frac{1}{p}\left(\frac{1}{p}-1\right)}{2!}\sum_{i\leq k\leq j} t_{ik}t_{kj} + \cdots + \frac{\frac{1}{p}\left(\frac{1}{p}-1\right)\cdots\left(\frac{1}{p}-n+2\right)}{(n-1)!}\sum_{i\leq k_1\leq\cdots\leq k_{n-2}\leq j} t_{ik_1}t_{k_1k_2}\cdots t_{k_{n-2}j} & \text{if } i < j \end{cases} \tag{2. 3}$$

- *if A is a lower triangular matrix, we have*

$$\alpha_{i\,j} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ \frac{1}{p}t_{ij} + \frac{\frac{1}{p}\left(\frac{1}{p}-1\right)}{2!}\sum_{i\geq k\geq j} t_{ik}t_{kj} + \cdots + \frac{\frac{1}{p}\left(\frac{1}{p}-1\right)\cdots\left(\frac{1}{p}-n+2\right)}{(n-1)!}\sum_{i\geq k_1\geq\cdots\geq k_{n-2}\geq j} t_{ik_1}t_{k_1k_2}\cdots t_{k_{n-2}j} & \text{if } i > j \end{cases} \tag{2. 4}$$

To prove this theorem we need the following lemmas:

Lemma 2.2. *If $A = (a_{ij})_{n\times n}$ is an upper triangular matrix, then so is $A^p = (\alpha_{ij})_{n\times n}$ and*

$$\alpha_{ij} = \begin{cases} 0 & \textit{if } i > j \\ a_{ii}^p & \textit{if } i = j \\ \sum_{i\leq k_1\leq\cdots\leq k_{p-1}\leq j} a_{ik_1}a_{k_1k_2}\cdots a_{k_{p-1}j} & \textit{if } i < j \end{cases}$$

Lemma 2.3. *If $A = (a_{ij})_{n\times n}$ is a lower triangular matrix, then so is $A^p = (\alpha_{ij})_{n\times n}$ and*

$$\alpha_{ij} = \begin{cases} 0 & \textit{if } i < j \\ a_{ii}^p & \textit{if } i = j \\ \sum_{i\geq k_1\geq\cdots\geq k_{p-1}\geq j} a_{ik_1}a_{k_1k_2}\cdots a_{k_{p-1}j} & \textit{if } i > j \end{cases}$$

We will prove the Lemma 2.3 by induction on p.

Proof. For $p = 2$, we have

$$A^2 = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^2$$

$$= \begin{pmatrix} a_{11}^2 & 0 & \cdots & 0 \\ a_{21}a_{11} + a_{21}a_{22} & a_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}a_{11} + a_{n2}a_{21} + \cdots + a_{nn}a_{n1} & a_{n2}a_{22} + a_{n3}a_{32} + \cdots + a_{nn}a_{n2} & \cdots & a_{nn}^2 \end{pmatrix}$$

$$=(\alpha_{ij})_{1\leq i,j\leq n}.$$

So A^2 is a lower triangular matrix and we have for all i, j, $1 \leq i, j \leq n$,

$$\alpha_{ij} = \begin{cases} 0 & if \quad i < j, \\ a_{ii}^2 & if \quad i = j, \\ \sum_{i \geq k \geq j} a_{ik} a_{kj} & if \quad i > j. \end{cases} .$$

Suppose that the result is true for $p \geq 2$. Assume such that $A^p = (\alpha_{ij})_{1 \leq i,j \leq n}$ where

$$\alpha_{ij} = \begin{cases} 0 & if \quad i < j, \\ a_{ii}^p & if \quad i = j, \\ \sum_{i \geq k_1 \geq \cdots \geq k_{p-1} \geq j} a_{ik_1} a_{k_1 k_2} \cdots a_{k_{p-1} j} & if \quad i > j. \end{cases}$$

Consider case $p + 1$. By the induction hypothesis we have

$$\begin{aligned} A^{p+1} =& A^p A \\ =& \begin{pmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix} \times \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}. \\ =& \begin{pmatrix} \alpha_{11} a_{11} & 0 & \cdots & 0 \\ \alpha_{21} a_{11} + \alpha_{22} a_{21} & \alpha_{22} a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} a_{11} + \alpha_{n2} a_{21} + \cdots + \alpha_{nn} a_{n1} & \alpha_{n2} a_{22} + \alpha_{n3} a_{32} + \cdots + \alpha_{nn} a_{n2} & \cdots & \alpha_{nn} a_{nn} \end{pmatrix}. \\ =& (\beta_{ij})_{1 \leq i,j \leq n}. \end{aligned}$$

Where
For $i < j$, we have

$$\beta_{ij} = 0.$$

For $i = j$, we have

$$\beta_{ii} = \alpha_{ii} a_{ii} = a_{ii}^p a_{ii} = a_{ii}^{p+1}.$$

For $i > j$ we have

$$\begin{aligned} \beta_{ij} =& \sum_{k=j}^{i} \alpha_{ik} a_{kj} \\ =& \sum_{k=j}^{i} \left(\sum_{i \geq k_1 \geq \cdots \geq k_{p-1} \geq k} a_{ik_1} a_{k_1 k_2} \cdots a_{k_{p-1} k} \right) a_{kj} \\ =& \sum_{i \geq k_1 \geq \cdots \geq k_p \geq j} a_{ik_1} a_{k_1 k_2} \cdots a_{k_p j}. \end{aligned}$$

□

Proof of theorem
First we write A as follow

$$A = I + T \tag{2. 5}$$

where I is the identity matrix of order n and $T = (t_{i\,j})_{1\leq i,j\leq n}$ is a strictly triangular matrix which is nilpotent of degree at most n with $t_{i\,j} = a_{i\,j}$ for all $i \neq j$ and $t_{i\,j} = 0$ for $i = j$. The matrices T and I commute so, we can apply the Newton's theorem. Let p be a rational number, the Newton's theorem [1] can be expressed by the following expression:

$$(1 + x)^p = 1 + p_1 x + p_2 x^2 + p_3 x^3 + ...$$

with $p_m = \frac{p(p-1)\cdots(p-m+1)}{m!}$, for all $m \in \mathbb{N}$.

In the matrix case, the expression is:

$$A^p = (I + T)^p = I + pT + \frac{p\,(p-1)}{2!}T^2 + \cdots + \frac{p\,(p-1)\cdots(p-n+1)}{n!}T^n + \cdots \tag{2. 6}$$

Since T is a nilpotent matrix of degree at most n (this confirms the convergence of the series (2. 6) because $T^m = 0,\ \forall m \geq n$), A^p is exactly the sum of the first n terms of (2. 6).
We suppose now p is a positive integer number. Then a p^{th} root of A is given by:

$$\begin{aligned} A^{\frac{1}{p}} =& (I + T)^{\frac{1}{p}} \qquad (2.\ 7)\\ =& I + \frac{1}{p}T + \frac{\frac{1}{p}\left(\frac{1}{p}-1\right)}{2!}T^2 + \cdots + \frac{\frac{1}{p}\left(\frac{1}{p}-1\right)\cdots\left(\frac{1}{p}-n+2\right)}{(n-1)!}T^{n-1}. \end{aligned}$$

We end the proof by replacing $T,\quad T^2,\quad ...\quad, T^{n-1}$ in the last sum by their formulas in Lemma 2.2 and Lemma 2.3.

3. Examples

Example 3.1. *Let $p = 3$ and*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 0 & 5 & 1 \end{pmatrix} = I + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 5 & 0 \end{pmatrix}}_{T}$$

It is clear that T is a nilpotent matrix of degree 4 ($T^4 = T^5 = \cdots = 0$).

So by using Theorem 2.1 we have

$$\begin{aligned} A^{\frac{1}{3}} &= I + \frac{1}{3}T + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}T^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}T^3 \\ &= \begin{pmatrix} 1 & & & \\ \alpha_{21} & 1 & & \\ \alpha_{31} & \alpha_{32} & 1 & \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{pmatrix} \end{aligned}$$

Where

$$\alpha_{i\,j} = \begin{cases} 0 & \textit{if } i < j \\ 1 & \textit{if } i = j \\ \frac{1}{3}t_{ij} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\sum_{i \geq k \geq j} t_{ik}t_{kj} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{(3)!}\sum\limits_{i \geq k_1 \geq k_2 \geq j} t_{ik_1}t_{k_1k_2}t_{k_2j} & \textit{if } i > j \end{cases}$$

$$= \begin{cases} 0 & \textit{if } i < j \\ 1 & \textit{if } i = j \\ \frac{1}{3}t_{ij} - \frac{1}{9}\sum_{i \geq k \geq j} t_{ik}t_{kj} + \frac{5}{81}\sum\limits_{i \geq k_1 \geq k_2 \geq j} t_{ik_1}t_{k_1k_2}t_{k_2j} & \textit{if } i > j \end{cases}$$

Then

$$\alpha_{21} = \frac{1}{3} \times (2) - \frac{1}{9}(0 \times 2 + 2 \times 0) + \frac{5}{81} \times (0) = \frac{2}{3}.$$

$$\alpha_{31} = \frac{1}{3} \times (1) - \frac{1}{9} \times (6) + \frac{5}{81} \times (0) = -\frac{1}{3}$$

$$\alpha_{32} = \frac{1}{3} \times (3) - \frac{1}{9}(0) + \frac{5}{81}(0) = 1$$

$$\alpha_{41} = \frac{1}{3} \times (1) - \frac{1}{9} \times (5) + \frac{5}{81} \times (30) = \frac{44}{27}.$$

$$\alpha_{42} = \frac{1}{3} \times (0) - \frac{1}{9} \times 15 + \frac{5}{81} \times (0) = -\frac{5}{3}$$

$$\alpha_{43} = \frac{1}{3} \times 5 - \frac{1}{9}(0) + \frac{5}{81}(0) = \frac{5}{3}.$$

Then a third root of A is given by:

$$A^{\frac{1}{3}} = \begin{pmatrix} 1 & & & \\ \frac{2}{3} & 1 & & \\ -\frac{1}{3} & 1 & 1 & \\ \frac{44}{27} & -\frac{5}{3} & \frac{5}{3} & 1 \end{pmatrix}$$

Example 3.2. *Let $p = 3$ and*

$$B = A^T = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I + \underbrace{\begin{pmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{T}$$

We note first that

$$(A^{\frac{1}{3}})^T = \left(I + \frac{1}{3}T + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}T^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}T^3 \right)^T = (A^T)^{\frac{1}{3}} = B^{\frac{1}{3}}$$

So by using Theorem 2.1 and after calculations similar to the previous example, we obtain

$$B^{\frac{1}{3}} = \begin{pmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{44}{27} \\ 0 & 1 & 1 & -\frac{5}{3} \\ 0 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Remark 3.3. *If we take a triangular matrix A with diagonal elements equal to $k \neq 1$ then we write $A = k\,(I + S)$ with S is a strictly triangular matrix. Then, a p^{th} root of A can be obtained.*

4. Conclusion

In this paper, we derive a general expression for a p^{th} root of a triangular matrix with diagonal elements equal to 1 for any positive integer p. Also we prove that it has the same form. We can use these results in the decomposition of matrices and to compute the roots of other classes of matrices.

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6. Author contributions.

Conceptualization, M T. M. and I. K; methodology, M. T. M.; validation, M. T. M. and I. K.; writing-original draft preparation, I. K.; review and editing, M T. M.; and B. R. M.; supervision, B. R. M.; All authors have read and agreed to the published version of the manuscript.

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