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Relative TL-ideals in Semigroups

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Abstract. In this paper, we investigate the notions of the left (right, two-sided) $\phi - TL$-ideals and $(\phi, \sigma) - TL$-ideals in a semigroup, as well as some of their related properties. Furthermore, the results of relative TL-ideals are applied to L-ideals of semigroups.

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1. Introduction

The concept of a fuzzy subset of a set to describe uncertainty was proposed by Lotfi A. Zadeh [27] in 1965. Goguen [9] generalized fuzzy sets to lattices by choosing values from a lattice instead of the unit interval in 1967. After that, Rosenfeld [21] gave the ideas of fuzzy subgroupoid and fuzzy subgroup in 1971, beginning the study of fuzzy group theory. Many researchers have since developed fuzzy subsemigroups, fuzzy subrings, fuzzy ideals, and other concepts (see, e.g [1, 5, 11, 12, 15, 16, 17, 19]). Especially, the concepts of TL-subgroup of a group, TL-subring and TL-ideal of a ring and TL-subsemigroup of a semigroup were proposed in [6, 7, 22, 25, 26] and their qualities were carefully researched. Mordeson et al. [18] provide a comprehensive presentation of fuzzy semigroups, including theoretical results on fuzzy semigroups and their applications.

Several researchers have investigated the subalgebraic structures of an L-subset. For instance, normal L-subgroup of L-group [2], L-ideal of L-ring [17], TL-ideal of TL-ring [25]. Furthermore, various researchers have carried research on this subject [3, 4].

Wallace [23, 24], defined left (right) H-ideals of a semigroup and developed the concept of relative ideals on semigroups in 1962. Hrmova [13] gave the definition of a (H_1, H_2)-ideal of a semigroup in 1967.

In this paper, using an L-subset on a semigroup, we define several TL-ideals taking into consideration the works of [23, 13]. We also discuss some fundamental principles related to such ideals. Furthermore, we characterize a relative of these ideals.

This article is organized as follows: In the second section, we give definitions and theorems available in the literature for a better understanding of our work. In section 3, the definitions of left ϕ- TL ideal, right ϕ- TL ideal, two sided ϕ- TL ideal and (ϕ, σ)-TL ideal are given using triangular norm on L. Then it was seen that when $\phi = 1_S$ is taken, these ideals coincide with L- left ideal, L-right ideal and L-two sided ideal respectively. Moreover, we have shown that the T-cartesian product and T-product of two left ϕ-TL ideals (right, two sided) are also left ϕ- TL ideal (right, two sided). In particular, we prove that $LIdeal(S, \phi, T, L), RIdeal(S, \phi, T, L), Ideal(S, \phi, T, L)$ and $Ideal(S, \phi, \sigma, T, L)$ are complete lattice. Using TL level sets, we explain relative TL-ideals. In conclusion of this study, it is proved that homomorphism image and inverse image of ϕ-TL ideal(right, two sided) are also left ϕ- TL ideal (right, two sided).

2. Preliminaries

We start with some basics that are used in this paper. For details one can see [8, 9, 10, 14, 20, 18, 27].

Throughout this paper, $(L, \leq, 0, 1)$ denotes a complete lattice with the bottom element 0 and the top element 1. For $u, v \in L$ and every family $\{b_\alpha \mid \alpha \in \Lambda\}$, we can denote some operations such as

$$u \vee v = \sup\{u, v\}, \quad u \wedge v = \inf\{u, v\},$$
$$\bigvee_{\alpha\in\Lambda} b_\alpha = \sup\{b_\alpha \mid \alpha \in \Lambda\},$$
$$\bigwedge_{\alpha\in\Lambda} b_\alpha = \inf\{b_\alpha \mid \alpha \in \Lambda\}.$$

Theorem 2.1. Let us L be an order set in which all subsets have an infimum. Then L is a complete lattice.

Definition 2.2. Let L be a complete lattice. Then L is called infinitely $\vee$-distributive lattice if

$$a \wedge (\textstyle\bigvee_{\alpha\in\Lambda} b_\alpha) = \textstyle\bigvee_{\alpha\in\Lambda}(a \wedge b_\alpha)$$

for all $a \in L$ and any $\{b_\alpha \mid \alpha \in \Lambda\} \subseteq L$.

Unless otherwise stated, L is a complete infinitely $\vee$-distributive lattice. The closed interval $[0, 1]$ together with the partial ordering ”$\leq$” form a complete infinitely $\vee$-distributive lattice.

Definition 2.3. A triangular norm is defined to be a two-place function on L which satisfies the axioms: monotone, commutative, has 1 as a neutral element and associative.

The set of all T-idempotent elements is denoted by $D_T = \{\alpha \in L \mid \alpha T \alpha = \alpha\}$. Some of the commonly used examples of t-norms are as follows:
Basic examples are the t-norms T_M, T_D on L are given by:

$$T_M(u, v) = u \wedge v,$$

$$T_D(u, v) = \begin{cases} v \wedge u & \text{iff } u = 1 \text{ or } v = 1, \\ 0 & \text{iff } otherwise. \end{cases}$$

Definition 2.4. Let T be a t-norm on L. Then T is called infinitely $\vee$-distributive if

$$aT(\bigvee_{\alpha\in\Lambda} b_\alpha) = \bigvee_{\alpha\in\Lambda}(aTb_\alpha),$$

for all $a \in L$ and any $\{b_\alpha \mid \alpha \in \Lambda\} \subseteq L$.

Proposition 2.5. Let T be any t-norm on L. Then the following hold:

(i) $uTv \leq u \wedge v$ for all $u, v \in L$,
(ii) $uT0 = 0Tu = 0$ for all $u \in L$,
(iii) $T = T_M$ iff $D_T = L$,
(iv) $(\bigwedge_{\alpha\in\Lambda} a_\alpha)T(\bigwedge_{\beta\in\mho} b_\beta) \leq \bigwedge_{\alpha\in\Lambda,\beta\in\mho} a_\alpha T b_\beta$.

Goguen [9] introduces a L-subset of X, which is a function from X to L, as a generalization of Zadeh's fuzzy subset [27]. The set of all L-subsets of X is called L-power set of X and indicated by $F(X, L)$. Also, all subsets of X is denoted by $F(X)$ when $L = [0, 1]$. Let $A \subseteq X$ and $\alpha \in L$. Then α_A will denote a L-subset of X with value α if $x \in A$ and 0 elsewhere. In particular, the L-subset 1_A is called the characteristic function of a set A.

Definition 2.6. Let $\pi \in F(X, L)$. Then for $\alpha \in L$, the set

$$\pi_\alpha = \{u \in X \mid \alpha \leq \pi(u)\}$$

is called a α-cut or α-level or level subset of π.

Definition 2.7. Let $\pi, \rho \in F(X, L)$. Then the union, intersection, and T-intersection of π and ρ are the L-subsets $\pi \cap \rho$, $\pi \cup \rho$ and $\pi T \rho$, defined as

$$\pi \cap \rho(u) = \pi(u) \wedge \rho(u),$$
$$\pi \cup \rho(u) = \pi(u) \vee \rho(u),$$
$$\pi T \rho(u) = \pi(u) T \rho(u).$$

respectively.

Definition 2.8. Let $\{\pi_i \mid i \in \Lambda\} \subseteq F(X, L)$. Then the intersection, union of the family $\{\pi_i \mid i \in \Lambda\}$ are the L-subsets $\bigcup_{i\in\Lambda} \pi_i$, $\bigcap_{i\in\Lambda} \pi_i$ defined as

$$(\bigcap_{i\in\Lambda} \pi_i)(u) = \bigwedge_{i\in\Lambda} \pi_i(u),$$
$$(\bigcup_{i\in\Lambda} \pi_i)(u) = \bigvee_{i\in\Lambda} \pi_i(u).$$

respectively.

Definition 2.9. Let $\pi_1 \in F(X_1, L)$ and $\pi_2 \in F(X_2, L)$. Then the T-cartesian product of π_1 and π_2 is a L-subset $\pi_1 \times_T \pi_2$ of $U_1 \times U_2$ defined as

$$\pi_1 \times_T \pi_2(u_1, u_2) = \pi_1(u_1) T \pi_2(u_2).$$

If $T = T_M$, then $\pi_1 \times_T \pi_2$ is simply referred to as the cartesian product of π_1 and π_2 and written as $\pi_1 \times \pi_2$.

Definition 2.10. Let f be a mapping from X into Y and $\pi \in F(X, L)$, $\rho \in F(Y, L)$. Then the image of π under f and the pre-image (or inverse image) of ρ under f are the L-subsets $f(\pi)$ and $f^{-1}(\rho)$ defined as

$$f(\pi)(v) = \bigvee_{f(u)=v} \pi(u)$$

and

$$f^{-1}(\rho)(u) = \rho(f(u)).$$

respectively.
In the definition of the $f(\pi)$, if v can not be written as $v = f(u)$, then $f(\pi)(v) = 0$.

Throughout this paper, let S be always a semigroup, unless otherwise stated. For $U, V \in P(S) \setminus \{\emptyset\}$, the set UV is defined by the set $\{uv \mid u \in U \quad \text{and} \quad v \in V\}$. We give the notion relative ideals of a semigroup that has introduced [23, 24, 13].

Definition 2.11. Let U, V, W be any non-empty subsets of S. Then

i) U is called a left V-ideal of S if $VU \subseteq U$,
ii) U is called a right V-ideal of S if $UV \subseteq U$,
iii) U is called a two sided V-ideal of S if $VU \subseteq U$ and $UV \subseteq U$,
iv) U is called a (V, W)-ideal of S if $VU \subseteq U$ and $UW \subseteq U$.

In addition the sets of left V-ideal of S, right V-ideal of S, two-sided V-ideal of S, and (V, W)-ideal of S are written as $LI_V(S)$, $RI_V(S)$, $I_V(S)$, $I_{V,W}(S)$ respectively.

Let S_1 and S_2 be two semigroups. Under the coordinatewise multiplication, the Cartesian product $S_1 \times S_2$ of S_1 and S_2 forms a semigroup. The semigroup $S_1 \times S_2$ is said the Cartesian product of the semigroups S_1 and S_2.

Definition 2.12. Let S_1 and S_2 be two semigroups. A function $f : S_1 \to S_2$ is called a homomorphism if

$$f(a \cdot b) = f(a) \cdot f(b)$$

for all $a, b \in S_1$.

Definition 2.13. Let $\pi, \rho \in F(S, L)$. Then the T-product of π and ρ is an L-subset $\pi \circ_T \rho$ defined as

$$\pi \circ_T \rho(u) = \bigvee_{u=ab} \pi(a) T \rho(b).$$

If $T = T_M$, then $\pi \circ_T \rho$ are simply referred to as the product of π and ρ and written as $\pi \circ \rho$.

We can give the following lemma given by Chon [7], which is very well known in the literature.

Lemma 2.14. Let $\pi, \phi, \rho \in F(S, L)$ and T be an infinitely $\vee$-distributive t- norm. Then

$$(\pi \circ_T \rho) \circ_T \phi = \pi \circ_T (\rho \circ_T \phi).$$

Definition 2.15. Let $\pi \in F(S, L)$. Then

(i) π is called a left L-ideal of S if $\pi(v) \leq \pi(uv)$ for all $u, v \in S$,
(ii) π is called a right L-ideal of S if $\pi(u) \leq \pi(uv))$ for all $u, v \in S$,
(iii) π is called a two-sided L-ideal of S if $\pi(u) \vee \pi(v) \leq \pi(uv)$ for all $u, v \in S$.

3. RELATIVE TL-IDEALS IN SEMIGROUPS

In this chapter, we give the definition of the notion of left $\phi - TL$-ideals, right $\phi - TL$-ideals, two sided $\phi - TL$- ideals, and $(\phi, \sigma) - TL$- ideals of a semigroup S and give some fundamental properties of such ideals.

Definition 3.1. Let $\pi, \phi, \sigma \in F(S, L)$. Then

(i) π is called a left $\phi - TL$-ideal of S if $\phi(u)T\pi(v) \leq \pi(uv)$ for all $u, v \in S$,
(ii) π is called a right $\phi - TL$-ideal of S if $\pi(u)T\phi(v) \leq \pi(uv))$ for all $u, v \in S$,
(iii) π is called a two-sided $\phi{-}TL$-ideal of S if $\phi(u)T\pi(v) \leq \pi(uv)$ and $\pi(u)T\phi(v) \leq \pi(uv))$ for all $u, v \in S$,
(iv) π is called a $(\phi, \sigma) - TL$-ideal of S if $\phi(u)T\pi(v) \leq \pi(uv)$ and $\pi(u)T\sigma(v) \leq \pi(uv))$ for all $u, v \in S$.

T t-norm **L complete lattice**	$T = T_M$ L comlplete lattice	T t-norm $L = [0,1]$	$T = T_M$ $L = [0,1]$
LIdeal(S, ϕ, T, L) left $\phi - TL$-ideals	LIdeal(S, ϕ, L) left $\phi - L$-ideals	LIdeal(S, ϕ, T) left $\phi - T$-ideals	LIdeal(S, ϕ) fuzzy left ϕ-ideals
RIdeal(S, ϕ, T, L) right $\phi - TL$-ideals	RIdeal(S, ϕ, L) right $\phi - L$-ideals	RIdeal(S, ϕ, T) fuzzy right $\phi - T$-ideals	RIdeal(S, ϕ) fuzzy right ϕ-ideals
Ideal(S, ϕ, T, L) two-sided $\phi - TL$-ideals	Ideal(S, ϕ, L) two-sided $\phi - L$-ideals	Ideal(S, ϕ, T) fuzzy two-sided $\phi - T$-ideals	Ideal(S, ϕ) fuzzy two-sided ϕ-ideals
Ideal(S, ϕ, σ, T, L) $(\phi - \sigma) - TL$-ideals	Ideal(S, ϕ, σ, L) $(\phi - \sigma) - L$-ideals	Ideal(S, ϕ, σ, T) fuzzy $(\phi - \sigma) - T$-ideals	Ideal(S, ϕ, σ) fuzzy $(\phi - \sigma)$-ideals

FIGURE 1. Changes and naming of symbols

In particular, when $\phi = 1_S$, the sets LIdeal(S, ϕ, T, L), RIdeal(S, ϕ, T, L), and Ideal(S, ϕ, T, L) are written as LIdeal(S, L), RIdeal(S, L), and Ideal(S, L), respectively.
It is easy to verify the following remark by Definition 2.15 and Definition 3.1

Remark 3.2. Let $\pi \in F(S, L)$. Then

(i) $\pi \in LIdeal(S, 1_S, T, L)$ iff $\pi \in LIdeal(S, L)$,
(ii) $\pi \in RIdeal(S, 1_S, T, L)$ iff $\pi \in RIdeal(S, L)$,
(iii) $\pi \in LIdeal(S, 1_S, T, L)$ iff $\pi \in Ideal(S, L)$.

Example 3.3. Let $(S = \{0, 1\}, \cdot)$ be a semigroup. Then the sets Ideal(S, L), Ideal(S, ϕ, T, L) and Ideal (S, ϕ, σ, T, L) are can be easily as follows:

$$\text{Ideal}(S, L) = \{\pi \in F(S, L) \mid \pi(1) \leq \pi(0)\},$$
$$\text{Ideal}(S, \phi, T, L) = \{\pi \in F(S, L) \mid \pi(1)T\phi(0) \leq \pi(0)\},$$
$$\text{Ideal}(S, \phi, \sigma, T, L) = \{\pi \in F(S, L) \mid \pi(1)T\phi(0) \leq \pi(0), \quad \pi(1)T\sigma(0) \leq \pi(0)\}.$$

One may easily verify that the following propositions are true by Proposition 2.5 (i)-(ii).

Proposition 3.4. Let $U, V, W \in P(S) \setminus \{\emptyset\}$ and $\alpha, \beta, \delta \in L$. Then

(i) If $U \in LI_V(S)$, then $\alpha_U \in LIdeal(S, \beta_V, T, L)$,

(ii) If $U \in RI_V(S)$, then $\alpha_U \in RIdeal(S, \beta_V, T, L)$,
(iii) If $U \in I_V(S)$, then $\alpha_U \in Ideal(S, \beta_V, T, L)$,
(iv) If $U \in I_{V,W}(S)$, then $\alpha_U \in Ideal(S, \beta_V, \delta_W, T, L)$.

Proposition 3.5. Let $U, V, W \in P(S) \setminus \{\emptyset\}$ and $\alpha, \beta, \delta \in L$, $\alpha T \beta \neq 0$, $\alpha T \delta \neq 0$. Then

(i) If $\alpha_U \in LIdeal(S, \beta_V, T, L)$, then $U \in LI_V(S)$,
(ii) If $\alpha_U \in RIdeal(S, \beta_V, T, L)$, then $U \in RI_V(S)$,
(iii) If $\alpha_U \in Ideal(S, \beta_V, T, L)$, then $U \in I_V(S)$,
(iv) If $\alpha_U \in Ideal(S, \beta_V, \delta_W, T, L)$, then $U \in I_{V,W}(S)$.

Corollary 3.6. Let $U, V, W \in P(S) \setminus \{\emptyset\}$. Then

(i) $U \in LI_V(S)$ iff $1_U \in LIdeal(S, 1_V, T, L)$,
(ii) $U \in RI_V(S)$ iff $1_U \in RIdeal(S, 1_V, T, L)$,
(iii) $U \in I_V(S)$ iff $1_U \in Ideal(S, 1_V, T, L)$,
(iv) $U \in I_{V,W}(S)$ iff $1_U \in Ideal(S, 1_V, 1_W, T, L)$.

Proposition 3.7. Let $\pi, \phi \in F(S, L)$. Then the following properties hold.

(i) If $\pi \in LIdeal(S, L)$, then $\pi \in LIdeal(S, \phi, T, L)$,
(ii) If $\pi \in RIdeal(S, L)$, then $\pi \in RIdeal(S, \phi, T, L)$,
(iii) If $\pi \in Ideal(S, L)$, then $\pi \in Ideal(S, \phi, T, L)$.

Proof The proof follows from Proposition 2.5 (i).
The following proposition can be simply obtained using Definition 2.13.
Proposition 3.8. Let $\pi, \phi, \sigma \in F(S, L)$. Then

(i) $\pi \in LIdeal(S, \phi, T, L)$ iff $\phi \circ_T \pi \subseteq \pi$,
(ii) $\pi \in RIdeal(S, \phi, T, L)$ iff $\pi \circ_T \phi \subseteq \pi$,
(iii) $\pi \in Ideal(S, \phi, T, L)$ iff $\phi \circ_T \pi \subseteq \pi$ and $\pi \circ_T \phi \subseteq \pi$,
(iv) $\pi \in Ideal(S, \phi, \sigma, T, L)$ iff $\phi \circ_T \pi \subseteq \pi$ and $\pi \circ_T \sigma \subseteq \pi$.

Proof (i) Let $\pi \in LIdeal(S, \phi, T, L)$ and $u, v \in S$. Then

$$\begin{aligned} \phi \circ_T \pi(x) &= \bigvee_{u=ab} \phi(a) T \pi(b) \\ &\leq \bigvee_{u=ab} \pi(ab) \\ &= \pi(u). \end{aligned}$$

Thus, $\phi \circ_T \pi \subseteq \pi$. Conversely, suppose that $\phi \circ_T \pi \subseteq \pi$. Let $u, v \in S$

$$\begin{aligned} \phi(u) T \pi(v) &\leq \bigvee_{uv=ab} \phi(a) T \pi(b) \\ &= \phi \circ_T \pi(uv) \\ &\leq \pi(uv). \end{aligned}$$

Therefore $\pi \in LIdeal(S, \phi, T, L)$.
(ii) It is proved similarly to (i).
(iii) This is an immediate consequence of (i) and (ii).

(iv) $\pi \in Ideal(S, \phi, \sigma, T, L)$ and $u, v \in S$. Then

$$\begin{aligned} \phi \circ_T \pi(u) &= \bigvee_{u=ab} \phi(a)T\pi(b) \\ &\leq \bigvee_{u=ab} \pi(ab) \\ &= \pi(u) \end{aligned}$$

and

$$\begin{aligned} \pi \circ_T \sigma(u) &= \bigvee_{u=ab} \pi(a) \circ_T \sigma(b) \\ &\leq \bigvee_{u=ab} \pi(ab) \\ &= \pi(u). \end{aligned}$$

Thus $\phi \circ_T \pi \subseteq \pi$ and $\pi \circ_T \sigma \subseteq \pi$. Conversely, suppose that $\phi \circ_T \pi \subseteq \pi$ and $\pi \circ_T \sigma \subseteq \pi$. Let $u, v \in S$

$$\begin{aligned} \phi(u)T\pi(v) &\leq \bigvee_{uv=ab} \phi(a)T\pi(b) \\ &= \phi \circ_T \pi(uv) \\ &\leq \pi(uv) \end{aligned}$$

and

$$\begin{aligned} \pi(u) \circ_T \sigma(v) &\leq \bigvee_{uv=ab} \pi(a) \circ_T \sigma(b) \\ &\leq \bigvee_{u=ab} \pi(ab) \\ &= \pi(u). \end{aligned}$$

Thus, we obtain that $\pi \in Ideal(S, \phi, \sigma, T, L)$.

Theorem 3.9. Let $\phi, \sigma \in F(S, L)$ and $\{\pi_i \mid i \in \Lambda\} \subseteq F(S, L)$. Then the following holds:

(i) If $\pi_i \in LIdeal(S, \phi, T, L)$ for every $i \in \Lambda$, then $\bigcap_{i\in\Lambda} \pi_i \in LIdeal(S, \phi, T, L)$,
(ii) If $\pi_i \in RIdeal(S, \phi, T, L)$ for every $i \in \Lambda$, then $\bigcap_{i\in\Lambda} \pi_i \in RIdeal(S, \phi, T, L)$,
(iii) If $\pi_i \in Ideal(S, \phi, T, L)$ for every $i \in \Lambda$, then $\bigcap_{i\in\Lambda} \pi_i \in Ideal(S, \phi, T, L)$,
(iv) If $\pi_i \in Ideal(S, \phi, \sigma, T, L)$ for every $i \in \Lambda$, then $\bigcap_{i\in\Lambda} \pi_i \in Ideal(S, \phi, \sigma, T, L)$.

Proof. (i) Let $u, v \in S$. Then by hypothesis, we have

$$\begin{aligned}
\phi(u)T(\bigcap_{i\in\Lambda}\pi_i)(v) &= \phi(u)T\bigwedge_{i\in\Lambda}\pi_i(v)\\
&\leq \bigwedge_{i\in\Lambda}(\phi(u)T\pi_i(v))\\
&\leq \bigwedge_{i\in\Lambda}\pi_i(uv)\\
&= (\bigcap_{i\in\Lambda}\pi_i)(uv).
\end{aligned}$$

Hence $\bigcap_{i\in\Lambda}\pi_i$ is a left $\phi - TL$-ideal of S. (ii), (iii), and (iv) are proved similarly (i).

It is easy to verify the following corollary by Theorem 3.9 and Theorem 2.1.

Corollary 3.10. Let $\phi \in F(S, L)$. Then the following properties hold.

(i) (LIdeal(S, ϕ, T, L),$\subseteq$) is a complete lattice,
(ii) (RIdeal(S, ϕ, T, L),$\subseteq$) is a complete lattice,
(iii) (Ideal(S, ϕ, T, L),$\subseteq$) is a complete lattice,
(iv) (Ideal(S, ϕ, σ, T, L),$\subseteq$) is a complete lattice.

Theorem 3.11. Let $\phi, \sigma \in F(S, L)$ and $\{\pi_i \mid i \in \Lambda\} \subseteq F(S, L)$ and T be an infinitely $\vee$-distributive t-norm. Then the following holds:

(i) If $\pi_i \in LIdeal(S, \phi, T, L)$ for every $i \in \Lambda$, then $\bigcup_{i\in\Lambda}\pi_i \in LIdeal(S, \phi, T, L)$,
(ii) If $\pi_i \in RIdeal(S, \phi, T, L)$ for every $i \in \Lambda$, then $\bigcup_{i\in\Lambda}\pi_i \in RIdeal(S, \phi, T, L)$,
(iii) If $\pi_i \in Ideal(S, \phi, T, L)$ for every $i \in \Lambda$, then $\bigcup_{i\in\Lambda}\pi_i \in Ideal(S, \phi, T, L)$,
(iv) If $\pi_i \in Ideal(S, \phi, \sigma, T, L)$ for every $i \in \Lambda$, then $\bigcup_{i\in\Lambda}\pi_i \in Ideal(S, \phi, \sigma, T, L)$.

Proof (i) Let $u, v \in S$. Then by hypothesis, we have

$$\begin{aligned}
\phi(u)T(\bigcup_{i\in\Lambda}\pi_i)(v) &= \phi(u)T\bigvee_{i\in\Lambda}\pi_i(v)\\
&= \bigvee_{i\in\Lambda}\phi(u)T\pi_i(v)\\
&\leq \bigvee_{i\in\Lambda}\pi_i(uv)\\
&= (\bigcup_{i\in\Lambda}\pi_i)(uv).
\end{aligned}$$

Hence $\bigcup_{i\in\Lambda}\pi_i \in LIdeal(S, \phi, T, L)$. Similarly, we can see that the proofs (ii), (iii), and (iv) are proved similarly (i).

Theorem 3.11 (i-iv) may not be true in general when T has no infinitely $\vee$-distributive.

Example 3.12. Consider the complete lattice $L = \{0, a, b, c, 1\}$, where $0 < a < 1$, $0 < b < c < 1$, T_D be drastic t-norm on L and $S = \{0, 1\}$ is a semigroup according to the product binary relation. Let us define L-subsets ϕ, π, ϕ of S as follows:

u	0	1
$\phi(u)$	b	1
$\pi(u)$	a	a
$\phi(u)$	a	b

Then $\pi, \phi \in Ideal(S, \phi, T_D, L)$. But $\pi \cup \phi$ is not a two sided $\phi - T_D$ L-ideal of S.
Theorem 3.13. Let $\pi, \phi, \sigma \in F(S, L)$. Then the following holds:

(i) If $\pi_\alpha \in LI_{\phi_\alpha}(S)$ for all $\alpha \in L$, then $\pi \in LIdeal(S, \phi, T, L)$,
(ii) If $\pi_\alpha \in RI_{\phi_\alpha}(S)$ for all $\alpha \in L$, then $\pi \in RIdeal(S, \phi, T, L)$,
(iii) If $\pi_\alpha \in I_{\phi_\alpha}(S)$ for all $\alpha \in L$, then $\pi \in Ideal(S, \phi, T, L)$,
(iv) If $\pi_\alpha \in LI_{\phi_\alpha}(S)$ and $\pi_\beta \in RI_{\sigma_\beta}(S)$ for every $\alpha, \beta \in L$, then $\pi \in Ideal(S, \phi, \sigma, T, L)$.

Proof.(i) Let $u, v \in S$ and $\alpha = \phi(u)T\pi(v)$. In this case, $\alpha \leq \phi(u)$ and $\alpha \leq \pi(v)$. It follows that $u \in \phi_\alpha$ and $v \in \pi_\alpha$. Thus, π_α is a left ϕ_α-ideal of S and hence $uv \in \pi_\alpha$. Hence $\alpha \leq \pi(uv)$. Therefore, π is a left $\phi - TL$-ideal of S. (ii), (iii), and (iv) are proved similarly (i).
Theorem 3.14. Let $\pi, \phi, \sigma \in F(S, L)$. Then the following holds:

(i) If $\pi \in LIdeal(S, \phi, T, L)$, then $\pi_\alpha \in LI_{\phi_\alpha}(S)$ for every non-empty sets π_α and ϕ_α for all $\alpha \in D_T$,
(ii) If $\pi \in RIdeal(S, \phi, T, L)$, then $\pi_\alpha \in RI_{\phi_\alpha}(S)$ for every non-empty sets π_α and ϕ_α for all $\alpha \in D_T$,
(iii) If $\pi \in RIdeal(S, \phi, T, L)$, then $\pi_\alpha \in RI_{\phi_\alpha}(S)$ for every non-empty sets π_α and ϕ_α for all $\alpha \in D_T$,
(iv) If $\pi \in Ideal(S, \phi, \sigma, T, L)$, then $\pi_\alpha \in LI_{\phi_\alpha}(S)$ for every non-empty sets and $\pi_\beta \in RI_{\sigma_\beta}$ for every non-empty sets π_β, σ_β for every $\alpha, \beta \in D_T$.

Proof (i) Let $u \in \phi_\alpha$ and $v \in \pi_\alpha$. By hypothesis,

$$\alpha = \alpha T\alpha \leq \phi(u)T\pi(v) \leq \pi(uv).$$

Thus $uv \in \pi_\alpha$. Therefore $\pi_\alpha \in LI_{\phi_\alpha}$.
The same motivation is used for the others (ii), (iii), and (iv) are similar to (i).
It is easy to verify the following corollary by Theorem 3.13 and Theorem 3.14.
Corollary 3.15. Let $\pi, \phi, \sigma \in F(S, L)$. Then the following holds:

(i) $\pi_\alpha \in LI_{\phi_\alpha}$ for every non-empty sets π_α and ϕ_α, for all $\alpha \in L$ iff $\pi \in LIdeal(S, \phi, T, L)$,
(ii) $\pi_\alpha \in RI_{\phi_\alpha}$ for every non-empty sets π_α and ϕ_α, for all $\alpha \in L$ iff $\pi \in RIdeal(S, \phi, T, L)$,
(ii) $\pi_\alpha \in RI_{\phi_\alpha}$ for every non-empty sets π_α and ϕ_α, for all $\alpha \in L$ iff $\pi \in Ideal(S, \phi, T, L)$,
(iv) If π_α is a left ϕ_α-ideal of S for every non-empty sets π_α, ϕ_α and π_β is a right σ_β-ideal of S for every non-empty sets π_β, σ_β for every $\alpha, \beta \in L$ iff $\pi \in Ideal(S, \phi, \sigma, T, L)$.

Proposition 3.16. Let $\pi, \rho, \phi, \sigma \in F(S, L)$ be L-subsets of S, and T be an infinitely $\vee$-distributive t-norm. Then the following holds:

(i) If $\pi \in LIdeal(S, \phi, T, L)$, then $\pi \circ_T \rho \in LIdeal(S, \phi, T, L)$,
(ii) If $\rho \in RIdeal(S, \phi, T, L)$, then $\pi \circ_T \rho \in RIdeal(S, \phi, T, L)$,
(iii) $\pi \in LIdeal(S, \phi, T, L)$ and $\rho \in RIdeal(S, \phi, T, L)$, then $\pi \circ_T \rho \in Ideal(S, \phi, T, L)$,
(iv) $\pi \in LIdeal(S, \phi, T, L)$ and $\rho \in RIdeal(S, \sigma, T, L)$, then $\pi \circ_T \rho \in Ideal(S, \phi, \sigma, T, L)$.

Proof (i) Let $u, v \in S$. Then by hypothesis, we have

$$\begin{aligned}\phi(u)T(\pi \circ_T \rho)(v) &= \phi(u)T(\bigvee_{v=ab} \pi(a)T\rho(b)) \\ &= \bigvee_{v=ab} \phi(u)T\pi(a)T\rho(b) \\ &\leq \bigvee_{v=ab} \pi(ua)T\rho(b) \\ &\leq \bigvee_{uv=c.d} \pi(c)T\rho(d) = (\pi \circ_T \rho)(uv).\end{aligned}$$

Thus $\pi \circ_T \rho$ is is a left $\phi - TL$-ideal of S. The proofs of assertions (ii),(iii), and (iv) are similar to (i).

Proposition 3.17. Let $\pi, \rho, \phi, \sigma \in F(S, L)$ be which Im$\phi \subseteq D_T$, Im$\sigma \subseteq D_T$. Then the following holds:

(i) If $\pi, \rho \in LIdeal(S, \phi, T, L)$, then $\pi T\rho \in LIdeal(S, \phi, T, L)$,
(ii) If $\pi, \rho \in RIdeal(S, \phi, T, L)$, then $\pi T\rho \in RIdeal(S, \phi, T, L)$,
(iii) If $\pi, \rho \in Ideal(S, \phi, T, L)$, then $\pi T\rho \in Ideal(S, \phi, T, L)$,
(iv) If $\pi, \rho \in LIdeal(S, \phi, \sigma, T, L)$, then $\pi T\rho \in Ideal(S, \phi, \sigma, T, L)$.

Proof. Let $u, v \in S$. Then by hypothesis, we have

$$\begin{aligned}\phi(u)T(\pi T\rho)(v) &= \phi(u)T\pi(v)T\rho(v) \\ &= \phi(u)T\phi(u)T\pi(v)T\rho(v) \\ &= \phi(u)T\pi(v)T\phi(u)T\rho(v) \\ &\leq \pi(uv)T\rho(uv) = \pi T\rho(uv).\end{aligned}$$

Hence $\pi T\rho \in LIdeal(S, \phi, T, L)$. The proofs of assertions (ii-iv) are similar to (i).

Theorem 3.18. Let f is a semigroup homomorphism from S to S'. Let $\pi, \phi, \sigma \in F(S, L)$, T be an infinitely $\vee$-distributive t-norm. Then the following holds.

(i) If $\pi \in LIdeal(S, \phi, T, L)$, then $f(\pi) \in LIdeal(S, f(\phi), T, L)$,
(ii) If $\pi \in RIdeal(S, \phi, T, L)$, then $f(\pi) \in RIdeal(S, f(\phi), T, L)$,
(iii) If $\pi \in Ideal(S, \phi, T, L)$, then $f(\pi) \in Ideal(S, f(\phi), T, L)$,
(iv) If $\pi \in LIdeal(S, \phi, \sigma, T, L)$, then $f(\pi) \in LIdeal(S, f(\phi), f(\sigma), T, L)$.

Proof (i) Let $u, v \in S'$. Then by hypothesis, we have

$$\begin{aligned}f(\phi)(u)Tf(\pi)(v) = \bigvee_{f(a)=u} \phi(a)T \bigvee_{f(b)=v} \pi(b) &= \bigvee_{f(a)=u, f(b)=v} \phi(a)T\pi(b) \\ &\leq \bigvee_{f(a)=u, f(b)=v} \pi(ab) \\ &\leq \bigvee_{f(u)=uv} \pi(u) \\ &= f(\pi)(uv).\end{aligned}$$

Hence $f(\pi) \in LIdeal(S, \phi, T, L)$. The same motivation is used for the others (ii-iv) are similar to (i).

Theorem 3.19. Let f is a semigroup homomorphism from S to S'. Let $\pi, \phi, \sigma \in F(S', L)$. Then the following holds.

(i) If $\pi \in LIdeal(S', \phi, T, L)$, then $f^{-1}(\pi) \in LIdeal(S, f^{-1}(\phi), T, L)$,
(ii) If $\pi \in RIdeal(S', \phi, T, L)$, then $f^{-1}(\pi) \in RIdeal(S, f^{-1}(\phi), T, L)$,
(iii) If $\pi \in Ideal(S', \phi, T, L)$, then $f^{-1}(\pi) \in Ideal(S, f^{-1}(\phi), T, L)$,
(iv) If $\pi \in Ideal(S', \phi, \sigma, T, L)$, then $f^{-1}(\pi) \in Ideal(S, f^{-1}(\phi), f^{-1}(\sigma), T, L)$.

Proof(i) Let $u, v \in S$. Then by hypothesis, we have

$$\begin{aligned} f^{-1}(\phi)(u)Tf^{-1}(\pi)(v) &= \phi(f(u))T\pi(f(v)) \\ &\leq \pi(f(u).f(v)) \\ &= \pi(f(uv)) \\ &= f^{-1}(\pi)(uv). \end{aligned}$$

Thus $f^{-1}(\pi) \in LIdeal(S, f^{-1}(\phi), T, L)$. The same motivation is used for the others (ii), (iii), and (iv) are similar to (i).

Theorem 3.20. Let S_1, S_2 be semigroups such that $\pi_i, \phi_i, \sigma_i \in F(S_i, L)$ for each $i = 1, 2$. Then the following assertions hold:

(i) If $\pi_i \in LIdeal(S_i, \phi_i, T, L)$ for each $i = 1, 2$, then $\pi_1 \times_T \pi_2 \in LIdeal(S_1 \times S_2, \phi_1 \times_T \phi_2 - TL, T, L)$,
(ii) If $\pi_i \in RIdeal(S_i, \phi_i, T, L)$ for each $i = 1, 2$, then $\pi_1 \times_T \pi_2 \in RIdeal(S_1 \times S_2, \phi_1 \times_T \phi_2 - TL, T, L)$,
(iii) If $\pi_i \in Ideal(S_i, \phi_i, T, L)$ for each $i = 1, 2$, then $\pi_1 \times_T \pi_2 \in Ideal(S_1 \times S_2, \phi_1 \times_T \phi_2 - TL, T, L)$,
(iv) If $\pi_i \in Ideal(S_i, \phi_i, \sigma_i, T, L)$ for each $i = 1, 2$, for each $i = 1, 2$, then $\pi_1 \times_T \pi_2 \in Ideal(S_1 \times S_2, \phi_1 \times_T \phi_2, \sigma_1 \times_T \sigma_2, T, L)$.

Proof. (i) Let $u = (u_1, u_2), v = (v_1, v_2) \in S_1 \times S_2$. Then by hypothesis, we have

$$\begin{aligned} (\phi_1 \times_T \phi_2)(u)T(\pi_1 \times_T \pi_2)(v) &= \phi_1(u_1)T\phi_2(u_2)T\pi_1(v_1)T\pi_2(v_2) \\ &= \phi_1(u_1)T\pi_1(v_1)T\phi_2(u_2)T\pi_2(v_2) \\ &\leq \pi_1(u_1v_1)T\pi_2(u_2v_2) \\ &= (\pi_1 u_T \pi_2)(u_1v_1, u_2v_2) \\ &= (\pi_1 \times_T \pi_n)(uv). \end{aligned}$$

Thus, $\pi_1 \times_T \pi_2 \in LIdeal(S_1 \times S_2, \phi_1 \times_T \phi_2, T, L)$. The proofs of (ii)-(iv) are similar to (i).

4. Conclusion

Fuzzy set theory has an important place in mathematics and many other fields. The motivation of this paper is to extend the structure of relative ideals existing in the literature to L-subsets with the help of triangular norms to give the reader a better understanding of these structures. In this study, we have initiated a new concept relative TL- ideals and we have given some of their properties.

This paper is a contribution to future works in fuzzy semigroup theory. In future research, other ideals such as the $\phi - TL$ bi-ideal, $\phi - TL$ quasi ideal, and $\phi - TL$ interior ideal can be studied on this algebraic structure.

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Conceptualization, E.F.O.; validation, S.Y.; writingoriginal draft preparation, E.F.O; review and editing, E.F.O. and S.Y.; supervision, S.Y.; project administration, E.F.O.; The final manuscript has been read and approved by all writers.

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8. Conflicts of interest

The authors declare no conflict of interest.

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