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Domain of Binomial Matrix in Some Spaces of Double Sequences

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Abstract.: The target of the existing paper is to acquaint the new spaces

$\mathcal{B}^{r,s}_{\infty}$, $\mathcal{B}^{r,s}$, $\mathcal{B}^{r,s}_{bp}$ and $\mathcal{B}^{r,s}_{reg}$ which consist of all double sequences whose binomial-transforms are in the spaces $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, respectively. Besides these, some properties of them are investigated and inclusion relations are given. Moreover, the $\alpha-$, $\beta(\vartheta)-$ and $\gamma-$duals are determined. Finally, some 4-dimensional matrix mapping classes are characterized.

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1. Introduction and Preliminaries

The term sequence has a great role in real and functional analysis. The generalization of single sequences are the double sequences. Every double sequence is an infinite matrix. Double sequences and various types of linear spaces of double sequences are constructed and studied their properties. The earlier work on double sequences is found in Browmich [6]. Furthermore, it was studied by Hardy [14], Moricz [16], Başarır and Sonalcan [2], Mursaleen and Mohiuddine [18], Mursaleen and Başar [19] and many others. Hardy [14] introduced the concept of regular convergence for double sequences. Hill [15] was the first who applied methods of functional analysis to double sequences. A good account of the study of double sequences can be found in monograph by Mursaleen and Mohiuddine [20].

At the beginning of the study, let us present some fundamental concepts which are going to be used in the rest of the article. The function F described with $F:\mathbb{N}\times\mathbb{N}\to\varpi$, $(i,j)\mapsto F(i,j)=u_{ij}$ is entitled as *double sequence*, where ϖ denotes any nonempty set and $\mathbb{N}=\{0,1,2,...\}$. $\mathbb{C}$ means the complex field. Ω stands for the linear space of all complex valued double sequences. Any linear subspace of Ω is entitled as *double sequence* *space*. The set of all bounded complex valued double sequences is symbolized by $\mathcal{M}_u$. It is said that $u = (u_{ij}) \in \Omega$ is *convergent* in the *Pringsheim's sense* provided that for every positive number ε, there exists $n_\varepsilon \in \mathbb{N}$ such that $|u_{ij} - L| < \varepsilon$ whenever $i, j > n_\varepsilon$. $L \in \mathbb{C}$ is called the *Pringsheim limit* of u and stated by $p - \lim_{i,j\to\infty} u_{ij} = L$. $\mathcal{C}_p$ represents the space of all such u which are called shortly as *p-convergent*. Of particular interest is unlike single sequences, p-convergence does not require boundedness in double sequences. If we take $u = (u_{ij}) \in \Omega$ identified by

$$u_{ij} = \begin{pmatrix} 0 & 1 & 2 & 3 & \cdots & j & \cdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 2 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 3 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \ldots \\ i & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \ldots \end{pmatrix},$$

it can easily seen that $p - \lim u_{ij} = 0$ but $\|u\|_\infty = \sup_{i,j\in\mathbb{N}} |u_{ij}| = \infty$. As a conclusion $u \in \mathcal{C}_p \backslash \mathcal{M}_u$. The bounded sequences which are also p-convergent are indicated by $\mathcal{C}_{bp}$, that is, $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. A double sequence $u = (u_{ij}) \in \mathcal{C}_p$ is called as *regularly convergent* (such sequences belong to $\mathcal{C}_r$) if the limits $u_i := \lim_j u_{ij}$, $(i \in \mathbb{N})$ and $u_j := \lim_i u_{ij}$, $(j \in \mathbb{N})$ exist, and the limits $\lim_i \lim_j u_{ij}$ and $\lim_j \lim_i u_{ij}$ exist and are equivalent to the $p - \lim$ of u. A sequence $u = (u_{ij})$ is called *double null sequence* if it converges to zero. It is known from Boos [5] and Móricz [16] that $\mathcal{M}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_r$ are Banach spaces with the norm $\|.\|_\infty$.

Let us take any $u \in \Omega$ and describe the sequence $S = (s_{kl})$ as

$$s_{kl} := \sum_{i=0}^{k} \sum_{j=0}^{l} u_{ij}, \quad (k, l \in \mathbb{N}).$$

Thus, the pair $((u_{kl}), (s_{kl}))$ is named as *double series*.

Let Ψ be a space of double sequences, converging with respect to some linear convergence rule $\vartheta - \lim : \Psi \to \mathbb{C}$. The sum of a double series $\sum_{i,j} u_{ij}$ relating to this rule is defined by $\vartheta - \sum_{i,j} u_{ij} = \vartheta - \lim_{k,l\to\infty} s_{kl}$. Here and thereafter, when needed we will use the summation $\sum_{i,j}$ instead of $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty}$, assume that $\vartheta \in \{p, bp, r\}$ and p' denotes the conjugate of p, that is, $p' = p/(p-1)$ for $1 < p < \infty$. With the notation of Zeltser [34], we describe the double sequences $e^{kl} = (e_{ij}^{kl})$ and e by $e_{ij}^{kl} = 1$ if $(k, l) = (i, j)$ and

$e_{ij}^{kl} = 0$ for other cases, and $e = \sum_{k,l} e^{kl}$ for every $i, j, k, l \in \mathbb{N}$. The sets

$$\begin{aligned}
\Psi^{\alpha} &:= \left\{ t = (t_{ij}) \in \Omega : \sum_{i,j} |t_{ij} u_{ij}| < \infty \quad \text{for all} \quad (u_{ij}) \in \Psi \right\}, \\
\Psi^{\beta(\vartheta)} &:= \left\{ t = (t_{ij}) \in \Omega : \vartheta - \sum_{i,j} t_{ij} u_{ij} \quad \text{exists for all} \quad (u_{ij}) \in \Psi \right\}, \\
\Psi^{\gamma} &:= \left\{ t = (t_{ij}) \in \Omega : \sup_{k,l \in \mathbb{N}} \left| \sum_{i,j=0}^{k,l} t_{ij} u_{ij} \right| < \infty \quad \text{for all} \quad (u_{ij}) \in \Psi \right\}
\end{aligned}$$

are $\alpha-$, $\beta(\vartheta)-$ and $\gamma-$*duals* of $\Psi \subset \Omega$, respectively. It is well known that if $\Psi \subset \Lambda$, then $\Lambda^{\alpha} \subset \Psi^{\alpha}$ and $\Psi^{\alpha} \subset \Psi^{\gamma}$ for the double sequence spaces Ψ and Λ.

Let us remember the definition of triangle matrix. If $d_{klij} = 0$ for $i > k$ or $j > l$ or both for every $k, l, i, j \in \mathbb{N}$, it is said that $D = (d_{klij})$ is a *triangular matrix* and also if $d_{klkl} \neq 0$ for every $k, l \in \mathbb{N}$, then the 4-dimensional matrix D is called *triangle*.

Now, we shall deal with the matrix mapping. Let us consider double sequence spaces Ψ and Λ and the 4-dimensional complex infinite matrix $D = (d_{klij})$. If for every $u = (u_{ij}) \in \Psi$

$$(Du)_{kl} = \vartheta - \sum_{i,j} d_{klij} u_{ij},$$

is exists and is in Λ, then it is said that D is a matrix mapping from Ψ into Λ and is written as $D : \Psi \to \Lambda$.

Let $(\Psi, \Lambda) = \{D = (d_{klij}) | D : \Psi \to \Lambda\}$. Here, $D \in (\Psi, \Lambda)$ iff $D_{kl} \in \Psi^{\beta(\vartheta)}$, where $D_{kl} = (d_{klij})_{i,j \in \mathbb{N}}$.

The domain $\Psi_D^{(\vartheta)}$ of D in a double sequence space Ψ consists of whose D-transforms are in Ψ is defined by the following way:

$$\Psi_D^{(\vartheta)} := \left\{ u = (u_{ij}) \in \Omega : Du := \left(\vartheta - \sum_{ij} d_{klij} u_{ij} \right)_{k,l \in \mathbb{N}} \text{exists and is in } \Psi \right\}. \tag{1. 1}$$

In the past, many authors were interested in double sequence spaces. Now, let us give some information about these studies. In her doctoral dissertation, Zeltser [33] has fundamentally examined the topological structure of double sequences.

Latterly, Başar and Sever [1] defined and examined

$$\mathcal{L}_p = \left\{ u = (u_{ij}) : \sum_{i,j} |u_{ij}|^p < \infty \right\},$$

which is a Banach space.

The space $\mathcal{L}_u$ which was defined by Zeltser [34] is the special case of $\mathcal{L}_p$ with $p = 1$.

Talebi [23] examined the space $\mathcal{E}_p^{r,s}$ for $1 \leq p < \infty$ and also Yeşilkayagil and Başar [32] examined for $0 < p < 1$ where $\mathcal{E}_p^{r,s} = \{u = (u_{ij}) : E(r, s)u \in \mathcal{L}_p\}$. Here, $E(r, s)$ indicates the double Euler matrix of orders r, s $(r, s \in (0, 1))$.

Tuğ and Başar [24] and Tuğ [25] have defined and examined some domains of the matrix $B(r,s,t,u)$.

For further information on double sequences the reader may refer to papers [1, 7, 8, 13, 16, 17, 21, 23, 24, 25, 28, 29, 30, 31, 32, 33, 34, 35] and references therein.

Bişgin [3, 4] have introduced the sequence spaces $b_0^{r,s}, b_c^{r,s}, b_p^{r,s}$ and $b_\infty^{r,s}$ of single sequences whose 2-dimensional binomial matrix $B^{r,s}$-transforms are convergent to zero, convergent, absolutely p-summable and bounded, respectively. Our main purpose in this study is to define the 4-dimensional Binomial matrix and to examine the matrix domains of this matrix on classical double sequence spaces.

Let us give a brief for the rest of the study: The second section dedicated for the production and examination of the new spaces. In section 3, we will try to state duals of the new double binomial sequence spaces. Finally, it will be given some classes of matrix mapping.

2. The new double binomial sequence spaces

In the current section, we acquaint the double sequence spaces $\mathcal{B}_\infty^{r,s}$, $\mathcal{B}^{r,s}$, $\mathcal{B}_{bp}^{r,s}$ and $\mathcal{B}_{reg}^{r,s}$ by using the 4-dimensional binomial matrix $B^{(r,s)}$.

Let r, s and $r+s$ are nonzero real numbers. We describe the 4-dimensional binomial matrix $B^{(r,s)} = (b_{klij}^{r,s})$ of orders r, s as follows:

$$b_{klij}^{r,s} := \begin{cases} \frac{1}{(r+s)^{k+l}} \binom{k}{i} \binom{l}{j} s^{k+j-i} r^{l+i-j} & , \quad 0 \le i \le k \ , \ 0 \le j \le l, \\ \\ 0 & , \quad \text{otherwise}, \end{cases} \tag{2. 2}$$

for every $k, l, i, j \in \mathbb{N}$, so the $B^{(r,s)}$-transform $u = (u_{ij}) \in \Omega$ is stated with

$$\nu_{kl} := (B^{(r,s)}u)_{kl} = \sum_{i,j}^{k,l} \frac{1}{(r+s)^{k+l}} \binom{k}{i} \binom{l}{j} s^{k+j-i} r^{l+i-j} u_{ij}. \tag{2. 3}$$

It will be assume unless stated otherwise that the double sequences (u_{ij}) and (ν_{ij}) are as in the equality (2. 3) and r, s and $r+s$ are nonzero real numbers. We would like touch on a point, while $r+s=1$, it is obtained 4-dimensional Euler matrix $E(r,s) = (e_{klij}^{r,s})$ from the 4-dimensional binomial matrix. So, $B^{(r,s)}$ is more general then $E(r,s)$. Consider that the 4-dimensional unit matrix $I = (\delta_{klij})$ defined by

$$\delta_{klij} = \begin{cases} 1 & , \quad (k,l) = (i,j), \\ \\ 0 & , \quad \text{otherwise}. \end{cases}$$

From the equality

$$\delta_{klij} = \sum_{m,n} b_{klmn}^{r,s} . c_{mnij}^{r,s},$$

the inverse $\{B^{(r,s)}\}^{-1} = C^{(r,s)} = (c_{klij}^{r,s})$ is found as follows:

$$c_{klij}^{r,s} := \begin{cases} (-1)^{k+l-(i+j)} \binom{k}{i} \binom{l}{j} s^{k-l-i} r^{l-k-j} (r+s)^{i+j} & , \quad 0 \le i \le k, 0 \le j \le l, \\ \\ 0 & , \quad \text{otherwise}, \end{cases}$$

for every $k, l, i, j \in \mathbb{N}$. Now, we may acquaint the spaces $\mathcal{B}_{\infty}^{r,s}$, $\mathcal{B}^{r,s}$, $\mathcal{B}_{bp}^{r,s}$ and $\mathcal{B}_{reg}^{r,s}$ described by the following way:

$$\begin{aligned}
\mathcal{B}_{\infty}^{r,s} &= \left\{u = (u_{ij}) \in \Omega : \sup_{k,l\in\mathbb{N}} \left|(B^{(r,s)}u)_{kl}\right| < \infty\right\}, \\
\mathcal{B}^{r,s} &= \left\{u = (u_{ij}) \in \Omega : \exists L \in \mathbb{C}, p - \lim_{k,l\to\infty} \left|(B^{(r,s)}u)_{kl} - L\right| = 0\right\}, \\
\mathcal{B}_{bp}^{r,s} &= \left\{u = (u_{ij}) \in \Omega : B^{(r,s)}u \in \mathcal{C}_{bp}\right\}, \\
\mathcal{B}_{reg}^{r,s} &= \left\{u = (u_{ij}) \in \Omega : B^{(r,s)}u \in \mathcal{C}_{r}\right\}.
\end{aligned}$$

$\mathcal{B}_{\infty}^{r,s}$, $\mathcal{B}^{r,s}$, $\mathcal{B}_{bp}^{r,s}$ and $\mathcal{B}_{reg}^{r,s}$ can be rewritten as $\mathcal{B}_{\infty}^{r,s} = (\mathcal{M}_u)_{B^{(r,s)}}$, $\mathcal{B}^{r,s} = (\mathcal{C}_p)_{B^{(r,s)}}$, $\mathcal{B}_{bp}^{r,s} = (\mathcal{C}_{bp})_{B^{(r,s)}}$ and $\mathcal{B}_{reg}^{r,s} = (\mathcal{C}_r)_{B^{(r,s)}}$, respectively, with the notation of (1. 1).

Here, $\Psi_{B^{(r,s)}}$ is entitled as double binomial sequence space.

Definition 2.1 ([13],[22]). *$D = (d_{klij})$ is entitled RH regular if D transforms all bounded p-convergent sequence into a p-convergent sequence with the same p-limit.*

Lemma 2.2 ([13],[22]). *A triangle $D = (d_{klij})$ is RH regular iff*

$$\begin{aligned}
RH_1 &: \quad p - \lim_{k,l\to\infty} d_{klij} = 0 \quad \textit{for each} \quad i, j \in \mathbb{N}, \\
RH_2 &: \quad p - \lim_{k,l\to\infty} \sum_{i,j} d_{klij} = 1, \\
RH_3 &: \quad p - \lim_{k,l\to\infty} \sum_{i} |d_{klij}| = 0 \quad \textit{for each} \quad j \in \mathbb{N}, \\
RH_4 &: \quad p - \lim_{k,l\to\infty} \sum_{j} |d_{klij}| = 0 \quad \textit{for each} \quad i \in \mathbb{N}, \\
RH_5 &: \quad \sum_{i,j>\mu} |d_{klij}| < \eta, \quad \textit{where } \mu \textit{ and } \eta \textit{ finite positive integers}.
\end{aligned}$$

We would like touch on a point $B^{(r,s)}$ described by (2. 2) is RH regular whenever $r.s > 0$. In the rest of the article, it will be assumed that $r.s > 0$.

Theorem 2.3. *The sets $\mathcal{B}_{\infty}^{r,s}$, $\mathcal{B}_{bp}^{r,s}$ and $\mathcal{B}_{reg}^{r,s}$ are linearly norm isomorphic to the spaces $\mathcal{M}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, respectively, and are Banach spaces with the norm*

$$\|u\|_{\mathcal{B}_{\infty}^{r,s}} = \|B^{(r,s)}u\|_{\infty} = \sup_{k,l\in\mathbb{N}} \left|(B^{(r,s)}u)_{kl}\right|. \tag{2. 4}$$

Proof. Here, this theorem will be proved only for $\mathcal{B}_{\infty}^{r,s}$.

To confirm the fact that $\mathcal{B}_{\infty}^{r,s}$ is linearly norm isomorphic to the space $\mathcal{M}_u$, we need to be sure there is a linear bijection $T : \mathcal{B}_{\infty}^{r,s} \to \mathcal{M}_u$, $u \mapsto \nu = Tu$ which preserves norm. If we take $T = B^{(r,s)}$, it is precisely linear. Consider the equality $Tu =$

$$\begin{bmatrix} u_{00} & \vdots & \sum_{j=0}^{l} \frac{1}{(r+s)^l} \binom{l}{j} s^j r^{l-j} u_{0j} & \vdots \\ \frac{su_{00}+ru_{10}}{r+s} & \vdots & \sum_{i=0,j=0}^{1,l} \frac{1}{(r+s)^{1+l}} \binom{1}{i} \binom{l}{j} s^{1+j-i} r^{l+i-j} u_{ij} & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=0}^{k} \frac{1}{(r+s)^k} \binom{k}{i} s^{k-i} r^i u_{ij} & \vdots & \sum_{i=0,j=0}^{k,l} \frac{1}{(r+s)^{k+l}} \binom{k}{i} \binom{l}{j} s^{k+j-i} r^{l+i-j} u_{ij} & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$= \theta$ which yields us that $u = \theta$(injectivity). Consider $\nu \in \mathcal{M}_u$ and

$$\begin{aligned} u_{kl} &= \{(B^{(r,s)})^{-1}\nu\}_{kl} \\ &= \sum_{i,j=0}^{k,l} (-1)^{k+l-(i+j)} \binom{k}{i} \binom{l}{j} s^{k-l-i} r^{l-k-j} (r+s)^{i+j} \nu_{ij} \qquad (2.\ 5) \end{aligned}$$

for every $k, l = 0, 1, ...$ Hence, it can be easily seen by the following equality

$$\begin{aligned} \|u\|_{\mathcal{B}_\infty^{r,s}} &= \|B^{(r,s)}u\|_\infty \\ &= \sup_{k,l\in\mathbb{N}} \left| \sum_{i,j=0}^{k,l} \frac{1}{(r+s)^{k+l}} \binom{k}{i} \binom{l}{j} s^{k+j-i} r^{l+i-j} u_{ij} \right| \\ &= \sup_{k,l\in\mathbb{N}} \left| \sum_{i,j=0}^{k,l} \delta_{klij} \nu_{ij} \right| \\ &= \sup_{k,l\in\mathbb{N}} |\nu_{kl}| = \|\nu\|_\infty < \infty \end{aligned}$$

that $u \in \mathcal{B}_\infty^{r,s}$(surjectivity) and T preserved the norm. Thus, the initial assertion of the theorem has been proved. From the Corollary 6.3.41 in [5]: "Let (Ψ, p) and (Λ, q) be semi-normed spaces and $T : (\Psi, p) \to (\Lambda, q)$ be an isometric isomorphism. Then, (Ψ, p) is complete if and only if (Λ, q) is complete. In particular, (Ψ, p) is a Banach space if and only if (Λ, q) is a Banach space", we reach what we want. □

Theorem 2.4. *The set $\mathcal{B}^{r,s}$ is linearly isomorphic to the space $\mathcal{C}_p$ and is a complete semi-normed space with the semi-norm*

$$\|u\|_{\mathcal{B}^{r,s}} = \lim_{i\to\infty} \left[\sup_{k,l\geq i} \left| (B^{(r,s)}u)_{kl} \right| \right].$$

Proof. It is analogous with previous one, so we ignore it. □

Now, we shall give our results about inclusion relations.

Theorem 2.5. *$\mathcal{M}_u \subset \mathcal{B}_\infty^{r,s}$ strictly holds.*

Proof. Suppose that $u = (u_{ij}) \in \mathcal{M}_u$ is an arbitrary double sequence. At that time, $\|u\|_\infty = \sup_{i,j\in\mathbb{N}} |u_{ij}| \leq \eta$, where $\eta \in \mathbb{R}^+$. Thus, it can be immediately seen from the inequality

$$\|u\|_{\mathcal{B}_\infty^{r,s}} = \sup_{k,l\in\mathbb{N}} \left| \frac{1}{(r+s)^{k+l}} \sum_{i,j=0}^{k,l} \binom{k}{i}\binom{l}{j} s^{k+j-i} r^{l+i-j} u_{ij} \right| \leq \|u\|_\infty$$

that the inclusion $\mathcal{M}_u \subset \mathcal{B}_\infty^{r,s}$ is valid. To show that the inclusion is strict, let us consider the sequence $u = (u_{kl}) = \frac{(-s-r)^{k+l}}{r^k s^l}$ for every $k, l \in \mathbb{N}$. Then, $u = (u_{kl}) \notin \mathcal{M}_u$ but $B^{(r,s)}u = \frac{(-1)^{k+l} r^k s^l}{(r+s)^{k+l}} \in \mathcal{M}_u$, that is $u = (u_{kl}) \in \mathcal{B}_\infty^{r,s}$. □

Theorem 2.6. $\mathcal{C}_{bp} \subset \mathcal{B}^{r,s}$.

Proof. Consider the sequence $u = (u_{ij}) \in \mathcal{C}_{bp}$ with $p - \lim_{i,j\to\infty} u_{ij} = L$. Since the 4-dimensional binomial matrix is RH-regular, then $p - \lim_{k,l\to\infty} \nu_{kl} = L$, where $\nu = (\nu_{kl}) = (B^{(r,s)}u)_{kl}$. Hence, we see that $\mathcal{C}_{bp} \subset \mathcal{B}^{r,s}$, as desired. □

3. Duals

Current section is dedicated with $\{\mathcal{B}_\infty^{r,s}\}^{\kappa_1}$, $\left\{\mathcal{B}_{bp}^{r,s}\right\}^{\kappa_2}$, $\{\mathcal{B}^{r,s}\}^{\beta(\vartheta)}$ and $\{\mathcal{B}_{reg}^{r,s}\}^{\beta(\vartheta)}$, where $\kappa_1 \in \{\alpha, \beta(bp), \beta(p), \gamma\}$ and $\kappa_2 \in \{\beta(\vartheta), \gamma\}$.

Theorem 3.1. $\{\mathcal{B}_\infty^{r,s}\}^\alpha = \mathcal{L}_u$.

Proof. Let us take $t = (t_{kl}) \in \mathcal{L}_u$ and $u = (u_{kl}) \in \mathcal{B}_\infty^{r,s}$. In that case, $\nu = (\nu_{kl}) \in \mathcal{M}_u$ from the equality (2. 3) and $\sup_{k,l\in\mathbb{N}} |\nu_{kl}| < \eta$, where $\eta \in \mathbb{R}^+$. Therefore, it is clear with the following

$$\begin{aligned}
\sum_{k,l} |t_{kl}u_{kl}| &= \sum_{k,l} |t_{kl}| \left| \sum_{i,j=0}^{k,l} (-1)^{k+l-(i+j)} \binom{k}{i}\binom{l}{j} s^{k-l-i} r^{l-k-j} (r+s)^{i+j} \nu_{ij} \right| \\
&\leq \sum_{k,l} |t_{kl}| \left| \frac{1}{r^k s^l} \sum_{i,j=0}^{k,l} \binom{k}{i}\binom{l}{j} (-s)^{k-i}(r+s)^i(-r)^{l-j}(r+s)^j \right| |\nu_{ij}| \\
&\leq \eta \sum_{k,l} |t_{kl}| \left| \frac{1}{r^k s^l} \sum_{i=0}^{k} \binom{k}{i} (-s)^{k-i}(r+s)^i \sum_{j=0}^{l} \binom{l}{j} (-r)^{l-j}(r+s)^j \right| \\
&= \eta \sum_{k,l} |t_{kl}|
\end{aligned}$$

that $t = (t_{kl}) \in \{\mathcal{B}_\infty^{r,s}\}^\alpha$. Hence, the inclusion $\mathcal{L}_u \subset \{\mathcal{B}_\infty^{r,s}\}^\alpha$ is valid.

For the sufficiency part, let us assume the sequence $t = (t_{kl}) \in \{\mathcal{B}_\infty^{r,s}\}^\alpha - \mathcal{L}_u$. So, $\sum_{k,l} |t_{kl}u_{kl}| < \infty$ for every $u = (u_{kl}) \in \mathcal{B}_\infty^{r,s}$. If we consider $e \in \mathcal{B}_\infty^{r,s}$, then it is clear that $te = t \notin \mathcal{L}_u$, that is $t \notin \{\mathcal{B}_\infty^{r,s}\}^\alpha$. It is a contradiction. Thus, it is seen that $\{\mathcal{B}_\infty^{r,s}\}^\alpha \subset \mathcal{L}_u$ and this completes the proof. □

Now, we give some lemmas which will be used in both this and next sections.(see [13], [34] and [35]).

Lemma 3.2. *Suppose that* $D = (d_{klij})$ *be a 4-dimensional infinite matrix. Then,* $D = (d_{klij}) \in (\mathcal{C}_{bp}, \mathcal{C}_{\vartheta})$ *if and only if following conditions hold:*

$$\sup_{k,l\in\mathbb{N}} \sum_{i,j} |d_{klij}| < \infty, \tag{3. 6}$$

$$\exists d_{ij} \in \mathbb{C} \ni \vartheta - \lim_{k,l\to\infty} d_{klij} = d_{ij} \textit{ for all } \quad i,j \in \mathbb{N}, \tag{3. 7}$$

$$\exists L \in \mathbb{C} \ni \vartheta - \lim_{k,l\to\infty} \sum_{i,j} d_{klij} = L \quad \textit{exists}, \tag{3. 8}$$

$$\exists i_0 \in \mathbb{N} \ni \vartheta - \lim_{k,l\to\infty} \sum_{j} |d_{kli_0j} - d_{i_0j}| = 0, \tag{3. 9}$$

$$\exists j_0 \in \mathbb{N} \ni \vartheta - \lim_{k,l\to\infty} \sum_{i} |d_{klij_0} - d_{ij_0}| = 0. \tag{3. 10}$$

In the case of (3. 10), $d = (d_{ij}) \in \mathcal{L}_u$ *and*

$$\vartheta - \lim_{k,l\to\infty} [Du]_{kl} = \sum_{i,j} d_{ij}u_{ij} + \left(L - \sum_{i,j} d_{ij}\right) bp - \lim_{k,l\to\infty} u_{kl}$$

satisfies for $u \in \mathcal{C}_{bp}$.

Lemma 3.3. *Suppose that* $D = (d_{klij})$ *be a 4-dimensional infinite matrix. Then,* $D = (d_{klij}) \in (\mathcal{C}_p, \mathcal{C}_{\vartheta})$ *if and only if (3. 6)-(3. 8) and followings are provided:*

$$\forall i \in \mathbb{N}, \quad \exists j_0 \in \mathbb{N} \ni d_{klij} = 0 \quad \textit{for every} \quad j > j_0 \quad \textit{and} \quad k,l \in \mathbb{N}, \tag{3. 11}$$

$$\forall j \in \mathbb{N}, \quad \exists i_0 \in \mathbb{N} \ni d_{klij} = 0 \quad \textit{for every} \quad i > i_0 \quad \textit{and} \quad k,l \in \mathbb{N}. \tag{3. 12}$$

In the case of (3. 12), $\exists i_0, j_0 \in \mathbb{N}$ *such that* $d = (d_{ij}) \in \mathcal{L}_u$ *and* $(d_{ij_0})_{i\in\mathbb{N}}$, $(d_{i_0j})_{j\in\mathbb{N}} \in \wp$, *where* $\wp$ *represents the space of every finitely sequences which are non-equivalent zero and*

$$\vartheta - \lim_{k,l\to\infty} [Du]_{kl} = \sum_{i,j} d_{ij}u_{ij} + \sum_{i} \left(L - \sum_{i,j} d_{ij}\right) p - \lim_{k,l\to\infty} u_{kl}$$

satisfies for $u \in \mathcal{C}_p$.

Lemma 3.4. *Suppose that* $D = (d_{klij})$ *be a 4-dimensional infinite matrix. Then,* $D = (d_{klij}) \in (\mathcal{C}_r, \mathcal{C}_{\vartheta})$ *if and only if (3. 6)-(3. 8) and followings are provided:*

$$\exists j_0 \in \mathbb{N} \ni \vartheta - \lim_{k,l\to\infty} \sum_{i} d_{klij_0} = x_{j_0}, \tag{3. 13}$$

$$\exists i_0 \in \mathbb{N} \ni \vartheta - \lim_{k,l\to\infty} \sum_{j} d_{kli_0j} = y_{i_0}. \tag{3. 14}$$

In the case of (3. 14), $d = (d_{ij}) \in \mathcal{L}_u$ and $x_j, y_i \in \ell_1$ and

$$\begin{aligned}\vartheta - \lim_{k,l\to\infty}[Du]_{kl} &= \sum_{i,j} d_{ij}u_{ij} + \sum_i \left(y_i - \sum_j d_{ij}\right)u_i + \sum_j \left(x_j - \sum_i d_{ij}\right)u_j \\ &+ \left(L + \sum_{i,j} d_{ij} - \sum_i y_i - \sum_j x_j\right) r - \lim_{k,l\to\infty} u_{kl}\end{aligned}$$

satisfies for $u \in \mathcal{C}_r$.

Lemma 3.5. [25] *Suppose that $D = (d_{klij})$ be a 4-dimensional infinite matrix. Then, $D = (d_{klij}) \in (\mathcal{C}_{bp}, \mathcal{M}_u)$ if and only if the condition (3. 6) hold.*

Lemma 3.6. [7] *Suppose that $D = (d_{klij})$ be a 4-dimensional infinite matrix. Then, $D = (d_{klij}) \in (\mathcal{M}_u, \mathcal{C}_{bp})$ if and only if the conditions (3. 6), (3. 7) and followings are provided:*

$$\exists d_{ij} \in \mathbb{C} \ni bp - \lim_{k,l\to\infty} \sum_{ij} |d_{klij} - d_{ij}| = 0, \tag{3. 15}$$

$$bp - \lim_{k,l\to\infty} \sum_{j=0}^{l} d_{klij} \quad \textit{exists for each} \quad i \in \mathbb{N}, \tag{3. 16}$$

$$bp - \lim_{k,l\to\infty} \sum_{i=0}^{k} d_{klij} \quad \textit{exists for each} \quad j \in \mathbb{N}, \tag{3. 17}$$

$$\sum_{i,j} |d_{klij}| \quad \textit{converges}. \tag{3. 18}$$

Lemma 3.7. [28] *Suppose that $D = (d_{klij})$ be a 4-dimensional infinite matrix. Then, $D = (d_{klij}) \in (\mathcal{M}_u, \mathcal{M}_u)$ if and only if the condition (3. 6) hold.*

Lemma 3.8. [29] *Suppose that $D = (d_{klij})$ be a 4-dimensional infinite matrix. Then, $D = (d_{klij}) \in (\mathcal{M}_u, \mathcal{C}_p)$ if and only if the conditions (3. 7), (3. 11) and (3. 12) hold.*

Lemma 3.9. [30] *Suppose that $D = (d_{klij})$ be a 4-dimensional infinite matrix. In that case:*

(i) *If $0 < p \leq 1$, then $D \in (\mathcal{L}_p, \mathcal{M}_u)$ if and only if*

$$\sup_{k,l,i,j\in\mathbb{N}} |d_{klij}| < \infty, \tag{3. 19}$$

(ii) *If $1 < p < \infty$, then $D \in (\mathcal{L}_p, \mathcal{M}_u)$ if and only if*

$$\sup_{k,l\in\mathbb{N}} \sum_{i,j} |d_{klij}|^{p'} < \infty. \tag{3. 20}$$

Lemma 3.10. [30] *Suppose that $D = (d_{klij})$ be a 4-dimensional infinite matrix. In that case:*

(i) *If $0 < p \leq 1$, then $D \in (\mathcal{L}_p, \mathcal{C}_{bp})$ if and only if the conditions (3. 7) and (3. 19) hold with $\vartheta = bp$,*

(ii) *If* $1 < p < \infty$*, then* $D \in (\mathcal{L}_p, \mathcal{C}_{bp})$ *if and only if the conditions (3. 7) and (3. 20) hold with* $\vartheta = bp$*.*

Consider the sets $w_1 - w_{13}$ defined by

$$
\begin{aligned}
w_1 &= \left\{t = (t_{ij}) \in \Omega : \sup_{k,l\in\mathbb{N}} \sum_{i,j} |\chi(k,l,i,j,m,n)| < \infty\right\}, \\
w_2 &= \left\{t = (t_{ij}) \in \Omega : \exists d_{ij} \in \mathbb{C} \ni \vartheta - \lim_{k,l\to\infty} \chi(k,l,i,j,m,n) = d_{ij}\right\}, \\
w_3 &= \left\{t = (t_{ij}) \in \Omega : \exists L \in \mathbb{C} \ni \vartheta - \lim_{k,l\to\infty} \sum_{i,j} \chi(k,l,i,j,m,n) = L \quad \text{exists}\right\}, \\
w_4 &= \left\{t = (t_{ij}) \in \Omega : \exists j_0 \in \mathbb{N} \ni \vartheta - \lim_{k,l\to\infty} \sum_{i} |\chi(k,l,i,j_0,m,n) - d_{ij_0}| = 0\right\}, \\
w_5 &= \left\{t = (t_{ij}) \in \Omega : \exists i_0 \in \mathbb{N} \ni \vartheta - \lim_{k,l\to\infty} \sum_{j} |\chi(k,l,i_0,j,m,n) - d_{i_0 j}| = 0\right\}, \\
w_6 &= \left\{t = (t_{ij}) \in \Omega : \forall i \in \mathbb{N}, \exists j_0 \in \mathbb{N} \ni \chi(k,l,i,j,m,n) = 0, \forall j > j_0, \forall k,l \in \mathbb{N}\right\}, \\
w_7 &= \left\{t = (t_{ij}) \in \Omega : \forall j \in \mathbb{N}, \exists i_0 \in \mathbb{N} \ni \chi(k,l,i,j,m,n) = 0, \forall i > i_0, \forall k,l \in \mathbb{N}\right\}, \\
w_8 &= \left\{t = (t_{ij}) \in \Omega : \exists j_0 \in \mathbb{N} \ni \vartheta - \lim_{k,l\to\infty} \sum_{i} \chi(k,l,i,j_0,m,n) = d_{j_0}\right\}, \\
w_9 &= \left\{t = (t_{ij}) \in \Omega : \exists i_0 \in \mathbb{N} \ni \vartheta - \lim_{k,l\to\infty} \sum_{j} \chi(k,l,i_0,j,m,n) = d_{i_0}\right\}, \\
w_{10} &= \left\{t = (t_{ij}) \in \Omega : \exists d_{ij} \in \mathbb{C} \ni bp - \lim_{k,l\to\infty} \sum_{i,j} |\chi(k,l,i,j,m,n) - d_{ij}| = 0\right\}, \\
w_{11} &= \left\{t = (t_{ij}) \in \Omega : \forall i \in \mathbb{N} \ni bp - \lim_{k,l\to\infty} \sum_{j=0}^{l} \chi(k,l,i,j,m,n) \quad \text{exists}\right\}, \\
w_{12} &= \left\{t = (t_{ij}) \in \Omega : \forall j \in \mathbb{N} \ni bp - \lim_{k,l\to\infty} \sum_{i=0}^{k} \chi(k,l,i,j,m,n) \quad \text{exists}\}, \right. \\
w_{13} &= \left\{t = (t_{ij}) \in \Omega : \sum_{i,j} |\chi(k,l,i,j,m,n)| \quad \text{converges}\right\},
\end{aligned}
$$

where

$$\chi(k,l,i,j,m,n)=\sum_{m=i}^{k}\sum_{n=j}^{l}(-1)^{m+n-(i+j)}\binom{m}{i}\binom{n}{j}s^{m-n-i}r^{n-m-j}(r+s)^{i+j}t_{mn}.$$

Theorem 3.11. *Following statements are satisfied:*

$$(i)\qquad \left\{\mathcal{B}_{bp}^{r,s}\right\}^{\beta(\vartheta)}=\bigcap_{k=1}^{5}w_k,$$

$$(ii)\qquad \{\mathcal{B}^{r,s}\}^{\beta(\vartheta)}=\bigcap_{k=1}^{3}w_k\cap w_6\cap w_7,$$

$$(iii)\qquad \left\{\mathcal{B}_{reg}^{r,s}\right\}^{\beta(\vartheta)}=\bigcap_{k=1}^{3}w_k\cap w_8\cap w_9,$$

$$(iv)\qquad \{\mathcal{B}_{\infty}^{r,s}\}^{\beta(bp)}=w_1\cap w_2\bigcap_{k=10}^{13}w_k,$$

$$(v)\qquad \{\mathcal{B}_{\infty}^{r,s}\}^{\beta(p)}=w_2\cap w_6\cap w_7,$$

$$(vi)\qquad \{\mathcal{B}_{\infty}^{r,s}\}^{\gamma}=w_1,$$

$$(vii)\qquad \left\{\mathcal{B}_{bp}^{r,s}\right\}^{\gamma}=w_1.$$

Proof. (i) Suppose that $t=(t_{kl})\in\Omega$ and $u=(u_{kl})\in\mathcal{B}_{bp}^{r,s}$. Then, we can conclude from (2. 3) that $\nu=(\nu_{kl})\in\mathcal{C}_{bp}$. Now, let us define the 4-dimensional matrix $O^{r,s}=\left(o_{klij}^{r,s}\right)$ by

$$o_{klij}^{r,s}=\begin{cases}\chi(k,l,i,j,m,n) & ,\quad 0\le i\le k,0\le j\le l,\\ 0 & ,\quad \text{otherwise},\end{cases}$$

for every $k,l,i,j\in\mathbb{N}$. Therefore, we obtain by the relation (2. 5) that,

$$\begin{aligned}
z_{kl} &= \sum_{i,j=0}^{k,l}t_{ij}u_{ij}\\
&= \sum_{i,j=0}^{k,l}t_{ij}\left\{\sum_{m,n=0}^{i,j}(-1)^{i+j-(m+n)}\binom{i}{m}\binom{j}{n}s^{i-j-m}r^{j-i-n}(r+s)^{m+n}\nu_{mn}\right\}\\
&= \sum_{i,j=0}^{k,l}\left\{\sum_{m=i}^{k}\sum_{n=j}^{l}(-1)^{m+n-(i+j)}\binom{m}{i}\binom{n}{j}s^{m-n-i}r^{n-m-j}(r+s)^{i+j}t_{mn}\right\}\nu_{ij}\\
&= (O^{r,s}\nu)_{kl} \qquad (3.\ 21)
\end{aligned}$$

for every $k,l\in\mathbb{N}$. In that case, we conclude from (3. 21) that $tu=(t_{kl}u_{kl})\in\mathcal{CS}_{\vartheta}$ whenever $u=(u_{kl})\in\mathcal{B}_{bp}^{r,s}$ iff $z=(z_{kl})\in\mathcal{C}_{\vartheta}$ whenever $\nu=(\nu_{kl})\in\mathcal{C}_{bp}$. This implies that $t = (t_{kl}) \in \left(\mathcal{B}_{bp}^{r,s}\right)^{\beta(\vartheta)}$ if and only if $O^{r,s} \in (\mathcal{C}_{bp}, \mathcal{C}_\vartheta)$ and the proof is completed in view of Lemma 3.2.

The other parts of the theorem can be done analogously by using the Lemmas 3.3, 3.4, 3.6, 3.8, 3.7 and 3.5, respectively. So, we pass them. □

4. Some matrix classes

Theorem 4.1. *Assume that $D = (d_{klij})$ and $H = (h_{klij})$ be as follows:*

$$h_{klij} = \sum_{a=i}^{\infty}\sum_{b=j}^{\infty}(-1)^{a+b-(i+j)}\binom{a}{i}\binom{b}{j}s^{a-b-i}r^{b-a-j}(r+s)^{i+j}d_{klab}. \tag{4. 22}$$

Then, $D \in (\Psi_{B(r,s)}, \Lambda)$ if and only if $D_{kl} \in [\Psi_{B(r,s)}]^{\beta(\vartheta)}$ for every $k,l \in \mathbb{N}$ and $H \in (\Psi, \Lambda)$, where Ψ and $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r\}$.

Proof. Let $D \in (\Psi_{B(r,s)}, \Lambda)$. Then, Du exists and is in Λ for every $u \in \Psi_{B(r,s)}$, which implies that the fact that $D_{kl} \in [\Psi_{B(r,s)}]^{\beta(\vartheta)}$ for every $k,l \in \mathbb{N}$. So, by bearing in mind (2. 5), the relation

$$\begin{aligned}\sum_{i,j}^{k,l} d_{klij}u_{ij} &= \sum_{i,j}^{k,l} d_{klij}\sum_{a,b=0}^{i,j}(-1)^{i+j-(a+b)}\binom{i}{a}\binom{j}{b}s^{i-j-a}r^{j-i-b}(r+s)^{a+b}\nu_{ab}\\ &= \sum_{i,j}^{k,l}\left[\sum_{a,b=i,j}^{k,l}(-1)^{a+b-(i+j)}\binom{a}{i}\binom{b}{j}s^{a-b-i}r^{b-a-j}(r+s)^{i+j}d_{klab}\right]\nu_{ij}\end{aligned} \tag{4. 23}$$

is obtained for every $k,l \in \mathbb{N}$. Then, by taking ϑ-limit on (4. 23) while $k,l \to \infty$, we have $Du = H\nu$. Therefore, we obtain that $H\nu \in \Lambda$ whenever $\nu \in \Psi$, that is $H \in (\Psi, \Lambda)$.

For the sufficiency part, assume the sequence $D_{kl} \in [\Psi_{B(r,s)}]^{\beta(\vartheta)}$ for every $k,l \in \mathbb{N}$, $H \in (\Psi, \Lambda)$ and $u = (u_{ij}) \in \Psi_{B(r,s)}$ such that $\nu = B^{(r,s)}u$. Then, Du exists and therefore, rectangular partial sums for $\sum_{i,j} d_{klij}u_{ij}$ obtained as

$$\begin{aligned}(Du)_{kl}^{[m,n]} &= \sum_{i,j}^{m,n} d_{klij}u_{ij}\\ &= \sum_{i,j}^{m,n}\left[\sum_{a,b=i,j}^{m,n}(-1)^{a+b-(i+j)}\binom{a}{i}\binom{b}{j}s^{a-b-i}r^{b-a-j}(r+s)^{i+j}d_{klab}\right]\nu_{ij}\end{aligned} \tag{4. 24}$$

for every $k,l,m,n \in \mathbb{N}$. By taking ϑ-limit in (4. 24) while $m,n \to \infty$, it can be easily obtain from the following equality

$$\sum_{i,j} d_{klij}u_{ij} = \sum_{i,j} h_{klij}\nu_{ij}$$

for every $k,l \in \mathbb{N}$ that $Du = H\nu$ which leads to fact that $D \in (\Psi_{B(r,s)}, \Lambda)$. □

Corollary 4.2. *Suppose that $D = (d_{klij})$ be a 4-dimensional matrix. In that case the following statements are satisfied:*

(i) $D \in (\mathcal{B}^{r,s}, \mathcal{C}_\vartheta)$ *if and only if the conditions (3. 6)-(3. 8), (3. 11) and (3. 12) are satisfied with h_{klij} in place of d_{klij},*

(ii) $D \in \left(\mathcal{B}^{r,s}_{bp}, \mathcal{C}_\vartheta\right)$ *if and only if the conditions (3. 6)-(3. 10) are satisfied with h_{klij} in place of d_{klij},*

(iii) $D \in \left(\mathcal{B}^{r,s}_{bp}, \mathcal{M}_u\right)$ *if and only if the condition (3. 6) is satisfied with h_{klij} in place of d_{klij},*

(iv) $D \in \left(\mathcal{B}^{r,s}_{reg}, \mathcal{C}_\vartheta\right)$ *if and only if the conditions (3. 6)-(3. 8), (3. 13) and (3. 14) are satisfied with h_{klij} in place of d_{klij},*

(v) $D \in (\mathcal{B}^{r,s}_\infty, \mathcal{C}_{bp})$ *if and only if the conditions (3. 6), (3. 7), (3. 15)-(3. 18) are satisfied with h_{klij} in place of d_{klij},*

(vi) $D \in (\mathcal{B}^{r,s}_\infty, \mathcal{C}_p)$ *if and only if the conditions (3. 7), (3. 11) and (3. 12) are satisfied with h_{klij} in place of d_{klij}.*

Lemma 4.3. [30] *Let $B = (b_{klij})$ be a triangle matrix. In that case, $D = (d_{klij}) \in (\Psi, \Lambda_B)$ if and only if $BD \in (\Psi, \Lambda)$.*

Now, let us define the 4-dimensional matrix $G = (g_{klij})$ by

$$g_{klij} = \sum_{m,n=0}^{k,l} b^{r,s}_{klmn} d_{mnij}$$

for every $k, l, i, j \in \mathbb{N}$ and give following corollary.

Corollary 4.4. *Suppose that $D = (d_{klij})$ be a 4-dimensional matrix. In that case the following statements are satisfied:*

(i) $D \in (\mathcal{C}_p, (\mathcal{C}_\vartheta)_{B(r,s)})$ *if and only if the conditions (3. 6)-(3. 8), (3. 11) and (3. 12) are satisfied with g_{klij} in place of d_{klij},*

(ii) $D \in (\mathcal{C}_{bp}, (\mathcal{C}_\vartheta)_{B(r,s)})$ *if and only if the conditions (3. 6)-(3. 10) are satisfied with g_{klij} in place of d_{klij},*

(iii) $D \in (\mathcal{C}_r, (\mathcal{C}_\vartheta)_{B(r,s)})$ *if and only if the conditions (3. 6)-(3. 8), (3. 13) and (3. 14) are satisfied with g_{klij} in place of d_{klij},*

(iv) $D \in \left(\mathcal{L}_p, \mathcal{B}^{r,s}_{bp}\right)$ *if and only if the conditions (3. 7) and (3. 19) are satisfied for $0 < p \leq 1$ and $\vartheta = bp$ with g_{klij} in place of d_{klij},*

(v) $D \in \left(\mathcal{L}_p, \mathcal{B}^{r,s}_{bp}\right)$ *if and only if the conditions (3. 7) and (3. 20) are satisfied for $1 < p < \infty$ and $\vartheta = bp$ with g_{klij} in place of d_{klij},*

(vi) $D \in (\mathcal{L}_p, \mathcal{B}^{r,s}_\infty)$ *if and only if the condition (3. 19) is satisfied for $0 < p \leq 1$ with g_{klij} in place of d_{klij},*

(vii) $D \in (\mathcal{L}_p, \mathcal{B}^{r,s}_\infty)$ *if and only if the condition (3. 20) is satisfied for $1 < p < \infty$ with g_{klij} in place of d_{klij},*

(viii) $D \in \left(\mathcal{M}_u, \mathcal{B}^{r,s}_{bp}\right)$ *if and only if the conditions (3. 6),(3. 7), (3. 15)-(3. 18) are satisfied with g_{klij} in place of d_{klij},*

(ix) $D \in (\mathcal{M}_u, \mathcal{B}^{r,s})$ *if and only if the conditions (3. 7), (3. 11) and (3. 12) are satisfied with g_{klij} in place of d_{klij},*

(x) $D \in (\mathcal{C}_{bp}, \mathcal{B}_{\infty}^{r,s})$ *if and only if the condition (3. 6) is satisfied with* g_{klij} *in place of* d_{klij}.

5. CONCLUSION

Each story on single sequences has been usually experienced over double sequences. Constructing a new set of single or double sequence spaces and relating them to determine the location of those spaces between the other sequence spaces, characterizing matrix transformations on one of these spaces into another one, are among the qualified problems. In this article we introduced some double sequence spaces by using the domains of 4-dimensional binomial matrix on some classical double sequence spaces. Moreover, it is investigated their some properties and inclusion relations, computed duals and characterized some matrix classes. We conclude that the results obtained from the matrix $B^{(r,s)}$ is more general and extensive than the existent results of the authors [23, 32]. We expect that our results might be a reference for further studies in this field.

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