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Hermite-Hadamard Type Integral Inequalities for Functions Whose Mixed Partial Derivatives Are Co-ordinated Preinvex

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Abstract. The main objective of this article is to establish integral identity relating the left side of Hermite- Hadamard type inequality. By using this identity, we establish some new Hermite-Hadamard type integral inequalities for functions whose mixed partial derivatives are co-ordinated preinvex. These consequences generalize numerous outcomes established in previous studies for these classes of functions.

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1. Introduction

The investigation on extended convex functions has become a hot research topic in recent years. The applications of various properties of extended convex functions in establishing and improving numerous inequalities have attracted the attention of many researchers. Suppose that J is a finite interval of real numbers. A function $h : J \to \mathbb{R}$ is said to be convex if,

$$h(\xi\phi + (1-\xi)\psi) \leq \xi h(\phi) + (1-\xi)h(\psi), \tag{1. 1}$$

where $\xi \in [0, 1]$, for all $\phi, \psi \in J$.

The most famous inequality in the literature for convex functions is known as Hadamards inequality. This inequality was proposed in 1893 by Hadamard (see [10]). This double inequality is stated as:
Suppose that h is convex function on $[\phi, \psi] \subset \mathbb{R}$. Then the well known Hermite-Hadamard inequality [1] states that

$$h\left(\frac{\phi+\psi}{2}\right) \leq \frac{1}{\psi-\phi}\int_{\phi}^{\psi} h(x)dx \leq \frac{h(\phi)+h(\psi)}{2} \tag{1. 2}$$

for all $\phi, \psi \in J$.
Hadamards inequalities play a crucial role in various branches of science, including engineering, economics, astronomy, and mathematics. Thus, due to its great utility in several areas of pure and applied mathematics, much attention has been paid, by many mathematicians, to Hadamards inequality. Consequently, such inequalities were studied extensively by many authors. Also, numerous generalizations and extensions have been reported in a number of papers [1, 3, 4, 5, 7, 9, 13, 15], [16]-[21], [25, 26, 34] and [35]-[39] the references cited therein.
In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [8]. Ben-Israel and Mond [11] introduced the concept of preinvex functions, which is a special case of invexity. Pini [30] introduced the concept of pre-quasi-invex functions as a generalization of invex functions. Noor [28] has presented some estimates of the right hand side of a Hermite- Hadamard type inequality in which some preinvex functions and log-preinvex are involved.

Theorem 1.1. *Let $h : [\phi, \phi + \mu(\psi, \phi)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers Ω^0 (the interior of Ω) and $\phi, \psi \in \Omega^0$ with $\phi \leq \phi + \mu(\psi, \phi)$. Then the following inequality holds:*

$$h\Big(\frac{2\phi+\mu(\psi,\phi)}{2}\Big) \leq \frac{1}{\mu(\psi,\phi)}\int_{\phi}^{\phi+\mu(\psi,\phi)} h(x)dx \leq \frac{h(\phi)+h(\psi)}{2}. \tag{1. 3}$$

The following Hermite-Hadamrd type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ was also proved in [6]:

Theorem 1.2. *Let $h : \Delta := [\phi, \psi] \times [\gamma, \varrho] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be convex on the co-ordinates on Δ with $\psi < \psi$ and $\gamma < \varrho$. Then, one has the inequalities:*

$$\begin{aligned}
& h \quad \frac{\phi+\psi}{2}, \frac{\gamma+\varrho}{2} \\
& \leq \frac{1}{2} \quad \frac{1}{\psi-\phi}\int_{\phi}^{\psi} h \quad x, \frac{\gamma+\varrho}{2} \quad dx + \frac{1}{\gamma-\varrho}\int_{\gamma}^{\varrho} h \quad \frac{\phi+\psi}{2}, y \quad dy \\
& \leq \frac{1}{(\psi-\phi)(\varrho-\gamma)}\int_{\phi}^{\psi}\int_{\gamma}^{\varrho} g(x,y)dxdy \\
& \leq \frac{1}{4} \quad \frac{1}{\psi-\phi} \quad \int_{\phi}^{\psi} h(x,\gamma)dx + \int_{\phi}^{\psi} h(x,\varrho)dx \quad + \frac{1}{\gamma-\varrho} \quad \int_{\gamma}^{\varrho} h(\phi,y)dy + \int_{\gamma}^{\varrho} h(\psi,y)dy \\
& \leq \frac{h(\phi,\gamma)+h(\psi,\gamma)+h(\phi,\varrho)+h(\psi,\varrho)}{4}.
\end{aligned}$$

For several recent results on Hermite-Hadamard type inequalities for functions that satisfy different kinds of convexity on the co-ordinates on the rectangle from the plane $\mathbb{R}^2$ we

refer the reader to [2, 12, 14, 23, 27, 29, 33].
The main aim of this present paper is to define preinvex functions on the co-ordinates and to establish some Hermite-Hadamard type inequalities for functions whose mixed partial derivatives in absolute value are preinvex on the co-ordinates. Our established results generalize those result proved in [24].

2. Preliminaries

For convenience of our discussion in subsequent sections, let us reproduce some relevant definitions and earlier results below and recall some well known results related to convexity and preinvexity on the co-ordinates.
A modification for convex functions on Δ, which are also known as co-ordinated convex functions, was introduced by Dragomir [6, 31] as follows

Definition 2.1. *Let us now consider a function $h : \Delta =: [\phi, \psi] \times [\gamma, \varrho] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is convex on Δ if the following inequality:*

$$h(\xi\phi + (1-\xi)\bar{z}, \xi\gamma + (1-\xi)\bar{w}) \leq \xi h(\phi, \gamma) + (1-\xi)h(\bar{z}, \bar{w})$$

hold for all $\xi \in [0,1]$ and $(\phi, \gamma), (\bar{z}, \bar{w}) \in \Delta$.

Definition 2.2. *A function $h : \Delta = [\phi, \psi] \times [\gamma, \varrho] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the co-ordinates Δ with $\phi < \psi$ and $\gamma < \varrho$ if the partial functions $h_y : [\phi, \psi] \to \mathbb{R}$, $h_y(\bar{u}) = h(\bar{u}, y)$ and $h_x : [\gamma, \varrho] \to \mathbb{R}$, $h_x(\bar{v}) = h(x, \bar{v})$ are convex for all $x \in (\phi, \psi)$ and $y \in (\gamma, \varrho)$.*

A formal definition for co-ordinated convex functions is stated below:

Definition 2.3. *A function $h : \Delta := [\phi, \psi] \times [\gamma, \varrho] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ with $\gamma < \psi$ and $\gamma < \varrho$ if the partial functions*

$$\begin{aligned} & h(\xi\phi + (1-\xi)\bar{z}, \delta\gamma + (1-\delta)\bar{w}) \\ & \leq \xi\delta h(\phi, \gamma) + \xi(1-\delta)h(\phi, \bar{w}) + \delta(1-\xi)h(\bar{z}, \gamma) + (1-\delta)(1-\xi)h(\bar{z}, \bar{w}) \end{aligned} \tag{2. 4}$$

holds for all $\xi, \delta \in [0,1]$ and $(\phi, \gamma), (\bar{z}, \bar{w}) \in \Delta$.

Ben-Israel and Mond [32], established the idea of preinvex function as a special case of invex function.

Definition 2.4. *Consider Ω be a closed set in $\mathbb{R}^n$ and let $h : \Omega \to \mathbb{R}$ and $\mu : \Omega \times \Omega \to \mathbb{R}$ be continuous functions. Let $\phi \in \Omega$, then the set Ω is said to be invex at ϕ with respect to μ, if*

$$\phi + \xi\mu(\psi, \phi) \in \Omega, \tag{2. 5}$$

holds for all $\phi, \psi \in \Omega$, $\xi \in [0,1]$, then Ω is called an invex set with respect to μ if Ω is invex at each $\phi \in \Omega$. The invex set Ω is also called a μ-connected set.

Definition 2.5. *The function h on the invex set Ω is said to be preinvex with respect to μ, if*

$$h(\phi + \xi\mu(\psi, \phi)) \leq (1-\xi)h(\phi) + \xi h(\psi), \tag{2. 6}$$

holds for all $\phi, \psi \in \Omega$, $\xi \in [0,1]$. The function h is said to be pre-concave if and only if $-h$ is preinvex.
Note that every convex function is preinvex with respect to the map $\mu(\psi, \phi) = \psi - \phi$ but the converse is not true.
Latif et al. [22] gave notion of preinvex functions on the co-ordinates which generalize the classical convexity on the co-ordinates.

Definition 2.6. *Let Ω_1 and Ω_2 be non-empty subsets of $\mathbb{R}^n$ and let $\mu_1 : \Omega_1 \times \Omega_1 \to \mathbb{R}^n$ and $\mu_2 : \Omega_2 \times \Omega_2 \to \mathbb{R}^n$. We say $\Omega_1 \times \Omega_2$ is invex with respect to μ_1 and μ_2 at $(\eta, \nu) \in \Omega_1 \times \Omega_2$ if for each $(x, z) \in \Omega_1 \times \Omega_2$ and $\xi, \delta \in [0,1]$, we have*

$$(\eta + \xi\mu_1(x, u_1), \nu + \delta\mu_2(z, \nu)) \in \Omega_1 \times \Omega_2. \tag{2. 7}$$

$\Omega_1 \times \Omega_2$ is said to be invex set with respect to μ_1 and μ_2 if $\Omega_1 \times \Omega_2$ is invex at each $(\eta, \nu) \in \Omega_1 \times \Omega_2$.

Definition 2.7. *Let $\Omega_1 \times \Omega_2$ be an invex set with respect to $\mu_1 : \Omega_1 \times \Omega_1 \to \mathbb{R}^n$ and $\mu_2 : \Omega_2 \times \Omega_2 \to \mathbb{R}^n$. A function $h : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is said to be preinvex if for every $(x, z), (\eta, \nu) \in \Omega_1 \times \Omega_2$ and $\xi \in [0,1]$, we have*

$$h(\eta + \xi\mu_1(x, \eta), \nu + \xi\mu_2(z, \nu)) \le (1 - \xi)h(x, z) + \xi h(\eta, \nu). \tag{2. 8}$$

Definition 2.8. *Let $\Omega_1 \times \Omega_2$ be an invex set with respect to $\mu_1 : \Omega_1 \times \Omega_1 \to \mathbb{R}^n$ and $\mu_2 : \Omega_2 \times \Omega_2 \to \mathbb{R}^n$. A function $h : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is said to preinvex on the co-ordinates if the partial functions $h_y : \Omega_1 \to \mathbb{R}$, $h_y(\eta) = h(\eta, y)$ and $h_y : \Omega_1 \to \mathbb{R}$, $h_x(\nu) = h(x, \nu)$ are preinvex with respect to μ_1 and μ_2 respectively for all $y \in \Omega_2$ and $x \in \Omega_1$.*

Remark 2.9. *If $\mu_1(x, \eta) = x - \eta$ and $\mu_2(y, \nu) = y - \nu$ then h will be a convex function on the co-ordinates.*

Definition 2.10. *Let $\Omega_1 \times \Omega_2$ be an invex set with respect to $\mu_1 : \Omega_1 \times \Omega_1 \to \mathbb{R}^n$ and $\mu_2 : \Omega_2 \times \Omega_2 \to \mathbb{R}^n$. A function $h : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is said to preinvex on the co-ordinates $\Omega_1 \times \Omega_2$, then*

$$\begin{aligned} &h(\eta + \xi\mu_1(x, \eta), \nu + \delta\mu_2(z, \nu)) \\ &\le (1 - \xi)(1 - \delta)h(\eta, \nu) + (1 - \xi)\delta h(\eta, z) + (1 - \delta)\xi h(x, \nu) + \xi\delta h(x, z), \end{aligned}$$

where $(\eta, \nu), (x, z) \in \Omega_1 \times \Omega_2$.

Remark 2.11. *Every convex function on the co-ordinates is preinvex on the co-ordinates but the converse in not true. For example the function $h(\eta, \nu) = -|\eta||\nu|$ is not convex on the co-ordinates but it is a preinvex function with respect to the functions*

$$\mu_1(\eta, \bar{z}) = \begin{cases} \eta - \bar{z}, & \eta \ge 0, \bar{z} \ge 0 \quad and \quad \eta \le 0, \bar{z} \le 0, \\ \bar{z} - \eta, & otherwise. \end{cases}$$

$$\mu_2(\nu, \bar{w}) = \begin{cases} \nu - \bar{w}, & \nu \ge 0, \bar{w} \ge 0 \quad and \quad \nu \le 0, \bar{w} \le 0, \\ \bar{w} - \nu, & otherwise. \end{cases}$$

3. A KEY LEMMA

In this section, we present an identity associated with mixed partial differentiable function on co-ordinates, which plays an important role in establishing our main results.

Lemma 3.1. *Let $\Omega_1 \times \Omega_2$ be non-empty subsets of $\mathbb{R}^2$ and let $\mu_1 : \Omega_1 \times \Omega_1 \to \mathbb{R}$ and $\mu_2 : \Omega_2 \times \Omega_2 \to \mathbb{R}$. Suppose that $h : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a mixed partial differentiable function such that $\frac{\partial^2 h}{\partial\delta\partial\xi} \in L([\phi + \xi\mu_1(\psi, \phi)] \times [\gamma + \delta\mu_2(\varrho, \gamma)])$ with $\mu_1(\psi, \phi) \neq 0$ and $\mu_2(\varrho, \gamma) \neq 0$, where $\phi, \psi \in \Omega_1$ and $\gamma, \varrho \in \Omega_2$. Then the following equality holds:*

$$\begin{aligned}
&\Gamma(\phi, \psi, \gamma, \varrho)(h) = \frac{1}{\mu_1(\psi, \phi)\mu_2(\varrho, \gamma)} \int_{\phi}^{\phi+\mu_1(\psi,\phi)} \int_{\gamma}^{\gamma+\mu_2(\varrho,\gamma)} h(x, y)dydx \\
&+ h\left(\frac{2\phi + \mu_1(\psi, \phi)}{2}, \frac{2\gamma + \mu_2(\varrho, \gamma)}{2}\right) \\
&- \frac{1}{\mu_1(\psi, \phi)} \int_{\phi}^{\phi+\mu_1(\psi,\phi)} h\left(x, \frac{2\gamma + \mu_2(\varrho, \gamma)}{2}\right) dx \\
&- \frac{1}{\mu_2(\varrho, \gamma)} \int_{\gamma}^{\gamma+\mu_2(\varrho,\gamma)} h\left(\frac{2\phi + \mu_1(\psi, \phi)}{2}, y\right) dy \qquad (3.\ 9) \\
&= \mu_1(\psi, \phi)\mu_2(\varrho, \gamma) \int_0^1 \int_0^1 K(\xi, \delta) \frac{\partial^2}{\partial\xi\partial\delta} h(\phi + \xi\mu_1(\psi, \phi), \gamma + \delta\mu_2(\varrho, \gamma))d\delta d\xi,
\end{aligned}$$

where,

$$K(\xi, \delta) = \begin{cases} \xi\delta, & (\xi, \delta) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}], \\ \xi(\delta - 1), & (\xi, \delta) \in [0, \frac{1}{2}] \times (\frac{1}{2}, 1], \\ \delta(\xi - 1), & (\xi, \delta) \in (\frac{1}{2}, 1] \times [0, \frac{1}{2}], \\ (\xi - 1)(\delta - 1), & (\xi, \delta) \in (\frac{1}{2}, 1] \times (\frac{1}{2}, 1]. \end{cases}$$

Proof. Since

$$\begin{aligned}
&\mu_1(\psi, \phi)\mu_2(\varrho, \gamma) \int_0^1 \int_0^1 K(\xi, \delta) \frac{\partial^2}{\partial\xi\partial\delta} h(\phi + \xi\mu_1(\psi, \phi), \gamma + \delta\mu_2(\varrho, \gamma))d\delta d\xi \\
&= \mu_1(\psi, \phi)\mu_2(\varrho, \gamma) \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \xi\delta \frac{\partial^2}{\partial\xi\partial\delta} h(\phi + \xi\mu_1(\psi, \phi), \gamma + \delta\mu_2(\varrho, \gamma))d\delta d\xi \\
&+ \mu_1(\psi, \phi)\mu_2(\varrho, \gamma) \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \xi(\delta - 1) \frac{\partial^2}{\partial\xi\partial\delta} h(\phi + \xi\mu_1(\psi, \phi), \gamma + \delta\mu_2(\varrho, \gamma))d\delta d\xi \\
&+ \mu_1(\psi, \phi)\mu_2(\varrho, \gamma) \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \delta(\xi - 1) \frac{\partial^2}{\partial\xi\partial\delta} h(\phi + \xi\mu_1(\psi, \phi), \gamma + \delta\mu_2(\varrho, \gamma))d\delta d\xi \\
&+ \mu_1(\psi, \phi)\mu_2(\varrho, \gamma) \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (\delta - 1)(\xi - 1) \frac{\partial^2}{\partial\xi\partial\delta} h(\phi + \xi\mu_1(\psi, \phi), \gamma + \delta\mu_2(\varrho, \gamma))d\delta d\xi \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned} \qquad (3.\ 10)$$

Now by integration by parts, we have

$$
\begin{aligned}
J_1 &= \mu_1(\psi,\phi)\mu_2(\varrho,\gamma)\int_0^{\frac{1}{2}} \xi \left[\int_0^{\frac{1}{2}} \delta \frac{\partial^2}{\partial\xi\partial\delta} h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))d\delta\right] d\xi \\
&= \frac{1}{4}h\left(\frac{2\phi+\mu_1(\psi,\phi)}{2},\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}\right) - \frac{1}{2}\int_0^{\frac{1}{2}} h\left(\phi+\xi\mu_1(\psi,\phi),\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}\right) d\xi \\
&- \frac{1}{2}\int_0^{\frac{1}{2}} h\left(\frac{2\phi+\mu_1(\psi,\phi)}{2},\gamma+\delta\mu_2(\varrho,\gamma)\right) d\delta + \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))d\delta d\xi.
\end{aligned} \tag{3. 11}
$$

If we make use of the substitutions $x=\phi+\xi\mu_1(\psi,\phi)$ and $y=\gamma+\delta\mu_2(\varrho,\gamma)$, $(\xi,\delta)\in[0,1]\times[0,1]$, in (3. 11), we observe that

$$
\begin{aligned}
&= \frac{1}{4}h\left(\frac{2\phi+\mu_1(\psi,\phi)}{2},\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}\right) - \frac{1}{2\mu_1(\psi,\phi)}\int_{\frac{2\phi+\mu_1(\psi,\phi)}{2}}^{\psi} h\left(x,\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}\right) dx \\
&- \frac{1}{2\mu_2(\varrho,\gamma)}\int_{\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}}^{\varrho} h\left(\frac{2\phi+\mu_1(\psi,\phi)}{2},y\right) dy + \frac{1}{\mu_1(\psi,\phi)\mu_2(\varrho,\gamma)}\int_{\frac{2\phi+\mu_1(\psi,\phi)}{2}}^{\psi}\int_{\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}}^{\varrho} h(x,y)dydx.
\end{aligned}
$$

Similarly, by integration by parts, we also have that

$$
\begin{aligned}
J_2 &= \frac{1}{4}h\left(\frac{2\phi+\mu_1(\psi,\phi)}{2},\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}\right) - \frac{1}{2\mu_1(\psi,\phi)}\int_{\frac{2\phi+\mu_1(\psi,\phi)}{2}}^{\psi} h\left(x,\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}\right) dx \\
&- \frac{1}{2\mu_2(\varrho,\gamma)}\int_{\gamma}^{\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}} h\left(\frac{2\phi+\mu_1(\psi,\phi)}{2},y\right) dy + \frac{1}{\mu_1(\psi,\phi)\mu_2(\varrho,\gamma)}\int_{\frac{2\phi+\mu_1(\psi,\phi)}{2}}^{\psi}\int_{\gamma}^{\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}} h(x,y)dydx,
\end{aligned}
$$

$$
\begin{aligned}
J_3 &= \frac{1}{4}h\left(\frac{2\phi+\mu_1(\psi,\phi)}{2},\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}\right) - \frac{1}{2\mu_1(\psi,\phi)}\int_{\phi}^{\frac{2\phi+\mu_1(\psi,\phi)}{2}} h\left(x,\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}\right) dx \\
&- \frac{1}{2\mu_2(\varrho,\gamma)}\int_{\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}}^{\varrho} h\left(\frac{2\phi+\mu_1(\psi,\phi)}{2},y\right) dy + \frac{1}{\mu_1(\psi,\phi)\mu_2(\varrho,\gamma)}\int_{\phi}^{\frac{2\phi+\mu_1(\psi,\phi)}{2}}\int_{\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}}^{\varrho} h(x,y)dydx
\end{aligned}
$$

and

$$
\begin{aligned}
J_4 &= \frac{1}{4}h\left(\frac{2\phi+\mu_1(\psi,\phi)}{2},\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}\right) - \frac{1}{2\mu_1(\psi,\phi)}\int_{\phi}^{\frac{2\phi+\mu_1(\psi,\phi)}{2}} h\left(x,\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}\right) dx \\
&- \frac{1}{2\mu_2(\varrho,\gamma)}\int_{\varrho}^{\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}} h\left(\frac{2\phi+\mu_1(\psi,\phi)}{2},y\right) dy + \frac{1}{\mu_1(\psi,\phi)\mu_2(\varrho,\gamma)}\int_{\phi}^{\frac{2\phi+\mu_1(\psi,\phi)}{2}}\int_{\varrho}^{\frac{2\gamma+\mu_2(\varrho,\gamma)}{2}} h(x,y)dydx.
\end{aligned}
$$

Substitution of the J_1, J_2, J_3 and J_4 in (3. 10). We get our desired identity. □

4. Main Results

We are in a condition to establish the integral inequalities of Hermite-Hadamard type for functions whose mixed partial derivatives are co-ordinated preinvex

Theorem 4.1. *Let $\Omega_1\times\Omega_2$ be an open invex subsets of $\mathbb{R}^2$ with respect to the functions $\mu_1:\Omega_1\times\Omega_1\to\mathbb{R}$ and $\mu_2:\Omega_2\times\Omega_2\to\mathbb{R}$. Suppose that $h:\Omega_1\times\Omega_2\to\mathbb{R}$ be a mixed partial differentiable function such that $\frac{\partial^2 h}{\partial\delta\partial\xi}\in L([\phi+\xi\mu_1(\psi,\phi)]\times[\gamma+\delta\mu_2(\varrho,\gamma)])$ with $\mu_1(\psi,\phi)\neq 0$ and $\mu_2(\varrho,\gamma)\neq 0$, where $\phi,\psi\in\Omega_1$ and $\gamma,\varrho\in\Omega_2$. If $\left|\frac{\partial^2 h}{\partial\delta\partial\xi}\right|$ is preinvex on* *the co-ordinates on $\Omega_1 \times \Omega_1$, then the following inequality holds:*

$$|\Gamma(\phi,\psi,\gamma,\varrho)(h)| \le \frac{\mu_1(\psi,\phi)\mu_2(\varrho,\gamma)}{16}\left[\frac{\left|\frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\right| + \left|\frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\right| + \left|\frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\right| + \left|\frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\right|}{4}\right]. \tag{4. 12}$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}&|\Gamma(\phi,\psi,\gamma,\varrho)(h)| \\ &\le \mu_1(\psi,\phi)\mu_2(\varrho,\gamma)\int_0^1\int_0^1 |K(\xi,\delta)|\left|\frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\right| d\delta d\xi.\end{aligned} \tag{4. 13}$$

We know that $|\frac{\partial^2 h}{\partial\delta\partial\xi}|$ is preinvex on the co-ordinates on $\Omega_1 \times \Omega$, we have

$$\begin{aligned}&\left|\frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\right| \le \xi\delta\left|\frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\right| \\ &+\xi(1-\delta)\left|\frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\right| + \delta(1-\xi)\left|\frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\right| + (1-\xi)(1-\delta)\left|\frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\right|.\end{aligned} \tag{4. 14}$$

If we put (4. 14) into (4. 13), we have

$$\begin{aligned}&|\Gamma(\phi,\psi,\gamma,\varrho)(h)| \\ &\le \mu_1(\psi,\phi)\mu_2(\varrho,\gamma)\int_0^1\int_0^1 |K(\xi,\delta)\Bigg[\xi\delta\ \frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\ +\xi(1-\delta)\ \frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi} \\ &+\delta(1-\xi)\ \frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\ +(1-\xi)(1-\delta)\ \frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\ \Bigg]d\delta d\xi \\ &= \mu_1(\psi,\phi)\mu_2(\varrho,\gamma)\Bigg\{\int_0^{\frac12}\int_0^{\frac12}\xi\delta\Bigg[\xi\delta\ \frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\ +\xi(1-\delta)\ \frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\ +\delta(1-\xi)\ \frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi} \\ &+(1-\xi)(1-\delta)\ \frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\ \Bigg]d\delta d\xi + \int_0^{\frac12}\int_{\frac12}^1 \xi(\delta-1)\Bigg[\xi\delta\ \frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\ +\xi(1-\delta)\ \frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi} \\ &+\delta(1-\xi)\ \frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\ +(1-\xi)(1-\delta)\ \frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\ \Bigg]d\delta d\xi + \int_{\frac12}^1\int_0^{\frac12}\delta(\xi-1)\Bigg[\xi\delta\ \frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi} \\ &+\xi(1-\delta)\ \frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\ +\delta(1-\xi)\ \frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\ +(1-\xi)(1-\delta)\ \frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\ \Bigg]d\delta d\xi \\ &+\int_{\frac12}^1\int_{\frac12}^1(\xi-1)(\delta-1)\Bigg[\xi\delta\ \frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\ +\xi(1-\delta)\ \frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\ +\delta(1-\xi)\ \frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi} \\ &+(1-\xi)(1-\delta)\ \frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\ \Bigg]d\delta d\xi\Bigg\}.\end{aligned} \tag{4. 15}$$

Evaluating each integral in (4. 15) and simplifying, we get (4. 12). □

Theorem 4.2. *Let $\Omega_1 \times \Omega_2$ be an open invex subsets of $\mathbb{R}^2$ with respect to the functions $\mu_1 : \Omega_1 \times \Omega_1 \to \mathbb{R}$ and $\mu_2 : \Omega_2 \times \Omega_2 \to \mathbb{R}$. Suppose that $h : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a mixed partial differentiable function such that $\frac{\partial^2 h}{\partial\xi\partial\delta} \in L\left([\phi+\xi\mu_1(\psi,\phi)] \times [\gamma+\delta\mu_2(\varrho,\gamma)]\right)$ with*

$\mu_1(\psi,\phi)\neq 0$ and $\mu_2(\varrho,\gamma)\neq 0$, where $\phi,\psi\in\Omega_1$ and $\gamma,\varrho\in\Omega_2$. If $\left|\frac{\partial^2 h}{\partial\delta\partial\xi}\right|^q$ is preinvex on the co-ordinates on $\Omega_1\times\Omega_1$, $r,q>1$ and $\frac{1}{r}+\frac{1}{q}=1$, then the following inequality holds:

$$
\begin{aligned}
&|\Gamma(\phi,\psi,\gamma,\varrho)(h)| \\
&\leq \frac{\mu_1(\psi,\phi)\mu_2(\varrho,\gamma)}{4(r+1)^{\frac{2}{r}}}\left[\frac{\left|\frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\right|^q+\left|\frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\right|^q+\left|\frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\right|^q+\left|\frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\right|^q}{4}\right]^{\frac{1}{q}}.
\end{aligned}
\tag{4. 16}
$$

Proof. From Lemma 3.1, we have

$$
\begin{aligned}
&|\Gamma(\phi,\psi,\gamma,\varrho)(h)|\leq\mu_1(\psi,\phi)\mu_2(\varrho,\gamma)\\
&\times\int_0^1\int_0^1|K(\xi,\delta)|\left|\frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\right|d\delta d\xi.
\end{aligned}
\tag{4. 17}
$$

Now using the well-known Hölder's inequality for double integrals, we obtain

$$
\begin{aligned}
&\mu_1(\psi,\phi)\mu_2(\varrho,\gamma)\int_0^1\int_0^1|K(\xi,\delta)|\ \frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\ d\delta d\xi\\
&\leq\mu_1(\psi,\phi)\mu_2(\varrho,\gamma)\ \int_0^1\int_0^1|K(\xi,\delta)|^r d\delta d\xi\ ^{\frac{1}{r}}\ \left.\int_0^1\int_0^1\ \frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\ ^q d\delta d\xi\right)^{\frac{1}{q}}.
\end{aligned}
\tag{4. 18}
$$

We know that $\left|\frac{\partial^2 h}{\partial\delta\partial\xi}\right|^q$ is preinvex on the co-ordinates on $\Omega_1\times\Omega_2$, we have

$$
\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),c+\delta\mu_2(\varrho,\gamma))\right|^q d\delta d\xi\\
&\leq\int_0^1\int_0^1\left[\xi\delta\left|\frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\right|^q+\xi(1-\delta)\left|\frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\right|^q\right.\\
&\left.+\delta(1-\xi)\left|\frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\right|^q+(1-\xi)(1-\delta)\left|\frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\right|^q\right]d\delta d\xi.
\end{aligned}
$$

After some calculations,

$$
\begin{aligned}
&\int_0^1\int_0^1\xi\delta d\delta d\xi=\int_0^1\int_0^1\xi(1-\delta)d\delta d\xi=\int_0^1\int_0^1\delta(1-\xi)d\delta d\xi\\
&=\int_0^1\int_0^1(1-\xi)(1-\delta)d\delta d\xi=\frac{1}{4}.
\end{aligned}
\tag{4. 19}
$$

Also, we notice that

$$\int_0^1\int_0^1 |K(\xi,\delta)d\delta d\xi|^r = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \xi^r\delta^r d\delta d\xi + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 \xi^r(1-\delta)^r d\delta d\xi$$
$$\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} \delta^r(1-\xi)^r d\delta d\xi + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (1-\xi)^r(1-\delta)^r d\delta d\xi = \frac{4}{(r+1)^2}\left(\frac{1}{2}\right)^{2(r+1)}. \tag{4. 20}$$

Using (4. 19) and (4. 20) into (4. 18), we obtain

$$\int_0^1\int_0^1 |K(\xi,\delta)|\left|\frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\right| d\delta d\xi$$
$$\leq \frac{1}{4(r+1)^{\frac{2}{r}}}\left[\frac{\left|\frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\right|^q + \left|\frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\right|^q + \left|\frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\right|^q + \left|\frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\right|^q}{4}\right]^{\frac{1}{q}}. \tag{4. 21}$$

This completes the proof of the theorem. □

Remark 4.3. *Since* $2^r > r+1$ *if* $r>1$ *and accordingly*

$$\frac{1}{4} < \frac{1}{2(r+1)^{\frac{1}{r}}}$$

and hence we have that the following inequality

$$\frac{1}{16} < \frac{1}{4}.\frac{1}{4} < \frac{1}{2(r+1)^{\frac{1}{r}}}.\frac{1}{2(r+1)^{\frac{1}{r}}} = \frac{1}{4(r+1)^{\frac{2}{r}}},$$

and as a consequence we get an improvement of the constant in Theorem 4.2.

Theorem 4.4. *Let* $\Omega_1\times\Omega_2$ *be an open invex subsets of* $\mathbb{R}^2$ *with respect to the functions* $\mu_1:\Omega_1\times\Omega_1\to\mathbb{R}$ *and* $\mu_2:\Omega_2\times\Omega_2\to\mathbb{R}$*. Suppose that* $h:\Omega_1\times\Omega_2\to\mathbb{R}$ *be a mixed partial differentiable function such that* $\frac{\partial^2 h}{\partial\xi\partial\delta}\in L([\phi+\xi\mu_1(\psi,\phi)]\times[\gamma+\delta\mu_2(\varrho,\gamma)])$ *with* $\mu_1(\psi,\phi)\neq 0$ *and* $\mu_2(\varrho,\gamma)\neq 0$*, where* $\phi,\psi\in\Omega_1$ *and* $\gamma,\varrho\in\Omega_2$*. If* $\left|\frac{\partial^2 h}{\partial\delta\partial\xi}\right|^q$ *is preinvex on the co-ordinates on* $\Omega_1\times\Omega_1$ *and* $q\geq 1$*, then the following inequality holds:*

$$|\Gamma(\phi,\psi,\gamma,\varrho)(h)|$$
$$\leq \frac{\mu_1(\psi,\phi)\mu_2(\varrho,\gamma)}{16}\left[\frac{\left|\frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\right|^q + \left|\frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\right|^q + \left|\frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\right|^q + \left|\frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\right|^q}{4}\right]^{\frac{1}{q}}. \tag{4. 22}$$

Proof. From Lemma 3.1, we have

$$|\Gamma(\phi,\psi,\gamma,\varrho)(h)| \leq \mu_1(\psi,\phi)\mu_2(\varrho,\gamma)$$
$$\times\int_0^1\int_0^1 |K(\xi,\delta)|\left|\frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\right| d\delta d\xi. \tag{4. 23}$$

By the power mean inequality, we have

$$\begin{aligned}
&\int_0^1\int_0^1 |K(\xi,\delta)| \left|\frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\right| d\delta d\xi \\
&\leq \left(\int_0^1\int_0^1 |K(\xi,\delta)|\, d\delta d\xi\right)^{1-\frac{1}{q}} \left(\int_0^1\int_0^1 |K(\xi,\delta)| \left|\frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\right|^q d\delta d\xi\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{16}\right)^{1-\frac{1}{q}} \left(\int_0^1\int_0^1 |K(\xi,\delta)| \left|\frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\right|^q d\delta d\xi\right)^{\frac{1}{q}}. \qquad (4.\ 24)
\end{aligned}$$

Using the fact $|\frac{\partial^2 h}{\partial\delta\partial\xi}|^q$ is preinvex on the co-ordinates on $\Omega_1\times\Omega_2$, we have

$$\begin{aligned}
&\left|\frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\right|^q \leq \xi\delta\left|\frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\right|^q \\
&+\xi(1-\delta)\left|\frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\right|^q+\delta(1-\xi)\left|\frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\right|^q+(1-\xi)(1-\delta)\left|\frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\right|^q
\end{aligned}$$

and hence, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1 |K(\xi,\delta)|\left|\frac{\partial^2}{\partial\delta\partial\xi}h(\phi+\xi\mu_1(\psi,\phi),\gamma+\delta\mu_2(\varrho,\gamma))\right|^q d\delta d\xi \\
&\leq \int_0^1\int_0^1 |K(\xi,\delta)|\left[\xi\delta\left|\frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\right|^q+\xi(1-\delta)\left|\frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\right|^q\right. \\
&\left.+\delta(1-\xi)\left|\frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\right|^q+(1-\xi)(1-\delta)\left|\frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\right|^q\right] \\
&=\frac{1}{64}\left[\left|\frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\right|^q+\left|\frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\right|^q+\left|\frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\right|^q+\left|\frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\right|^q\right].
\end{aligned}$$

Therefore (4. 24) becomes

$$\leq \frac{\mu_1(\psi,\phi)\mu_2(\varrho,\gamma)}{16}\left[\frac{\left|\frac{\partial^2 h(\phi,\gamma)}{\partial\delta\partial\xi}\right|^q+\left|\frac{\partial^2 h(\phi,\varrho)}{\partial\delta\partial\xi}\right|^q+\left|\frac{\partial^2 h(\psi,\gamma)}{\partial\delta\partial\xi}\right|^q+\left|\frac{\partial^2 h(\psi,\varrho)}{\partial\delta\partial\xi}\right|^q}{4}\right]^{\frac{1}{q}}. \qquad (4.\ 25)$$

Substituting (4. 25) into (4. 23), we obtain (4. 22). □

Remark 4.5. *If we takes $\mu_1(\psi,\phi)=\psi-\phi$ and $\mu_2(\varrho,\gamma)=\varrho-\gamma$ in Theorem 4.1-Theorem 4.4, then h will be a convex functions on the co-ordinates and we recapture all those results proved in* [24].

5. CONCLUSIONS

This paper has presented some new results of the Hermite-Hadamard integral inequalities type for functions whose mixed partial derivatives are co-ordinated preinvex. In addition, the obtained results in this paper would be useful for generalization of inequalities that were proved in previous work.

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REFERENCES

[1] R. F. Bai, F. Qi and B. Y. Xi, *Hermite-Hadamard type inequalities for the m-and (α,m)-logarithmically convex functions*, Filomat **27**, No. 1(2013)1-7.

[2] S. P. Bai, F. Qi and S. H. Wang, *Some new integral inequalities of HermiteHadamard type for (α, m, P)-convex functions on co-ordinates*, J. Appl. Anal. Comput. **6**, 1(2016)171-178.

[3] L. Chun and F. Qi, *Integral inequalities of Hermite-Hadamard type for functions whose 3rd derivatives are s-convex*, Appl. Math. **12**, No. 3(2012)1898-1902.

[4] S. S. Dragomir, and S. Fitzpatrick, *The Hadamard inequalities for s-convex functions in the second sense*, Demonstratio Math. **32**, No. 4(1999)687-696.

[5] S. S. Dragomir, *On some new inequalities of Hermite-Hadamard type for m-convex functions*, Tamkang J. Math. **33**, 1(2002) 45-56.

[6] S. S. Dragomir, *On the Hadamard's inequlality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese J. Math. (2001) 775-88.

[7] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions*, Aequ. Math. **48**, (1994)100-111.

[8] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80**, 2(1981) 545-50.

[9] S. Hussain and S. Qaisar, *New integral inequalities of the type of Hermite-Hadamard through quasi convexity*, Punjab Univ. j. math. **45**, (2013) 33-38.

[10] J. Hadamard, *Étude sur les Propriétés des Fonctions Entières en Particulier d'une Fonction Considérée par Riemann*, J. Math. Pures Appl. **58**, (1893) 171-215.

[11] B. I. Isreal and B. Mond, *What is invexity?*, J. Australian Math. Soc., Ser. B. **28**, 1(1986) 1-9.

[12] H. Kalsoom, J. Wu, S. Hussain and M. A. Latif, *Simpsons Type Inequalities for Co-ordinated Convex Functions on Quantum Calculus*, symmetry. **11**, (2019) 768 pages.

[13] H. Kalsoom, S. Hussain, *Some Hermite-Hadamard type integral inequalities whose n-times differentiable functions are s-logarithmically convex functions*, Punjab Univ. j. math. **2019**, (2019)65-75.

[14] H. Kalsoom, M. A. Latif, S. Hussain, M. D. Junjua and G. Shahzadi, *Some (p, q)-estimates of Hermite Hadamard's type inequalities for co-ordinated convex and quasi convex functions,* Mathematics. **8** , (2019)683 pages.

[15] y. Deng, H. Kalsoom and S. Wu, *Some New Quantum Hermite-Hadamard Type Estimates Within a Class of Generalized (s, m)-Preinvex Functions*, Symmetry. **11**, (2019)1283 pages.

[16] M. A. Khan, Y. Khurshid, S. S. Dragomir and R. Ullah, *New Hermite-Hadamard type inequalities with applications*, Punjab Univ. j. math. **50**, No. 3 (2018) 1-12.

[17] M. A. Khan, Y. Khurshid, T. Ali and N. Rehman, *Inequalities for three times differentiable functions,* Punjab Univ. j. math. **48**, No. 2 (2016) 35-48.

[18] M. A. Latif, New Fejer and *Hermite-Hadamard type inequalities for differentiable p-convex mappings*, Punjab Univ. j. math. **51**, No. 2 (2019) 39-59.

[19] M. A. Latif, *Estimates of Hermite-Hadamard inequality for twice differentiable harmonically-convex functions with applications*, Punjab Univ. j. math. **50**, No. 1 (2018) 1-13.

[20] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some ϕ-analogues of Hermite-Hadamard inequality for sconvex functions in the second sense and related estimates*, Punjab Univ. j. math. **48**, No. 2 (2016) 147-166.

[21] M. A. Latif and S. Hussain, *New HermiteHadamard type Inequalities for Harmonically-Convex Functions*, Punjab Univ. j. math. **51**, No. 6 (2019) 1-17.

[22] M. A. Latif and S. S. Dragomir *Some Hermite-Hadamard type inequalities for functions whose partial derivatives in absloute value are preinvex on the co-oordinates*, Facta Univ. Ser. Math. Inform. **28**, 25(2013)257-270.

[23] M. A. Latif and M. Alomari *Hadamard-type inequalities for product two convex functions on the co-ordinates*, InInt. Math. **47**, 4(2009) 2327-2338.

[24] M. A. Latif and S.S. Dragomir, *On some new inequalities for differentiable co-ordinated convex functions*, J. Inequal. Appl. **28**, (2012)1-13.

[25] M. Muddassar and A. Ali, *New integral inequalities through generalized convex functions*, Punjab Univ. j. math. **46**, No. 2 (2014) 47-51.

[26] M. Muddassar and M. I. Bhatti, *Some generalizations of Hermite-Hadamard type integral inequalities and their applications*, Punjab Univ. j. math. **46**, No. 1 (2014) 9-18.

[27] M. Matloka, *On some Hadamard-type inequalities for (h_1, h_2)-preinvex functions on the co-ordinates*, J. Inequal. Appl. **227**, No. 1(2013).

[28] M. A. Noor, *On Hadamard integral inequalities involving two log-preinvex functions*, J. In-equal. Pure Appl. Math. **8**, 3(2007).

[29] M. E. Özdemir, A. O. Akdemir and M. Tunc, *On some Hadamard-type inequalities for co-ordinated convex functions*, arXiv preprint arXiv. **4327**, 1203(2012).

[30] R. Pini, *Invexity and generalized convexity*, Optimization **22** 4(1991)513-25.

[31] M. Z. Sarikaya, E. Set M. E. Özdemir and S. S. Dragomir, *New some Hadamard's type inequalities for co-ordinated convex functions*, arXiv preprint arXiv: 1005.0700. (2010).

[32] T. Weir and B. Mond *Preinvex functions in multiple bjective optimization*, J. Math. Anal. Appl. **136**, (1998) 29-38.

[33] B. Y. Xi and F. Qi, *Some new integral inequalities of Hermite-Hadamard type for (log,(α, m))-convex functions on co-ordinates*, Studia Universitatis Babeş-Bolyai Mathematica. **60**, 4(2015) 509-25.

[34] F. Zafar, H. Kalsoom and N. Hussain, *Some inequalities of Hermite-Hadamard type for n-times differentiable (ρ,m)-geometrically convex functions*, J. Nonlinear Sci. Appl. **8**, No.3(2015)201-217.

[35] S. Rashid, M. A. noor and K. I. Noor, F. Safdar, *Integral inequalities for generalized preinvex functions*, Punjab. Univ. J. Math. **51**, 10(2019)77-91.

[36] S. Rashid, M.A. Noor and K. I. Noor, Some generalize Riemann-Liouville fractional estimates involving functions having exponentially convexity property, Punjab. Univ. J. Math. **51**, 11(2019)01-15.

[37] S. Rashid, M. A. Noor, K. I. Noor, *New Estimates for Exponentially Convex Functions via Conformable Fractional Operator*, Fractal Fract. **3**, (2019)19.

[38] S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, Y.- M. Chu, *Hermite-Hadamard inequalities for the class of convex functions on time scale*, Mathematics. **7**, (2019) 956.

[39] M. A. Noor, K. I. Noor and S. Rashid, *Some new classes of preinvex functions and inequalities*, Mathematics. **7**, (2019)29.