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**On Pólya-Szegö Type Inequalities via $\mathcal{K}$-Fractional Conformable Integrals**

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**Abstract.** The studies of inequalities regarding the fractional differential and integral operators are considered to be essential because of their potential applications among researchers. This paper consigns to the generalizations of novel fractional integral inequalities. The Pólya-Szegö type variants are generalized by involving $\mathcal{K}$-fractional conformable integrals $(KFCI)$. This is the $\mathcal{K}$-analogue of the fractional conformable integrals. We discuss the implications and other consequences of the $\mathcal{K}$-fractional conformable fractional integrals.

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**Key Words:** Integral inequality; Pólya-Szegö inequality; Conformable $\mathcal{K}$-frcational integral.

## 1. Introduction

Fractional calculus is the calculus of integrals and derivatives of any arbitrary real or complex order. Recently fractional calculus has gained significant popularity because it provides several potential tools for solving various problems arising in different fields of pure and applied sciences. For example, it contributes very well to problems involving special functions of mathematical physics. Another reason for its involvement in other branches of science is that it can be considered a powerful tool for describing the long-memory process. For useful detail, see [14, 46, 47]. In many branches of pure and applied mathematics, the fractional differential and integral operators are very beneficial devices to perform the large variety of complicated amounts of powers of differentiation and integration. For an entire description of fractional calculus operators similarly to their features and applications, we refer the readers to the research manuscripts by Miller and Ross [12] and Kiryakova [9]. Numerous diverse definitions of fractional integrals at the side of their applications may be sought within the literature. Every description has its own benefits and suitable for applications to fantastic problems in various topics of sciences. Recently, Jarad et al. [6] contributed one greater element to the test of fractional operators with the useful resource of introducing new fractional integral and derivative operators which is probably based totally on the extremely-modern fractional calculus iteration procedure on conformable derivatives introduced by Abdeljawad [1].

Inequalities concerning fractional integrals are deemed to be critical as they may be valuable in the study of diverse differential and integral equations (see [13, 36, 37, 38, 39, 40, 41, 42]). This technique has drawn the attention of many mathematicians in the past several years. For inequalities associated with generalized fractional operators, we refer [5, 7, 8, 10, 11, 15, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. A variety of fractional integral operators is referred to in the literature for their fertile applications in every field of sciences. Sarikaya et al. [44] used the concepts of Riemann-Liouville fractional integrals and obtained a fractional analogue of Hermite-Hadamard's inequality. This idea compelled many researchers to use the concepts of fractional calculus in the theory of inequalities. Resultantly several new fractional analogues of classical results have been obtained using different novel and innovative approaches. The $\mathcal{K}$-analogues of numerous classical and fractional operators have been taken into consideration approximately a decade ago by few researchers, see the references [16, 19, 20, 21, 36, 43, 45]. We describe some $\mathcal{K}$-analogues of classical operators existing in the literature.

Mubeen and Habibullah [14] used this specific $\mathcal{K}$-functions concept in fractional calculus for very recently in literature in the form of $\mathcal{K}$-Riemann-Liouville fractional integral. Recently, many researchers are providing the new fractional differential operators and their generalized versions using iteration techniques and also for parameter $\mathcal{K} > 0$. Additionally, they determined the relationships of these generalized fractional operators with existing fractional and classical operators for the specific values of the parameters concerned. The study of fractional type inequalities is also of great importance. We refer the reader to [2, 17, 45] for further information and applications.

Pólya-Szegö integral inequality is one of the most intensively studied inequality. This inequality was introduced by Pólya-Szegö [18]:

$$\frac{\int_u^v f_1^2(\lambda)d\lambda \int_u^v h_1^2(\lambda)d\lambda}{\left(\int_u^v f_1(\lambda)h_1(\lambda)d\lambda\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{QR}{qr}}+\sqrt{\frac{qr}{QR}}\right)^2. \quad (1.\ 1)$$

Dragomir and Diamond proved that [4]

$$|\mathfrak{T}(f_1,h_1)| \leq \frac{(Q-q)(R-r)}{4(v-u)\sqrt{qrQR}}\int_u^v f_1(\lambda)d\lambda \int_u^v h_1(\lambda)d\lambda,$$

where $f_1$ and $h_1$ are two positive integrable functions on $[u,v]$ such that

$$0<q\leq f_1(\lambda)\leq Q<\infty, \quad 0<r\leq h_1(\lambda)\leq R<\infty.$$

The theory of special $\mathcal{K}$-functions was originated by Diaz and Pariguan in the form of $\mathcal{K}$-Pochhammer symbol $(x)_{n,\mathcal{K}}$, the $\mathcal{K}$-gamma function $\Gamma_{\mathcal{K}}$ and the $\mathcal{K}$-beta function $\delta_{\mathcal{K}}$ (see [3]):

$$(\mu)_{n,\mathcal{K}} := \mu(\mu+\mathcal{K})(\mu+2\mathcal{K})...(\mu+(n-1)\mathcal{K}), \quad (n\in\mathbb{N},\mathcal{K}>0),$$

and

$$\Gamma_{\mathcal{K}}(\mu)=\lim_{n\to\infty}\frac{n!\mathcal{K}^n(n\mathcal{K})^{\frac{\mu}{\mathcal{K}}-1}}{(\mu)_{n,\mathcal{K}}}, \quad \mathcal{K}>0, \quad (1.\ 2)$$

where $(\mu)_{n,\mathcal{K}}$ is the Pochammer $\mathcal{K}$-symbol for factorial function. The $\mathcal{K}$-gamma function can also be shown explicitly as the Mellin tranform of the exponential function $e^{-\frac{\vartheta^{\mathcal{K}}}{\mathcal{K}}}$ given by

$$\Gamma_{\mathcal{K}}(\mu)=\int_0^\infty \vartheta^{\mu-1}e^{-\frac{t^{\mathcal{K}}}{\mathcal{K}}}d\vartheta, \quad x>0.$$

Clearly,

$$\Gamma(\mu)=\lim_{\mathcal{K}\to 1}\Gamma_{\mathcal{K}}(\mu), \quad \Gamma_{\mathcal{K}}(\mu)=\mathcal{K}^{\frac{\mu}{\mathcal{K}}-1}\Gamma(\frac{\mu}{\mathcal{K}})$$
$$\Gamma_{\mathcal{K}}(\mu+\mathcal{K})=\mu\Gamma_{\mathcal{K}}(\mu).$$

Further, $\mathcal{K}$-delta function denoted by

$$\delta_{\mathcal{K}}(x,y)=\frac{1}{\mathcal{K}}\int_0^1 \vartheta^{\frac{x}{\mathcal{K}}-1}(1-\vartheta)^{\frac{y}{\mathcal{K}}-1}d\vartheta,$$

such that $\delta_{\mathcal{K}}(x,y)=\frac{1}{\mathcal{K}}\delta(\frac{x}{\mathcal{K}},\frac{y}{\mathcal{K}})$ and $\delta_{\mathcal{K}}(x,y)=\frac{\Gamma_{\mathcal{K}}(x)\Gamma_{\mathcal{K}}(y)}{\Gamma_{\mathcal{K}}(x+y)}$.

The main objective of this manuscript is to introduce the fractional conformable integrals said in [5] in the frame of $\mathcal{K}>0$ as well as its existence. We additionally generalize the Pólya-Szegö inequalities given in [18] for two positive functions involving innovative technique known as $\mathcal{K}$-fractional conformable integrals (KFCI). we have provided the inequalities and associated consequences concerning one and fractional parameters. The

work involved to the inequalities, their fertile application, and stability we refer the readers to [6, 46, 47].
Abdeljawad [1] introduced the left and right fractional conformable derivatives for a differentiable function $f_1$ in the form:

$$\mathfrak{I}^{\alpha}_{\mathfrak{d}_1^+} f_1(\lambda) = (\lambda - u)^{1-\alpha} f_1'(\lambda),$$

$$\mathfrak{I}^{\alpha}_{\mathfrak{d}_2^-} f_1(\lambda) = (v - \lambda)^{1-\alpha} f_1'(\lambda),$$

The corresponding left and right fractional conformable integrals for $0 < \alpha < 1$, by

$$\mathfrak{I}^{\alpha}_{\mathfrak{d}_1^+} f_1(x) = \int_u^x \frac{f_1(\lambda)}{(\lambda - u)^{1-\alpha}} d\lambda,$$

$$\mathfrak{I}^{\alpha}_{\mathfrak{d}_2^-} f_1(x) = \int_x^v \frac{f_1(\lambda)}{(v - \lambda)^{1-\alpha}} d\lambda.$$

Recalling the concept of the fractional conformable integral operator, which is mainly due to Jarad et al. [6].

**Definition 1.1.** *For $\delta \in \mathbb{C}$, and $Re(\delta) > 0$, then the left-sided and right-sided fractional conformable integral operator of order $\delta$ is defined as*

$${}^{\delta}\mathcal{J}^{\alpha}_{u^+} f_1(x) = \frac{1}{\Gamma(\delta)} \int_u^x \Big( \frac{(x-u)^{\alpha} - (\lambda - u)^{\alpha}}{\alpha} \Big)^{\delta - 1} f_1(\lambda) \frac{d\lambda}{(\lambda - u)^{1-\alpha}},$$

$${}^{\delta}\mathcal{J}^{\alpha}_{v^-} f_1(x) = \frac{1}{\Gamma(\delta)} \int_u^x \Big( \frac{(v-x)^{\alpha} - (v - \lambda)^{\alpha}}{\alpha} \Big)^{\delta - 1} f_1(\lambda) \frac{d\lambda}{(v - \lambda)^{1-\alpha}},$$

*where $\Gamma(\delta)$ is the Gamma function is defined as*

$$\Gamma(\delta) = \int_0^{\infty} e^{-\mu} \mu^{\delta - 1} du.$$

Next, we define the generalized left and right fractional conformable integrals in the frame of a new parameter $\mathcal{K} > 0$ introduced in [5].

**Definition 1.2.** *For $\delta \in \mathbb{C}$, $Re(\delta) > 0$, and let $f_1$ be a continuous function on a finite real interval $[u, v]$. Then generalized left-sided and right-sided $KFCI$ of order $\delta$ is defined as*

$${}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u^+} f_1(x) = \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)} \int_u^x \Big( \frac{(x-u)^{\alpha} - (\lambda - u)^{\alpha}}{\alpha} \Big)^{\frac{\delta}{\mathcal{K}} - 1} f_1(\lambda) \frac{d\lambda}{(\lambda - u)^{1-\alpha}}, \tag{1. 3}$$

$${}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{v^-} f_1(x) = \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)} \int_x^v \Big( \frac{(v-x)^{\alpha} - (v - \lambda)^{\alpha}}{\alpha} \Big)^{\frac{\delta}{\mathcal{K}} - 1} f_1(\lambda) \frac{d\lambda}{(v - \lambda)^{1-\alpha}} \tag{1. 4}$$

*where $\Gamma_{\mathcal{K}}$ is the Euler $\mathcal{K}$-Gamma function, $\mathcal{K} > 0, \alpha \in \mathbb{R} \setminus \{0\}$.*

## 2. MAIN RESULTS

In this section, we establish certain Pólya-Szegö type integral inequalities for positive integrable functions involving the generalized $\mathcal{K}$-fractional conformable integral operator (1. 3 ).

**Theorem 2.1.** *Let $f_1$ and $h_1$ be two positive integrable functions on $[0,\infty)$. Assume that there exist four positive integrable functions $\varphi_1, \varphi_2, \psi_1$ and $\psi_2$ on $[0,\infty)$ such that:*
$(I)\quad 0<\varphi_1(\lambda)\leq f_1(\lambda)\leq \varphi_2(\lambda),\ 0<\psi_1(\lambda)\leq h_1(\lambda)\leq\psi_2(\lambda),\quad (\lambda\in[u,x]).$
*Then for $\delta>0$, the following inequality holds:*

$$\frac{\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\Big)\big\{\psi_1\psi_2 f_1^2\big\}(x)\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\Big)\big\{\varphi_1\varphi_2 h_1^2\big\}(x)}{\Big(\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\Big)\big\{(\varphi_1\psi_1+\varphi_2\psi_2)f_1h_1\big\}(x)\Big)^2}\leq\frac{1}{4}. \qquad (2.\ 5)$$

*Proof.* From $(I)$, for $\lambda\in[u,x]$, we have

$$\Big(\frac{\varphi_2(\lambda)}{\psi_1(\lambda)}-\frac{f_1(\lambda)}{h_1(\lambda)}\Big)\geq 0. \qquad (2.\ 6)$$

Analogously, we have

$$\Big(\frac{f_1(\lambda)}{h_1(\lambda)}-\frac{\varphi_1(\lambda)}{\psi_2(\lambda)}\Big)\geq 0. \qquad (2.\ 7)$$

Multiplying (2. 6 ) and (2. 7 ), we obtain

$$\Big(\frac{\varphi_2(\lambda)}{\psi_1(\lambda)}-\frac{f_1(\lambda)}{h_1(\lambda)}\Big)\Big(\frac{f_1(\lambda)}{h_1(\lambda)}-\frac{\varphi_1(\lambda)}{\psi_2(\lambda)}\Big)\geq 0. \qquad (2.\ 8)$$

The inequality (2. 8 ) can be written as

$$\varphi_1(\lambda)\psi_1(\lambda)+\varphi_2(\lambda)\psi_2(\lambda)\ \ f_1(\lambda)h_1(\lambda)\geq\psi_1(\lambda)\psi_2(\lambda)f_1^2(\lambda)+\varphi_1(\lambda)\varphi_2(\lambda)h_1^2(\lambda). \qquad (2.\ 9)$$

Now, multiplying both sides of (2. 9 ) by $\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\ \ \frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\ \ ^{\frac{\delta}{\mathcal{K}}-1}\ \frac{1}{(\lambda-u)^{1-\alpha}}$,then integrating the resulting inequality with respect $\lambda$ from $u$ to $x$, we get

$$\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_u^x\ \frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\ ^{\frac{\delta}{\mathcal{K}}-1}\frac{\varphi_1(\lambda)\psi_1(\lambda)+\varphi_2(\lambda)\psi_2(\lambda)\ \ f_1(\lambda)h_1(\lambda)d\lambda}{(\lambda-u)^{1-\alpha}}$$
$$\geq\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_u^x\ \frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\ ^{\frac{\delta}{\mathcal{K}}-1}\frac{\psi_1(\lambda)\psi_2(\lambda)f_1^2(\lambda)}{(\lambda-u)^{1-\alpha}}d\lambda$$
$$+\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_u^x\ \frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\ ^{\frac{\delta}{\mathcal{K}}-1}\frac{\varphi_1(\lambda)\varphi_2(\lambda)h_1^2(\lambda)}{(\lambda-u)^{1-\alpha}}d\lambda, \qquad (2.\ 10)$$

from which, one has

$${}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\quad(\varphi_1\psi_1+\varphi_2\psi_2)f_1h_1\ \ (x)\geq\ \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\quad\psi_1\psi_2f_1^2\ \ (x)+\ \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\quad\varphi_1\varphi_2h_1^2\ \ (x).$$

Applying the $AM-GM$ inequality, that is., $\mu+\nu\geq\sqrt{\mu\nu},\ \mu,\nu\in\mathcal{R}^{+}$, we have

$${}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\quad(\varphi_1\psi_1+\varphi_2\psi_2)f_1h_1\ \ (x)\geq\sqrt{\ \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\quad\psi_1\psi_2f_1^2\ \ (x)\ \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\quad\varphi_1\varphi_2h_1^2\ \ (x)},$$

which leads to

$$ {}^{\delta}_{\mathcal{K}}\mathcal{J}_u^{\alpha} \quad \psi_1\psi_2 f_1^2 \;\; (x) \;\; {}^{\delta}_{\mathcal{K}}\mathcal{J}_u^{\alpha} \quad \varphi_1\varphi_2 h_1^2 \;\; (x) \leq \frac{1}{4} \quad {}^{\delta}_{\mathcal{K}}\mathcal{J}_u^{\alpha} \quad (\varphi_1\psi_1+\varphi_2\psi_2)f_1h_1 \;\; (x) \quad ^2. $$

Therefore, we obtain the inequality (2. 5 ) as requested. □

As a special case of Theorem 2.1, we obtain the following result:

**Corollary 2.2.** *Let $f_1$ and $h_1$ be two positive integrable functions on $[0,\infty)$ satisfying*
$(II) \quad 0 < q \leq f_1(\lambda) \leq Q < \infty, \;\; 0 < r \leq f_1(\lambda) \leq R < \infty, \;\; \forall\, \lambda \in [u,x].$
*Then for $\delta > 0$ and $\vartheta > 0$, we have*

$$ \frac{\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}_u^{\alpha}\Big)\big\{f_1^2\big\}(\lambda)\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}_u^{\alpha}\Big)\big\{h_1^2\big\}(\lambda)}{\bigg(\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}_u^{\alpha}\Big)\big\{f_1h_1\big\}(\lambda)\bigg)^2} \leq \frac{1}{4}\left(\sqrt{\frac{qr}{QR}}+\sqrt{\frac{QR}{qr}}\right)^2. $$

**Theorem 2.3.** *Let all assumptions of Theorem 2.1 hold. Then for $\delta > 0$ and $\vartheta > 0$, the following inequality holds:*

$$ \frac{\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}_u^{\alpha}\Big)\big\{\psi_1\psi_2\big\}(x)\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}_u^{\alpha}\Big)\big\{f_1^2\big\}(x)\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}_u^{\alpha}\Big)\big\{\varphi_1\varphi_2\big\}\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}_u^{\alpha}\Big)\big\{h_1^2\big\}(x)}{\bigg(\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}_u^{\alpha}\Big)\big\{\varphi_1f_1\big\}(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}_u^{\alpha}\Big)\big\{\psi_1h_1\big\}(x)+\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}_u^{\alpha}\Big)\big\{\varphi_2f_1\big\}(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}_u^{\alpha}\Big)\big\{\psi_2g\big\}(x)\bigg)^2} \leq \frac{1}{4}. $$

*Proof.* To prove (2. 11 ), using the condition $(I)$, we obtain

$$ \Big(\frac{\varphi_2(\lambda)}{\psi_1(\rho)}-\frac{f_1(\lambda)}{h_1(\rho)}\Big) \geq 0, $$

and

$$ \Big(\frac{f_1(\lambda)}{h_1(\rho)}-\frac{\varphi_1(\lambda)}{\psi_2(\rho)}\Big) \geq 0, $$

which imply that

$$ \left(\frac{\varphi_1(\lambda)}{\psi_2(\rho)}+\frac{\varphi_2(\lambda)}{\psi_1(\rho)}\right)\frac{f_1(\lambda)}{h_1(\rho)} \geq \frac{f_1^2(\lambda)}{h_1^2(\rho)}+\frac{\varphi_1(\lambda)\varphi_2(\lambda)}{\psi_1(\rho)\psi_2(\rho)}. \tag{2. 11} $$

Multiplying both sides of (2. 11 ) by $\psi_1(\rho)\psi_2(\rho)g^2(\rho)$, we have

$$ \varphi_1(\lambda)f_1(\lambda)\psi(\rho)g(\rho)+\varphi_2(\lambda)f_1(\lambda)\psi_2(\rho)g(\rho) \geq \psi_1(\rho)\psi_2(\rho)f_1^2(\lambda)+\varphi_1(\lambda)\varphi_2(\lambda)h_1^2(\rho). \tag{2. 12} $$

Multiplying both sides of (2. 12 ) by $\frac{\left(\frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\right)^{\frac{\delta}{\mathcal{K}}-1}}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)\mathcal{K}\Gamma_{\mathcal{K}}(\vartheta)} \frac{\left(\frac{(x-u)^{\alpha}-(\rho-u)^{\alpha}}{\alpha}\right)^{\frac{\vartheta}{\mathcal{K}}-1}}{(\lambda-u)^{1-\alpha}(\rho-u)^{1-\alpha}}$ and double integrating with respect to $\lambda$ and $\rho$ from $u$ to $x$, we have

$$\begin{aligned}
&\left(\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_u^x \left(\frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\right)^{\frac{\delta}{\mathcal{K}}-1}\frac{1}{(\lambda-u)^{1-\alpha}}\varphi_1(\lambda)f_1(\lambda)d\lambda\right)\\
&\times\left(\frac{1}{k\Gamma_{\mathcal{K}}(\vartheta)}\int_u^x \left(\frac{(x-u)^{\alpha}-(\rho-u)^{\alpha}}{\alpha}\right)^{\frac{\vartheta}{\mathcal{K}}-1}\frac{1}{(\rho-u)^{1-\alpha}}\psi(\rho)h_1(\rho)d\rho\right)\\
&+\left(\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_u^x \left(\frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\right)^{\frac{\delta}{\mathcal{K}}-1}\frac{1}{(\lambda-u)^{1-\alpha}}\varphi_2(\lambda)f_1(\lambda)d\lambda\right)\\
&\times\left(\frac{1}{k\Gamma_{\mathcal{K}}(\vartheta)}\int_u^x \left(\frac{(x-u)^{\alpha}-(\rho-u)^{\alpha}}{\alpha}\right)^{\frac{\vartheta}{\mathcal{K}}-1}\frac{1}{(\rho-u)^{1-\alpha}}\psi_2(\rho)h_1(\rho)d\rho\right)\\
\geq&\left(\frac{1}{k\Gamma_{\mathcal{K}}(\rho)}\int_u^x \left(\frac{(x-u)^{\alpha}-(\rho-u)^{\alpha}}{\alpha}\right)^{\frac{\vartheta}{\mathcal{K}}-1}\frac{1}{(\rho-u)^{1-\alpha}}\psi_1(\rho)\psi_2(\rho)d\rho\right)\\
&\times\left(\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_u^x \left(\frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\right)^{\frac{\delta}{\mathcal{K}}-1}\frac{1}{(\lambda-u)^{1-\alpha}}f_1^2(\lambda)d\lambda\right)\\
&+\left(\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_u^x \left(\frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\right)^{\frac{\delta}{\mathcal{K}}-1}\frac{1}{(\lambda-u)^{1-\alpha}}\varphi_1(\lambda)\varphi_2(\lambda)d\lambda\right)\\
&\times\left(\frac{1}{k\Gamma_{\mathcal{K}}(\vartheta)}\int_u^x \left(\frac{(x-u)^{\alpha}-(\rho-u)^{\alpha}}{\alpha}\right)^{\frac{\vartheta}{\mathcal{K}}-1}\frac{1}{(\rho-u)^{1-\alpha}}h_1^2(\rho)d\rho\right).
\end{aligned}$$

which leads to

$$\begin{aligned}
&\left({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\varphi_1 f_1\right)(x)\left({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\psi h_1\right)(x)+\left({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\varphi_2 f_1\right)(x)\left({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\psi_2 g\right)(x)\\
\geq&\left({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\psi_1\psi_2\right)(x)\left({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(f_1^2\right)(x)+\left({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\varphi_1\varphi_2\right)\left({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(h_1^2\right)(x).
\end{aligned}$$

Applying the $AM-GM$ inequality, we get

$$\begin{aligned}
&\left({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\varphi_1 f_1\right)(x)\left({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\psi h_1\right)(x)+\left({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\varphi_2 f_1\right)(x)\left({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\psi_2 h_1\right)(x)\\
\geq 2&\sqrt{\left({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\psi_1\psi_2\right)(x)\left({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(f_1^2\right)(x)\left({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(\varphi_1\varphi_2\right)\left({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\left(h_1^2\right)(x)},
\end{aligned}$$

which leads to the desired inequality (2. 11 ). This completes the proof. □

As a special case of Theorem 2.3, we get the following result.

**Corollary 2.4.** *Let $f_1$ and $h_1$ be two integrable functions on $[0,\infty)$ satisfying $(II)$. Then for $\delta>0$ and $\vartheta>0$, we have*

$$\frac{\alpha^{\frac{\delta+\vartheta}{\mathcal{K}}}\Gamma_{\mathcal{K}}(\delta+k)\Gamma_{\mathcal{K}}(\vartheta+k)}{(x-u)^{\frac{\alpha(\delta+k)}{\mathcal{K}}}}\frac{\left(\left({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\{f_1^2\}(x)\right)\left(\left({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\{h_1^2\}(x)\right)}{\left(\left(\left({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\{f_1\}(x)\right)\left(\left({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\right)\{h_1\}(x)\right)\right)^2}\leq\frac{1}{4}\left(\sqrt{\frac{pq}{PQ}}+\sqrt{\frac{PQ}{pq}}\right)^2.$$

**Theorem 2.5.** *Suppose that all assumptions of Theorem 2.1 are satisfied. Then for $\delta > 0$ and $\vartheta > 0$, the following inequality holds:*

$$ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} \quad f_1^2 \quad (x) \quad {}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} \quad h_1^2 \quad (x) \leq \quad {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} \quad \Big\{\frac{\varphi_2 h_1 f_1}{\psi_1}\Big\}(x) \quad {}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} \quad \Big\{\frac{\psi_2 f_1 h_1}{\varphi_1}\Big\}(x). \tag{2. 13}$$

*Proof.* From condition $(I)$, we have

$$\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_{u}^{x}\Big(\frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{1}{(\lambda-u)^{1-\alpha}}f_1^2(\lambda)d\lambda$$
$$\leq \frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_{u}^{x}\Big(\frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{1}{(\lambda-u)^{1-\alpha}}\frac{\varphi_2(\lambda)}{\psi_1(\lambda)}f_1(\lambda)h_1(\lambda)d\lambda,$$

which implies

$$\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\Big)\big\{f_1^2\big\}(x) \leq \Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\Big)\Big\{\frac{\varphi_2 f_1 h_1}{\psi_1}\Big\}(x). \tag{2. 14}$$

Analogously, we obtain

$$\frac{1}{k\Gamma_{\mathcal{K}}(\vartheta)}\int_{u}^{x}\Big(\frac{(x-u)^{\alpha}-(\rho-u)^{\alpha}}{\alpha}\Big)^{\frac{\vartheta}{\mathcal{K}}-1}\frac{1}{(\rho-a)^{1-\alpha}}h_1^2(\rho)d\rho$$
$$\leq \frac{1}{k\Gamma_{\mathcal{K}}(\vartheta)}\int_{u}^{x}\Big(\frac{(x-u)^{\alpha}-(\rho-u)^{\alpha}}{\alpha}\Big)^{\frac{\vartheta}{\mathcal{K}}-1}\frac{1}{(\rho-u)^{1-\alpha}}\frac{\psi_2(\rho)}{\varphi_1(\rho)}f_1(\rho)h_1(\rho)d\rho,$$

from which one has

$$\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\Big)\big\{h_1^2\big\}(x) \leq \Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\Big)\Big\{\frac{\psi_2 f_1 h_1}{\varphi_1}\Big\}(x). \tag{2. 15}$$

Multiplying (2. 14 ) and (2. 15 ), we get the desired inequality (2. 13 ) . □

**Corollary 2.6.** *Let $f_1$ and $h_1$ be two positive integrable functions on $[0,\infty)$ satisfying $(II)$. Then for $\delta$ and $\vartheta > 0$, we have*

$$\frac{\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\Big)\big\{f_1^2\big\}(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\Big)\big\{h_1^2\big\}(x)}{\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\Big)\big\{f_1h_1\big\}(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}\Big)\big\{f_1h_1\big\}(x)} \leq \frac{QP}{qp}.$$

## 3. Other integral inequalities for generalized $\mathcal{K}$-fractional conformable integrals

In this section, we give some new integral inequalities for generalized $\mathcal{K}$-fractional conformable integrals.

**Theorem 3.1.** *Let $f_1$ and $h_1$ be two positive function defined on $[0,\infty)$, and $p,\ q>1$ satisfying $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequalities hold:*

$$(a)\qquad q\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^p\Big)(x)\frac{(x-a)^{\frac{\alpha\vartheta}{\mathcal{K}}}}{\alpha^{\frac{\vartheta}{\mathcal{K}}}\Gamma_{\mathcal{K}}(\vartheta+k)}+p\frac{(x-a)^{\frac{\alpha\delta}{\mathcal{K}}}}{\alpha^{\frac{\delta}{\mathcal{K}}}\Gamma_{\mathcal{K}}(\delta+k)}\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1^q\Big)(x)$$
$$\geq pq\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1\Big)(x),$$

$$(b)\qquad q\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^p\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\vartheta}_{u}h_1^p\Big)(x)+p\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^q\Big)(x)\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1^q\Big)(x)$$
$$\geq pq\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1h_1\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1h_1\Big)(x),$$

$$(c)\qquad q\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^p\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1^q\Big)(x)+p\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\vartheta}_{u}f_1^q\Big)(x)\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\delta}_{u}h_1^p\Big)(x)$$
$$\geq pq\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1h_1^{p-1}\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1h_1^{q-1}\Big)(x),$$

$$(d)\qquad q\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1^q\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^p\Big)(x)+p\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\vartheta}_{u}f_1^p\Big)(x)\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\delta}_{u}h_1^q\Big)(x)$$
$$\geq pq\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^{p-1}h_1^{q-1}\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1h_1\Big)(x),$$

$$(e)\qquad q\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^p\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1^2\Big)(x)+p\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1^q\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^2\Big)(x)$$
$$\geq pq\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1h_1\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}(f_1h_1)^{\frac{2}{q}}\Big)(x),$$

$$(f)\qquad q\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^2\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1^{2-p}\Big)(x)+p\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1^2\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1^{2-q}\Big)(x)$$
$$\geq pq\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^{\frac{2}{q}}\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}(h_1)^{\frac{2}{q}}\Big)(x).$$

*Proof.* According to well-known Young's inequality, one obtains

$$\frac{1}{p}a^p+\frac{1}{q}b^q\geq ab\quad\forall a,b\geq 0,\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1. \tag{3. 16}$$

Putting $a=f_1(\lambda)$ and $b=h_1(\rho),\ \lambda,\rho>0$, we have

$$\frac{1}{p}f_1(\lambda)^p+\frac{1}{q}h_1(\rho)^q\geq f_1(\lambda)h_1(\rho),\quad f_1(\lambda)h_1(\rho)\geq 0. \tag{3. 17}$$

Multiplying both sides of (3. 17 ) by $\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\Big(\frac{(x-u)^\alpha-(\lambda-u)^\alpha}{\alpha}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{1}{(\lambda-u)^{1-\alpha}}$,then integrating the resulting inequality with respect $\lambda$ from $u$ to $x$, we get

$$\frac{1}{p}\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_u^x\Big(\frac{(x-u)^\alpha-(\lambda-u)^\alpha}{\alpha}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{1}{(\lambda-u)^{1-\alpha}}f_1(\lambda)^pd\lambda$$
$$+\frac{1}{q}h_1(\rho)^q\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_u^x\Big(\frac{(x-u)^\alpha-(\lambda-u)^\alpha}{\alpha}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{1}{(\lambda-u)^{1-\alpha}}d\lambda$$
$$\geq g(\rho)\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\int_u^x\Big(\frac{(x-u)^\alpha-(\lambda-u)^\alpha}{\alpha}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{1}{(\lambda-u)^{1-\alpha}}f_1(\lambda)d\lambda,$$

which leads to

$$\frac{1}{p}\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^p\Big)(x)+\frac{1}{q}\frac{(x-a)^{\frac{\alpha\delta}{\mathcal{K}}}}{\alpha^{\frac{\delta}{\mathcal{K}}}\Gamma_{\mathcal{K}}(\delta+k)}h_1(\rho)^q\geq h_1(\rho)\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1\Big)(x).$$

Multiplying both sides of (3. 17 ) by $\frac{1}{k\Gamma_{\mathcal{K}}(\vartheta)}\Big(\frac{(x-u)^{\alpha}-(\rho-u)^{\alpha}}{\alpha}\Big)^{\frac{\vartheta}{\mathcal{K}}-1}\frac{1}{(\rho-u)^{1-\alpha}}$, then integrating the resulting inequality with respect $\lambda$ from $u$ to $x$, we get

$$\begin{aligned}&q\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1^p\Big)(x)\frac{(x-u)^{\frac{\alpha\vartheta}{\mathcal{K}}}}{\alpha^{\frac{\vartheta}{\mathcal{K}}}\Gamma_{\mathcal{K}}(\vartheta+k)}+p\frac{(x-u)^{\frac{\alpha\delta}{\mathcal{K}}}}{\alpha^{\frac{\delta}{\mathcal{K}}}\Gamma_{\mathcal{K}}(\delta+k)}\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1^q\Big)(x)\\&\geq pq\Big({}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}f_1\Big)(x)\Big({}^{\vartheta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u}h_1\Big)(x),\end{aligned}$$

which implies $(a)$. □

The rest of inequalities can be shown in similar way by the following choice of parameters in the Young's inequality.

$$\begin{array}{lll}(b) & (a)=f_1(\lambda)h_1(\rho), & (b)=f_1(\rho)h_1(\lambda).\\(c) & (a)=\dfrac{f_1(\lambda)}{h_1(\lambda)}, & (b)=\dfrac{f_1(\rho)}{h_1(\rho)},\ h_1(\lambda),h_1(\rho)\neq 0.\\(d) & (a)=\dfrac{f_1(\rho)}{f_1(\lambda)}, & (b)=\dfrac{h_1(\rho)}{h_1(\lambda)},\ f_1(\lambda),h_1(\rho)\neq 0.\\(e) & (a)=f_1(\lambda)h_1^{\frac{2}{p}}(\rho), & (b)=f_1^{\frac{2}{q}}(\rho)h_1(\lambda).\\(f) & (a)=\dfrac{f_1^{\frac{2}{p}}(\lambda)}{h_1(\rho)}, & (b)=\dfrac{h_1^{\frac{2}{q}}(\lambda)}{h_1(\rho)},\ h_1(\rho),f_1(\rho)\neq 0.\end{array}$$

Repeating the foregoing argument, we obtain $(b)-(f)$.

**Theorem 3.2.** *Suppose that $f_1$ and $h_1$ are two positive function defined on $[0,\infty)$ such that for all $\lambda\in[u,x]$,*

$$q=\min_{u\leq\lambda\leq x}\frac{f_1(\lambda)}{h_1(\lambda)},\quad Q=\max_{u\leq\lambda\leq x}\frac{f_1(\lambda)}{h_1(\lambda)}. \tag{3. 18}$$

*Then the following inequalities hold:*

$$(a) \qquad 0 \leq \Big( {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1^2 \Big)(x) \Big( {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} h_1^2 \Big)(x) \leq \frac{(q+Q)^2}{4qQ} \Big( {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} fg \Big)(x),$$

$$(b) \qquad 0 \leq \sqrt{\Big( {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1^2 \Big)(x) \Big( {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} h_1^2 \Big)(x)} - \Big( {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1 h_1 \Big)(x)$$
$$\leq \frac{(\sqrt{Q}-\sqrt{q})^2}{2\sqrt{qQ}} \Big( {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1 h_1 \Big)(x),$$

$$(c) \qquad 0 \leq \Big( {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1^2 \Big)(x) \Big( {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} h_1^2 \Big)(x) - \bigg( \Big( {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1 h_1 \Big)(x) \bigg)^2$$
$$\leq \frac{(\sqrt{Q}-\sqrt{q})}{4qQ} \bigg( \Big( {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1 h_1 \Big)(x) \bigg)^2.$$

*Proof.* It follows from (3. 18 ) and

$$\Big(\frac{f_1(\lambda)}{h_1(\lambda)} - q\Big)\Big(Q - \frac{f_1(\lambda)}{h_1(\lambda)}\Big) h_1^2(\lambda) \geq 0, \quad u \leq \lambda \leq x, \tag{3. 19}$$

we can write as

$$f_1^2(\lambda) + qQ h_1^2(\lambda) \leq (q+Q) f_1(\lambda) h_1(\lambda). \tag{3. 20}$$

Multiplying (3. 20 ) by $\frac{1}{k\Gamma_{\mathcal{K}}(\delta)}\Big(\frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{1}{(\lambda-u)^{1-\alpha}}$, which is positive because $\lambda \in (u,x)$. Then by integrating with respect to $\lambda$, over $u$ to $x$, we get

$$\frac{1}{k\Gamma_{\mathcal{K}}(\delta)} \int_u^x \frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}^{\frac{\delta}{\mathcal{K}}-1} \frac{1}{(\lambda-u)^{1-\alpha}} f_1^2(\lambda) d\lambda$$
$$+qQ \frac{1}{k\Gamma_{\mathcal{K}}(\delta)} \int_u^x \frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}^{\frac{\delta}{\mathcal{K}}-1} \frac{1}{(\lambda-u)^{1-\alpha}} h_1^2(\lambda) d\lambda$$
$$\leq (q+Q) \frac{1}{k\Gamma_{\mathcal{K}}(\delta)} \frac{(x-u)^{\alpha}-(\lambda-u)^{\alpha}}{\alpha}^{\frac{\delta}{\mathcal{K}}-1} \frac{1}{(\lambda-u)^{1-\alpha}} f_1(\lambda) h_1(\lambda) d\lambda, \tag{3. 21}$$

implies that

$${}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1^2 \ (x) + qQ \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} h_1^2 \ (x) \leq (q+Q) \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1 h_1 \ (x). \tag{3. 22}$$

On the other hand, it follows from $qQ > 0$ and

$$\sqrt{\ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1^2 \ (x)} - \sqrt{qQ \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} h_1^2 \ (x)}^{\ 2} \geq 0,$$

observe that

$$2\sqrt{\ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1^2 \ (x)}\sqrt{qQ \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} h_1^2 \ (x)} \leq \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1^2 \ (x) + qQ \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} h_1^2 \ (x), \tag{3. 23}$$

then from (3. 22 ) and (3. 23 ), we obtain

$$4qQ \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1^2 \ (x) \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} h_1^2 \ (x) \leq (q+Q)^2 \ {}^{\delta}_{\mathcal{K}}\mathcal{J}^{\alpha}_{u} f_1 h_1 \ (x). \tag{3. 24}$$

Which implies $(a)$. By some transformation of $(a)$, similarly, we obtain $(b)$ and $(c)$. □

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