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An Analytic Approximation for Time-Fractional BBM-Burger Equation

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Abstract. This paper attempts to carry out a study on a specific type of time-fractional differential equation called Benjamin-Bona-Mahony-Burger (BBM-Burger). This equation describes the mathematical model of unidirectional transmission of low-amplitude long waves through frequency-dependent dispersive media. To go ahead the research, optimal homotopy asymptotic method (OHAM) and its new version (NOHAM) are applied to find analytic approximations to the BBM-Burger equation using the symbolic Maple package. Moreover, convergence of the both methods are addressed. The approximate results of the proposed methods are compared with the exact solution of BBM-Burger equation. What we learn from the results of applying OHAM and NOHAM on the BBM-Burger equation, is the high satisfactory accuracy. However, NOHAM produces an approximate solution with lower computational cost, than OHAM.

AMS (MOS) Subject Classification Codes: 34K28; 57T20.
Key Words: Optimal homotopy asymptotic method, New version of optimal homotopy asymptotic method, Convergence analysis, Benjamin-Bona-Mahony-Burger equation, Caputo derivative.

1. Introduction

Fractional differential equations (FDEs) play a crucial role in mathematical modeling of various problems relative to many scientific areas like mathematics and nonlinear dynamical systems [2, 10]. Leibniz and Hopital (1695) first introduced the basic idea of FDE, where the orders of derivatives and integrals can take integer or rational numbers in the interval $[0, 1]$. Since then, there have been published a lot of studies pertinent different aspects of fractional calculus. Lakshmikantham and Vatsala [15], examined the basic theory and initial value problem using the operators of Riemann-Liouville fractional calculus. Another research conducted by Diethelm and Ford [4], a series of analytic questions were proposed on the existence and uniqueness of numerical solutions obtained by FDEs. In addition to the studies on the theoretical aspects of FDEs, numerous researches have been conducted to develop various numerical approaches for finding analytic approximate solutions for non-linear systems of FDEs. Some of these approaches are as follows; finite difference [29], homotopy analysis [17], homotopy asymptotic [22], variational iteration [20], polynomial least squares [3], $(G^{'}/G)$-expansion [28], and others [13, 14, 25, 27]. Homotopy analysis method (HAM), introduced by S. Liao [16] is originated from homotopy concept, which relates to a fundamental concept in the mathematical area of topology. Many researchers have applied HAM to find an approximed solution to linear and non-linear functional problems, successfully [6, 8, 23]. Marinca and herisanu, used classical HAM and Homotopy Perturbation Method (HPM) to introduce a new approach called optimal homotopy asymptotic method (OHAM) [18], which removes any need to h-curves study. In comparison with HAM, its optimal form (OHAM) has two distinct advantages: the first one is to accelerated the convergence to the series solution, and the second one is to provide the possibility of adjusting of the convergence region, using an auxiliary function. In the similar area of research, Ali et al. published a paper which proposed an improvement of OHAM based on initial condition, auxiliary functions, controlling parameters of convergence, and homotopy theory [1]. It is important to mention that, hereafter this extended form of OHAM, will be called NOHAM. During recent years, many researchers have shown the effectiveness of homotopy based on utilizing of this concept on the methods for solving many differential and integral equations. [9, 21]. However, the application of NOHAM is not yet investigated for solving the systems of nonlinear FDEs like Benjamin-Bona-Mahony-Burger (BBM-Burger). Therefore, in this study, the time-fractional differential equation of BBM-Burger is analyzed to approximate its analytic solution by optimal homotopy asymptotic method (OHAM) and its new extended form (NOHAM). All calculations are performed by Maple symbolic package. Fundamentally, the BBM-Burger equation is served to model the propagation of long waves with small-amplitude in nonlinear frequency-dependent dispersive systems. In fact, BBBM-Burger is an extended form of the Kortewegde Vries (KdV) equation, a mathematical description of the wave propagation in shallow water surfaces. Generally, both of BBM-Burger and KdV equations are classified as wave breaking models [11, 26]. Since the KdV equation is not applicable to some physical systems, the BBM-Burger is developed to explain the mathematical unidirectional transmission of long-wave signals in a certain frequency-dependent (dispersive) medium [11, 13, 24]. The mathematical description of the BBM-Burger equation, of integer order, is given by

$$u_t - u_{xxt} - \beta u_{xx} + uu_x + \gamma u_x = 0, \quad x \in [x_L, x_R]. \tag{1. 1}$$

where the coefficients β and γ represent positive constants and $[x_L, x_R]$ is a domain interval. The time-fractional version of BBM-Burger equation is developed for analyzing the dynamic behavior of physical systems. Similar to [12], time-fractional form of BBM-Burger equation is as follows

$$D_t^{\alpha} u - u_{xxt} + u_x + (\frac{u^2}{2})_x = 0, \quad t > 0, \; 0 < \alpha \leq 1, \tag{1. 2}$$

where α represents the order of fractional derivative with respect to the time, ranging in the interval of $(0,1]$. For the time-fractional BBM-Burger equation, the initial condition and exact solution for $\alpha = 1$, are given by Eqs.(1. 3) and (1. 4), respectively [5]

$$u(x,0) = sech^2(\frac{x}{4}), \tag{1. 3}$$

$$u(x,t) = sech^2(\frac{x}{4} - \frac{t}{4}). \tag{1. 4}$$

The results of implementation of these approaches are compared with those of New homotopy analysis transform method (FHATM) [12], and the exact solutions.

2. Basic definitions of fractional calculus

In this section, some basic definitions of fractional calculus will be explained briefly.

2.1. **Definition.** A real-valued function $f(\kappa)$ with $\kappa > 0$ can be defined on the space C_μ, $\mu \in \mathbb{R}$ if there is a real number $\rho > \mu$ such that $f(\kappa) = \kappa^p f_1(\kappa)$, where $f_1(\kappa) \in C[0, +\infty)$ and it is defined on the space C_μ^n , if $f^{(n)} \in C_\mu$ for $n \in \mathbb{N}$.

2.2. **Definition.** The Riemann-Liouvilles integral of fractional order $\alpha \geq 0$ of a continuous function $f \in C_\mu$ with $\mu \geq -1$ is given as

$$J^\alpha f(\kappa) = \frac{1}{\Gamma(\alpha)} \int_0^\kappa (\kappa - s)^{\alpha-1} f(s) ds, \ \alpha > 0, \tag{2. 5}$$

$$J^0 f(\kappa) = f(\kappa). \tag{2. 6}$$

By considering $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$, the main properties of the operator J^α are listed as the following

$$J^\alpha J^\beta f(\kappa) = J^{\alpha+\beta} f(\kappa), \tag{2. 7}$$

$$J^\alpha J^\beta f(\kappa) = J^\beta J^\alpha f(\kappa), \tag{2. 8}$$

$$J^\alpha \kappa^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \kappa^{\alpha+\gamma}. \tag{2. 9}$$

2.3. **Definition.** The fractional-order derivative of $f(\kappa)$ is defined, in caputo sense, as follows

$$D^\alpha f(\kappa) = \frac{1}{\Gamma(m-\alpha)} \int_0^\kappa (\kappa - s)^{m-\alpha-1} f^m(s) ds, \tag{2. 10}$$

for $m - 1 < \alpha \leq m, m \in \mathbb{N}, \kappa > 0, f \in C_{-1}^m$.

2.4. **Lemma.** By assuming, $m-1 < \alpha \leq m, m \in \mathbb{N}, f \in C_{-1}^m$, and $\mu \geq -1$ the following properties will be set

$$D^\alpha J^\alpha f(\kappa) = f(\kappa), \tag{2. 11}$$

$$(J^\alpha D^\alpha) f(\kappa) = f(\kappa) - \Sigma_{n=0}^{m-1} f^{(n)}(0^+) \frac{\kappa^k}{n!}. \tag{2. 12}$$

3. BASIC PRINCIPLES OF THE PROPOSED TECHNIQUES

3.1. **OHAM.** The fundamental idea of OHAM is mainly based on the homotopy theory in topology [7,18,19]. Suppose the following fractional equation with the initial and boundary conditions

$$\Phi(u(\kappa)) + f(\kappa) + \Psi(u(\kappa)) = 0, \quad B(u(\kappa)) = 0, \tag{3. 13}$$

where κ is an independent variable, $\Phi, u(\kappa), f(\kappa)$, and Ψ represent a linear type operator, an unknown function, a known function, and a nonlinear type operator, respectively. Also, $B(u(\kappa)) = 0$ stands for a boundary operator.
An optimal homotopy $\mathbf{H}(\varphi(\kappa; q)) : \mathbb{R} \times [0, 1] \longrightarrow \mathbb{R}$ is constructed in order to satisfy a deformation equation of zero-order as the following

$$\begin{aligned} \mathbf{H}(\varphi(\kappa; q)) &= (1 - q)[\Phi(\varphi(\kappa; q)) + f(\kappa)] \\ &= H(q)[\Phi(\varphi(\kappa; q)) + f(\kappa) + \Psi(\varphi(\kappa; q))], \quad B(\varphi(\kappa; q)) = 0, \end{aligned} \tag{3. 14}$$

where q shows an embedding parameter in the interval $[0, 1]$, $H(\kappa, q, c_i)$ is an auxiliary function with nonzero and zero outputs, respectively for $q \neq 0$ and $q = 0$, $u_0(\kappa)$ represents the initial condition of $u(\kappa)$, and $\varphi(\kappa; q, c_i)$ is an unknown function. By inserting $q = 0$ and $q = 1$ into Eq. (3. 14), the following functions are obtained

$$\varphi(\kappa; 0, c_i) = u_0(\kappa), \tag{3. 15}$$

$$\varphi(\kappa; 1, c_i) = u(\kappa). \tag{3. 16}$$

Therefore, $\varphi(\kappa; q, c_i)$ will change continuously from the initial condition $u_0(\kappa)$ to $u(\kappa)$ with an increase in the amount of q from 0 to 1.
By putting $q = 0$ into Eq. (3. 14), the initial condition of $u_0(\kappa)$ is determined as a possible solution for the problem defined by
Zeroth order problem:

$$\Phi(u_0(\kappa)) + f(\kappa) = 0, \ B(u_0(\kappa)) = 0. \tag{3. 17}$$

Lets state the definition of auxiliary function, $H(\kappa, q, c_i)$ as the following

$$H(\kappa, q, c_i) = qH_1(\kappa, c_i) + q^2H_2(\kappa, c_i) + ..., \tag{3. 18}$$

one should notice that the auxiliary functions $H_i(\kappa, c_i), i = 1, 2, ...,$, depends only upon κ and c_i. By expanding $\varphi(\kappa; q, c_i)$ in powers of q, the following expansion is obtained

$$\varphi(\kappa; q, c_i) = u_0(\kappa) + \Sigma_{m=1}^{\infty} u_m(\kappa, c_1, c_2, ..., c_m)q^m. \tag{3. 19}$$

By putting Eqs.(3. 15)-(3. 19) into Eq.(3. 14) and equating the coefficients of the terms with identical powers of q, the mth-order problem subjected to $B(u_m(\kappa)) = 0$ is defined as

$$\begin{aligned} \Phi(u_m(\kappa)) &= \Phi(u_{m-1}(\kappa)) \\ &+\Sigma_{j=1}^{m} c_j[\Phi(u_{m-j}(\kappa)) + \Psi(u_{m-j}(\kappa)) + \delta_{jm} f(\kappa)], \quad m = 1, 2, 3, ..., \end{aligned} \tag{3. 20}$$

where δ_{jm} is the Kronecker delta.

Solving Eq. (3. 20) gives various approximates solutions $u_m(\kappa, c_1, c_2, ..., c_m)$,but there exist still m unknown auxiliary parameters $(c_1, c_2, c_3, ..., c_m)$ in the obtained solutions. It is assumed that the auxiliary parameters $(c_1, c_2, c_3, ..., c_m)$, are appropriately determined to establish the convergence of series (3. 19) at $q = 1$. Hence, putting Eqs. (3. 15) and (3. 16) into Eq. (3. 19) for $q = 1$ gives the solution $u(\kappa)$ as

$$u(\kappa, c_1, c_2, c_3, ...) = u_0(\kappa) + \Sigma_{m=1}^{\infty} u_m(\kappa, c_1, c_, c_3, ..., c_m). \tag{3. 21}$$

The mth order approximate solution can be calculated by

$$\widehat{u}(\kappa, c_1, c_2, c_3, ...) = u_0(\kappa) + \Sigma_{m=1}^{n} \widehat{u}_m(\kappa, c_1, c_, c_3, ..., c_m). \tag{3. 22}$$

After putting Eq. (3. 22) into Eq. (3. 13), the following residual is obtained as

$$R(\kappa, c_i) = \Phi(\widehat{u}(\kappa, c_i)) + \Psi(\widehat{u}(\kappa, c_i)) + f(\kappa),\ i = 1, 2, 3, \cdots . \tag{3. 23}$$

By supposing that $R(\kappa, c_i) = 0$, the exact solution will be $\widehat{u}(\kappa, c_i)$. However, such a case could not be true for a nonlinear equation. the functional $J(c_i)$ can be minimized using the least squares technique

$$J(c_i) = \int_a^b R^2(\kappa, c_i) d\kappa, \tag{3. 24}$$

where a and b are two real values relating to the given problem. The optimal values of $c_i (i = 1, 2, ..., m)$ will be determined based on the following conditions

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \frac{\partial J}{\partial c_3} = ... = \frac{\partial J}{\partial c_m} = 0. \tag{3. 25}$$

The approximate solution at the level m will be determined simply.

3.2. **NOHAM.** The main idea of NOHAM is based on OHAM. Consider the same boundary-value problem as defined

$$\Phi(u(\kappa)) + f(\kappa) + \Psi(u(\kappa)) = 0,\ B(u(\kappa)) = 0,\ \kappa \in \mathbb{R}, \tag{3. 26}$$

where $\kappa, f(\kappa), \Phi, u(\kappa), \Psi$, and B are as pre-defined. Suppose that $u_0(\kappa)$ is an initial approximation of $u(\kappa)$ such that

$$\Phi(u_0(\kappa)) + f(\kappa) = 0,\ B(u_0(\kappa)) = 0. \tag{3. 27}$$

The function $\varphi(\kappa; q, c_i)$ can be rewritten as the following form

$$\varphi(\kappa, q, c_i) = u_0(\kappa) + q u_1(\kappa, c_i), \tag{3. 28}$$

where q is an embedding parameter over the interval of $[0, 1]$. Thus, the approximate solution of the first order is calculated by

$$\widehat{u}(\kappa, c_i) = u(\kappa, c_i) = u_0(\kappa) + u_1(\kappa, c_i),\ B(\widehat{u}(\kappa, c_i)) = 0, \tag{3. 29}$$

where $c_1, c_2, ..., c_n$ represent auxiliary parameters, which can be determined eventually. A family of equations can be defined as follows

$$\begin{aligned} &\mathbf{H}[\Phi(\varphi(\kappa; q, c_i)) + f(\kappa), H(\kappa, c_i), \Psi(\varphi(\kappa; q, c_i))] = \\ &\qquad \Phi(u_0(\kappa)) + f(\kappa) + q[\Phi(u_1(\kappa, c_i)) - H(\kappa, c_i)\Psi(u_0(\kappa))]. \end{aligned} \tag{3. 30}$$

Now, an auxiliary function $H(\kappa, c_i)$ is defined as

$$H(\kappa, c_i) = H_1(\kappa, c_i) + H_2(\kappa, c_i) + H_3(\kappa, c_i) + \cdots . \tag{3. 31}$$

Note that Eq. (3. 30) satisfies the following properties

$$\begin{aligned} &\mathbf{H}[\Phi(\varphi(\kappa; 0, c_i)) + f(\kappa), H(\kappa, c_i), \Psi(\varphi(\kappa; 0, c_i))] \\ &\qquad = \Phi(u_0(\kappa)) + f(\kappa) = 0, \end{aligned} \tag{3. 32}$$

$$\begin{aligned}&\mathbf{H}[L(\varphi(\kappa;1,c_i))+f(\kappa),H(\kappa,c_i),N(\varphi(\kappa;1,c_i))]\\&\qquad=H(\kappa,c_i)[\Phi(\widehat{u}(\kappa,c_i))+f(\kappa)+\Psi(\widehat{u}(\kappa,c_i))]=0,\end{aligned} \tag{3. 33}$$

where $H(\kappa, c_i) \neq 0$ represents an auxiliary function needs some explanations.
From Eqs. (3. 28) and (3. 29), the following equations are obtained

$$\varphi(\kappa,0,c_i)=u_0(\kappa), \qquad \varphi(\kappa,1,c_i)=\widehat{u}(\kappa,c_i). \tag{3. 34}$$

The following functions can be defined from Eqs. (3. 30) and (3. 31)

$$\begin{aligned}&\Phi(u_1(\kappa,c_i))=H(\kappa,c_i)\Psi(u_0(\kappa)),\\&\qquad B(u_1(\kappa,c_i))=0, \quad i=1,2,...,n.\end{aligned} \tag{3. 35}$$

By comparing the coefficients of two terms q^0 and q^1 in Eq. (3. 30), we obtain equation $u_0(\kappa)$, determinate by Eq. (3. 27), and $u_1(\kappa, c_i)$ as following

$$\Phi(u_1(\kappa,c_i))=H(\kappa,c_i)\Psi(u_0(\kappa)), \quad B(u_1(\kappa,c_i))=0, \quad i=1,2,...,n. \tag{3. 36}$$

The general form of the nonlinear operator is rewritten as follows

$$\Psi(u_0(\kappa))=\Sigma_{i=1}^{m}h_i(\kappa)g_i(\kappa), \tag{3. 37}$$

where $h_i(\kappa)$ and $g_i(\kappa)$ represent known functions, which are defined based on the function $u_0(\kappa)$ and the nonlinear operator Ψ. Also, m is an integer. The equation of (3. 36) consists the summation of two solutions, the solution of homogeneous form and a particular solution of non-homogeneous. Hence, the solution of Eq. (3. 36) is found by summing the two mentioned solutions, although it may be selected only from particular solutions in exceptional cases. Let the unknown function $u_1(\kappa, c_j)$ be defined by

$$u_1(\kappa,c_j)=\Sigma_{i=1}^{m}H_i(\kappa,h_j(\kappa),c_j)g_i(\kappa), \ B(u_1(\kappa))=0, \tag{3. 38}$$

or

$$\begin{aligned}&u_1(\kappa,c_j)=\Sigma_{i=1}^{m}H_i(\kappa,g_j(\kappa),c_j)h_i(\kappa), \ B(u_1(\kappa))=0,\\&\qquad j=1,2,...,n,\end{aligned} \tag{3. 39}$$

where $H_i(\kappa, h_j(\kappa), c_j)$ is a linear mixture of several functions h_i, several terms from corresponding homogeneous equation, and some unknown parameters c_j for $j = 1, 2, ..., n$. Also m represents an arbitrary chosen integer. Now, if h_1 represents a polynomial function as $h_1 = \kappa^3$, then $H_i(\kappa, h_j(\kappa), c_j)$ can be considered as a mixture of polynomials: $H_i(\kappa, h_j(\kappa), c_j) = c_1\kappa + c_2\kappa^3 + c_3\kappa^5 + \cdots$. When h_1 represents a trigonometric function as $h_1 sin(\kappa)$, then $H_i(\kappa, h_j(\kappa), c_j) = c_1 sin(\kappa) + c_2 cos(\kappa) + c_3 sin(2\kappa) + \cdots$. In a similar way, if h_1 represents a logarithmic function, then $H_i(\kappa, h_j, c_j) = c_1 ln(\kappa) + c_2\kappa ln(\kappa) + c_3\kappa^2 ln(2\kappa) + \cdots$, where H_i and m can be determined using various ways. The solution $u_1(\kappa, c_j)$ calculated by Eq. (3. 38) is not considered as a complete approximate solution for Eq. (3. 26), but $\widehat{u}(\kappa, c_i)$ in Eq. (3. 29) is the approximate solution of Eq. (3. 26). The similar considerations could be valid for Eq. (3. 39), where the parameters h_i and g_i are interchangeable. Now, the complete solution of Eq. (3. 26) can be achieved by determining the optimal auxiliary parameters c_i (for $i = 1, 2, ..., n$) and substituting the values of $u_0(\kappa)$ and $u_1(\kappa, c_i)$ into Eq. (3. 29).

4. Convergence analysis of the OHAM and NOHAM

Suppose that $\{u_n\}_{n=0}^{\infty}$ is a sequence of approximate solutions. If $\exists 0 < \delta < 1$ the series $\Sigma_{k=0}^{m-1} u_k(\kappa)$ converges if $\|u_{k+1}\| \leq \delta \|u_k\|, \forall k \geq k_0$ for some $k_0 \in \mathbb{N}$.
in Eq.(3.23) $u_1(\kappa)$, $u_1(\kappa)$ can be decomposed as follows

$$u_1(\kappa) = c_1 Y_1(\kappa) + c_2 Y_2(\kappa) + \cdots . \tag{4. 40}$$

From Eq.(4.1), we have

$$V_1 = c_1 Y_1(\kappa), \tag{4. 41}$$

$$\mathrm{V}_2 = c_2 Y_2(\kappa),$$

$$\vdots$$

If $\exists 0 < \delta < 1$, the series $\Sigma_{k=0}^{m-1} V_k(\kappa)$ given in Eq.(3.19) converges if $\|V_{k+1}\| \leq \delta \|V_k\|, \forall k \geq k_0$ for some $k_0 \in \mathbb{N}$.

Proof. The sequence $\{S_n\}_{n=0}^{\infty}$ is defined as

$$\begin{aligned} S_0 &= u_0, \\ S_1 &= u_0 + V_1, \\ S_3 &= u_0 + V_1 + V_2, \\ &\vdots \\ S_n &= u_0 + V_1 + V_2 + ... + V_n, \\ &\vdots \end{aligned} \tag{4. 42}$$

In order to make $\{S_n\}_{n=0}^{\infty}$ a Cauchy sequence acting in the Hilbert space $\mathbb{R}$, the following relationships are considered

$$\|S_{n+1} - S_n\| = \|V_{n+1}\| \leq \delta \|V_n\| \leq \delta^2 \|V_{n-1}\| \leq \delta^{n-k_0+1} \|V_{k_0}\|. \tag{4. 43}$$

Now, for every $n, m \in \mathbb{N}, n \geq m > k_0$

$$\begin{aligned} \|S_n - S_m\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + ... + (S_{m+1} - S_m)\| \\ &\leq \|S_n - S_{n-1}\| + \|S_{n-1} + S_{n-2}\| + ... + \|S_{m+1} - S_m\| \end{aligned} \tag{4. 44}$$

(Triangle inequality)

$$\begin{aligned} &\leq \delta^{n-k_0} \|V_{k_0}\| + \delta^{n-k_0-1} \|V_{k_0}\| + ... + \delta^{m-k_0+1} \|V_{k_0}\| \\ &= \left(\frac{1-\delta^{n-m}}{1-\delta}\right) \delta^{m-k_0+1} \|V_{k_0}\|, \end{aligned} \tag{4. 45}$$

This implies that $lim_{n,m \longrightarrow \infty \|S_n - S_m\| = 0}$ (because $0 < \delta < 1$). Hence, it can be concluded that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence over the Hilbert space $\mathbb{R}$, and also the series $\Sigma_{k=0}^{m-1} V_k(\kappa)$ converges absolutely. A similar convergence analysis is carried out for

OHAM, but its only difference from NOHAM is that the partial summation is considered as $S_n = u_0 + u_1 + u_2 + ... + u_n$. □

5. Application of the proposed methods

Two approximation methods called OHAM and NOHAM are used to compute approximate solutions for time-fractional BBM-Burger equation with the initial condition (1. 3) and an exact solution (1. 4). The OHAM and NOHAM approximations will be studied in the following subsections.

5.1. **Numerical solution of BBMB equation by OHAM.** Having chosen the linear operator $\Phi = D_t^\alpha$, the zero-order equation (3. 17) is solved for obtaining an initial approximation $u_0(x,t)$ as follows

$$D_t^\alpha(u_0(x,t)) = 0, u(x,0) = sech^2(\frac{x}{4}). \tag{5. 46}$$

From Eq. (5. 46), we have

$$u_0(x,t) = sech^2(\frac{x}{4}). \tag{5. 47}$$

The following problems are resulted from Eq. (3. 20), of the order $m = 1, 2, 3, ...$

First-order problem:

$$D_t^\alpha(u_1(x,t)) = D_t^\alpha(u_0(x,t)) + c_1[D_t^\alpha(u_0(x,t)) - D_{xxt}(u_0(x,t)) + D_x(u_0(x,t)) + (\frac{u_0(x,t)^2}{2})_x], \tag{5. 48}$$

Second-order problem:

$$D_t^\alpha(u_2(x,t)) = D_t^\alpha(u_1(x,t)) + c_1[D_t^\alpha(u_1(x,t)) - D_{xxt}(u_1(x,t)) + D_x(u_1(x,t)) + (u_0(x,t)u_1(x,t))_x] + C_2[D_t^\alpha(u_0(x,t)) - D_{xxt}(u_0(x,t)) + D_x(u_0(x,t)) + (\frac{u_0(x,t)^2}{2})_x], \tag{5. 49}$$

$\vdots$

Having the solutions to Eqs. (5. 47)-(5. 49), first few approximations to the solution will be obtained

$$u_1(x,t) = c_1[-\frac{1}{2}sech^2(\frac{x}{4})tanh(\frac{x}{4}) - \frac{1}{2}sech^4(\frac{x}{4})tanh(\frac{x}{4})]\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{5. 50}$$

$$u_2(x,t) = -\frac{1}{8}\frac{1}{cosh^8(\frac{x}{8})\Gamma(2\alpha+1)\Gamma(\alpha+1)}[(-2\Gamma(\alpha+1)t^{2\alpha-1}\alpha c_1^2 + \tag{5. 51}$$

$4\, t^\alpha\Gamma(2\alpha+1)c_1^2 + 4t^\alpha\Gamma(2\alpha+1)c_1 + 4t^\alpha\Gamma(2\alpha+1)c_2)sinh(\frac{x}{4})cosh^5(\frac{x}{4}) - 2t^{2\alpha}\Gamma(\alpha+1)c_1^2cosh^6(\frac{x}{4}) + (-2t^{2\alpha-1}\alpha\Gamma(\alpha+1)c_1^2 + 4t^\alpha\Gamma(2\alpha+1)c_1^2 + 4t^\alpha\Gamma(2\alpha+1)c_1 + t^\alpha\Gamma(2\alpha+1)c_2)sinh(\frac{x}{4})cosh^3(\frac{x}{4}) - 5t^{2\alpha}\Gamma(\alpha+1)c_1^2cosh^4(\frac{x}{4}) + 15\Gamma(\alpha+1)t^{2\alpha-1}\alpha c_1^2 sinh(\frac{x}{4})cosh(\frac{x}{4}) +$

TABLE 1. The auxiliary parameters c_1 and c_2 for different values of α.

α	c_1	c_2
0.9	-0.4547763321	0.1192881889
0.95	-0.4798779592	0.1100369157
1	0.6636407613	-2.683876449

$$4\Gamma(\alpha+1)t^{2\alpha}c_1^2cosh^2(\frac{x}{4})+7\Gamma(\alpha+1)t^{2\alpha}c_1^2],$$

$$\vdots$$

In this research, a three terms approximation for $u(x,t)$ is considered. By substituting the solutions of zero−order, first−order, and second−order into Eq. (3. 24) and using the least square technique, the parameters c_1 and c_2 are determined for different values of α , as indicated in Table 1.

5.2. **Numerical solution of BBM-Burger equation by NOHAM.** Having chosen the linear operator $\Phi = D_t^{\alpha}$ and the nonlinear operator $\Psi = -u_{xxt} + u_x + (\frac{u^2}{2})_x$, the zero-order equation (3. 27) is solved to obtain the initial approximate solution $u_0(x,t)$ as follows

$$D_t^{\alpha}(u_0(x,t)) = 0,\ u(x,0) = sech^2(\frac{x}{4}). \tag{5. 52}$$

From Eq. (5. 52), we get

$$u_0(x,t) = sech^2(\frac{x}{4}). \tag{5. 53}$$

By substitution of Eq. (5. 53) into the nonlinear operator Ψ, one has

$$\Psi(u_0(x,t)) = (-\frac{1}{2}tanh(\frac{x}{4}))(sech^2(\frac{x}{4}) + sech^4(\frac{x}{4})). \tag{5. 54}$$

From Eq. (5. 54), two functions (g_1 and h_1) can be recognized as follows

$$g_1 = -\frac{1}{2}tanh(\frac{x}{4}), \tag{5. 55}$$

and

$$h_1 = sech^2(\frac{x}{4}) + sech^4(\frac{x}{4}). \tag{5. 56}$$

From Eqs. (5. 55) and (5. 56), we get

$$H(x,t,c_i) = H_1(x,t,h_1,c_i) = c_1 + c_2t^2sech^2(\frac{x}{4}) + c_3t^4sech^4(\frac{x}{4}) + c_4t^6sech^6(\frac{x}{6}) + \cdots . \tag{5. 57}$$

First order solution:

$$\Phi(u_1(x,t,c_i)) = H(x,t,c_i)\Psi(u_0(x,t)), \tag{5. 58}$$

TABLE 2. The auxiliary parameters c_i for different values of α.

α	c_1	c_2	c_3	c_4
0.2	0.	0.369303448	2.085133762	0.9495574777
0.5	0.	-1.952142501	3.505701573	-1.765458663
0.9	-0.4112405815	1.265729819	-2.253063080	1.254883075
0.95	-0.4315915721	1.402326431	-2.515599124	1.415785941
1	-0.4517697494	1.533422322	-2.767869457	1.573215223

By substituting four terms of Eq. (5. 57), and Eq. (5. 54) in Eq. (5. 58), we obtain

$$D_t^{\alpha}(u_1(x,t,c_i)) = -\frac{1}{2}c_1 tanh(\frac{x}{4})(sech^2(\frac{x}{4}) + sech^4(\frac{x}{4})) \quad (5.\ 59)$$
$$-\frac{1}{2}c_2t^2 tanh(\frac{x}{4})(sech^4(\frac{x}{4}) + sech^6(\frac{x}{4})) - \frac{1}{2}c_3t^4 tanh(\frac{x}{4})(sech^6(\frac{x}{4}) + sech^8(\frac{x}{4}))$$
$$-\frac{1}{2}c_4t^6 tanh(\frac{x}{4})(sech^8(\frac{x}{4}) + sech^{10}(\frac{x}{4})),$$

Therefore, we get

$$u_1(x,t,c_i) = -\frac{1}{2}c_1 tanh(\frac{x}{4})(sech^2(\frac{x}{4}) + sech^4(\frac{x}{4}))\frac{t^{\alpha}}{\Gamma(\alpha+1)} - c_2 tanh(\frac{x}{4})(sech^4(\frac{x}{4}) \quad (5.\ 60)$$
$$+sech^6(\frac{x}{4}))\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} - 12c_3 tanh(\frac{x}{4})(sech^6(\frac{x}{4}) + sech^8(\frac{x}{4}))\frac{t^{\alpha+4}}{\Gamma(\alpha+5)}$$
$$-360c_4 tanh(\frac{x}{4})(sech^8(\frac{x}{4}) + sech^{10}(\frac{x}{4}))\frac{t^{\alpha+6}}{\Gamma(\alpha+5)}.$$

By substituting $u_0(x,t)$ and $u_1(x,t)$ into Eq. (3. 29), approximate solution is obtained as follows

$$u(x,t,c_i) = sech^(\frac{x}{4}) - \frac{1}{2}c_1 tanh(\frac{x}{4})(sech^2(\frac{x}{4}) + sech^4(\frac{x}{4}))\frac{t^{\alpha}}{\Gamma(\alpha+1)} \quad (5.\ 61)$$
$$-c_2 tanh(\frac{x}{4})(sech^4(\frac{x}{4}) + sech^6(\frac{x}{4}))\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}$$
$$-12c_3 tanh(\frac{x}{4})(sech^6(\frac{x}{4}) + sech^8(\frac{x}{4}))\frac{t^{\alpha+4}}{\Gamma(\alpha+5)}$$
$$-360c_4 tanh(\frac{x}{4})(sech^8(\frac{x}{4}) + sech^{10}(\frac{x}{4}))\frac{t^{\alpha+6}}{\Gamma(\alpha+5)}.$$

Now, the auxiliary parameters c_1, c_2, c_3, and c_4 are determined by using least squares technique for different values of α, as in Table 2.

6. RESULTS, COMPARISON, AND DISCUSSION

To validate the accuracy of the proposed techniques (OHAM and NOHAM), their approximation results are evaluated with the exact solutions of BBMB equation. Figure 1 represents the OHAM and NOHAM approximate solutions for different values of α

TABLE 3. Exact and approximate solutions at different points $(x, 0.1)$ and $\alpha = 1$.

x	Exact	OHAM	Abs (OHAM)	NOHAM	Abs (NOHAM)
0	0.9993752604	0.9993686456	0.6614802e-5	1.	0.6247396e-3
0.1	1.	0.9999415132	0.5848681e-4	0.0004907742	0.4907743e-3
0.2	0.9993752604	0.9992686002	0.1066602e-3	0.9997275827	0.3523223e-3
0.3	0.9975041608	0.9973542007	0.1499601e-3	0.9977122604	0.2080996e-3
0.4	0.9943960268	0.9942086492	0.1873776e-3	0.9944529602	0.5693343e-3
0.5	0.9900662909	0.9898481679	0.2181230e-3	0.9899640996	0.1021913e-3

TABLE 4. Exact and approximate solutions by OHAM, NOHAM, and FHATM, for $\alpha = 1$.

x	t	Exact	OHAM	NOHAM	FHATM
20	0.01	0.0001825	0.0001825	0.0001824	0.0001828
15	0.01	0.0022210	0.0022210	0.0022210	0.0022247
10	0.01	0.0267237	0.0267305	0.0267002	0.0267690
20	0.001	0.0001817	0.0001817	0.0001817	0.0001817
15	0.001	0.0022110	0.0022110	0.0022111	0.0022114
10	0.001	0.0266053	0.0266060	0.0265401	0.0260985

($\alpha = 0.9, \alpha = 0.95$ and $\alpha = 1$) and the exact solution ($\alpha = 1$) at $t = 0.1$. In Fig.1 (a) and (b), it is seen that the accuracy of OHAM and NOHAM approximate solutions increases as the values of α approaches to 1. It is also seen, from Fig 1, that the approximate solutions of OHAM and NOHAM are in a very good agreement with the exact solution. Moreover, Figure 2 displays three-dimensional graphical plots of the exact and approximate solutions for $\alpha = 1$.

Also, the numerical values of the exact, approximate solutions, and absolute errors for $\alpha = 1$ are presented in Table 1. In figures 3 and 4, absolute errors of the two approaches, at a fixed value of t, $t = 0.1$, are plotted for different values of α. One can learned from these two figures that the methods are accurate at the beginning there are some perturbations, but as time passes the errors get close to zero. In comparison to OHAM, the solution convergence of NOHAM is achieved faster at low iterations because NOHAM uses an optimal auxiliary function with convergence control parameters (see section 5). In the OHAM and NOHAM approximations, the solution can be improved by considering more auxiliary parameters in the function $H(\xi, c_i)$.

Comparison of approximate solution with exact solution, and other techniques [12] is presented in Table 4. According to the Table 4, it is clear that the results obtined from two methods OHAM and NOHAM, have better convergence.

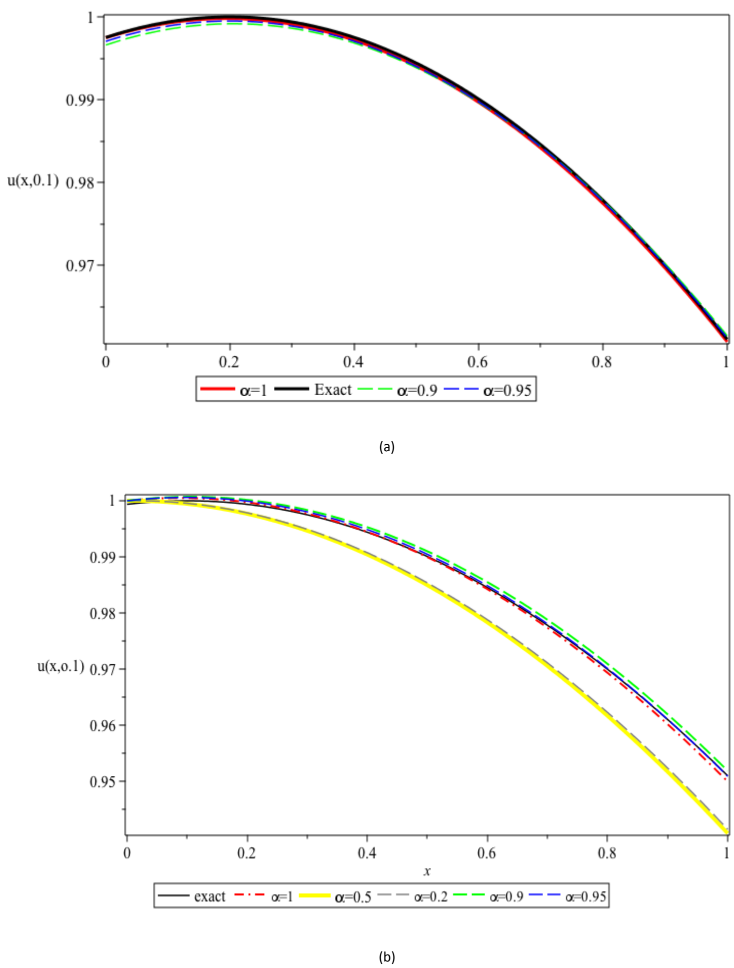

FIGURE 1. Comparison of the approximate solutions of OHAM (a) and NOHAM (b), with exact solution for different values of α.

7. CONCLUSION

In this paper, a time-fractional BBM-Burger equation was solved by two approximation techniques called OHAM and NOHAM. The effectiveness of the techniques (OHAM and NOHAM) are also validated by comparing their results with the exact solution. It is clear from the calculated results that the proposed techniques can approximate the BBM-Burgers

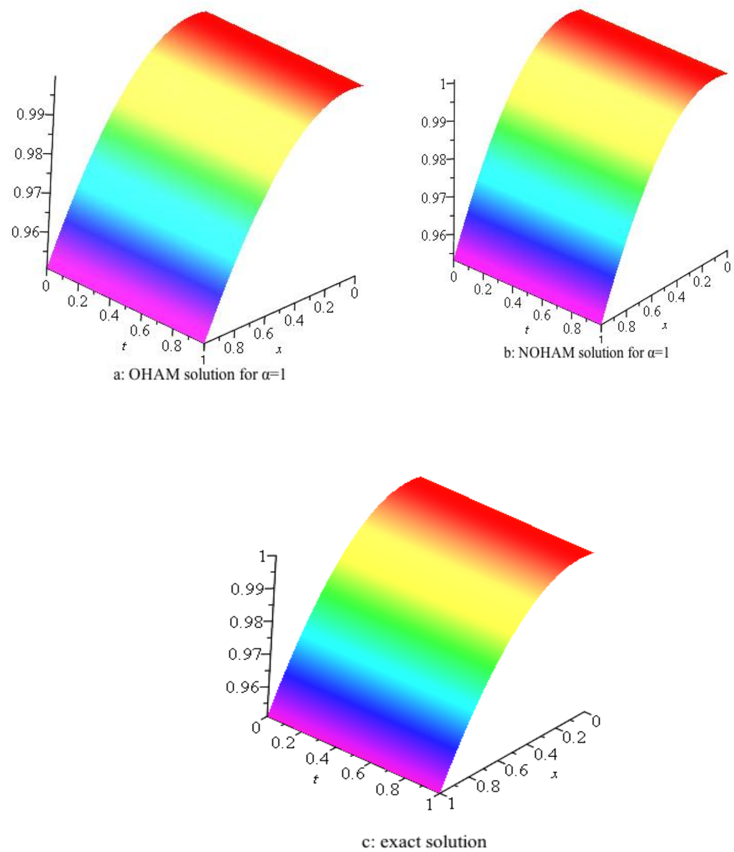

FIGURE 2. 3D graphical plots of the exact and approximate solutions for $\alpha = 1$.

exact solutions with high accuracy. However, NOHAM produces the approximation results with lower computational cost than OHAM. Finally, one can conclude that the proposed techniques can be considered as powerful tools for solving nonlinear partial differential equations, like BBM-Burger, with high accuracy and low computational cost.

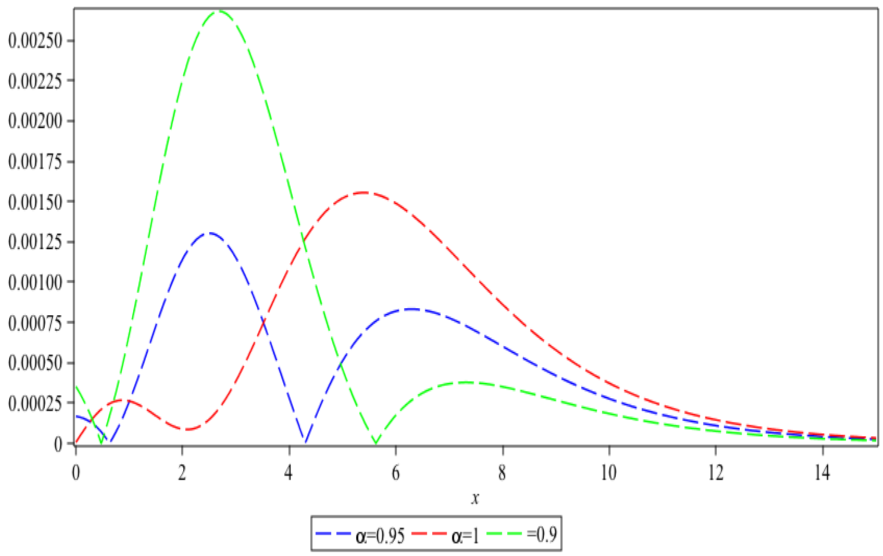

FIGURE 3. Absolut errors of OHAM for different values of α.

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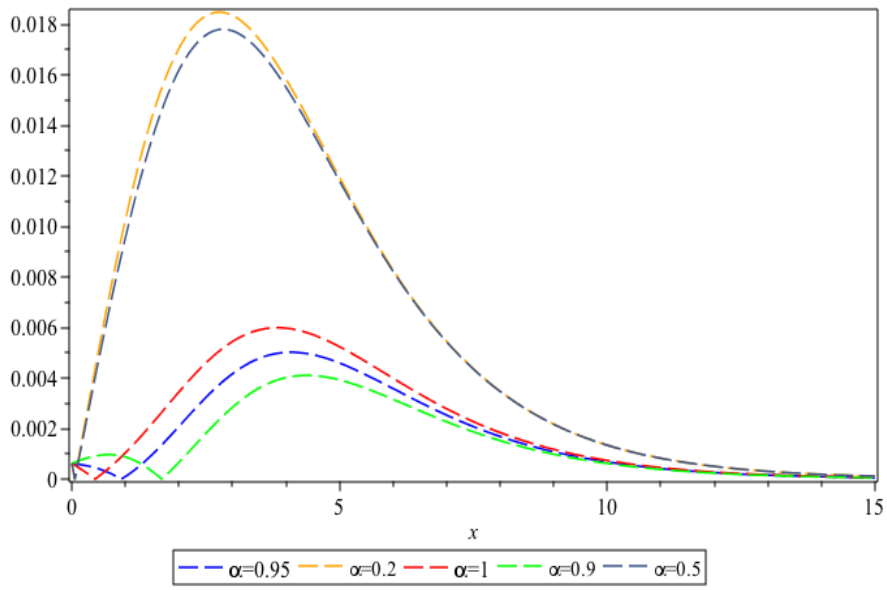

FIGURE 4. Absolut errors of NOHAM for some values of α.

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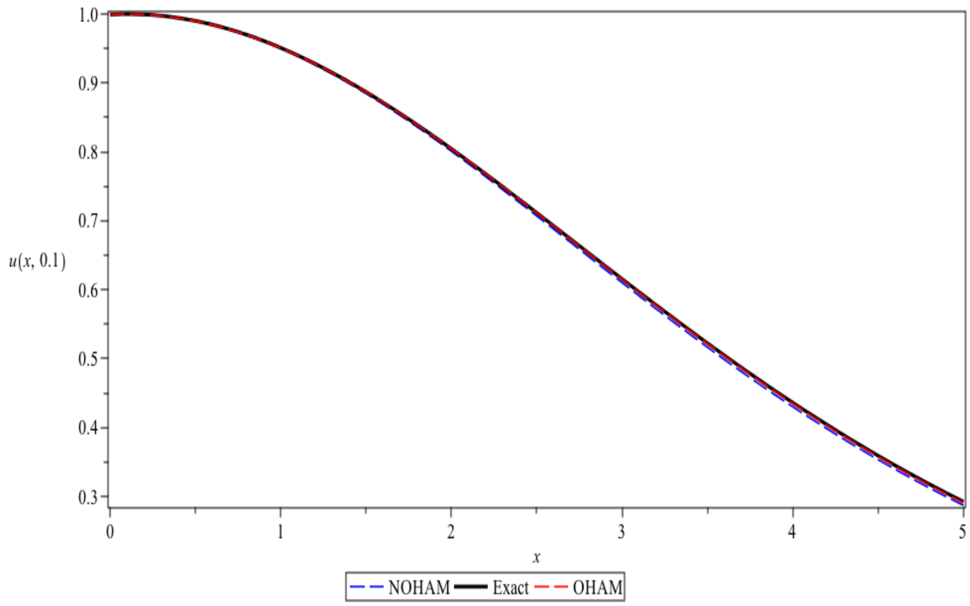

FIGURE 5. Exact solution, OHAM, and NOHAM for $\alpha = 1$.

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