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Some new generalizations for exponentially (s, m)-preinvex functions considering generalized fractional integral operators

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Abstract.:The generalized fractional integral has been one of the most useful operators for modelling non-local behaviors by fractional differential equations. It is considered, for several integral inequalities by introducing the concept of exponentially (s, m)-preinvexity. These variants derived via an extended Mittag-Leffler function based on boundedness, continuity and Hermite-Hadamard type inequalities. The consequences associated with fractional integral operators are more general and also present the results for convexity theory. Moreover, we point out that the variants are useful in solving the problems of science, engineering and technology where the Mittag-Leffler function occurs naturally.

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1. Introduction

The imperative study of derivatives and integrals has been done in calculus. The fractional calculus is evolved when he classical derivative was convoluted with the strength regulation kind kernel. For applications in bioengineering, financial economics, and pure mathematics, we refer the interested readers [1, 2]. The information in the context of fractional operators about rheological models can be investigated in [3, 4, 51, 52, 53, 54, 55, 56, 57, 58]. In perspective on the broad utilization of such frameworks, numerous scientists went to the examination of the hypothetical parts of fractional differential conditions. Specifically, there was unique consideration regarding demonstrating the integral inequalities for fractional systems enhanced with an assortment of classical and non-classical

(nonlocal) operators with the guide of advance strategies for functional investigation, see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 59, 60, 61, 62, 63, 64, 65]. Fractional integral inequalities involving convex functions played a significant role in the mathematical analysis due to their prominent features and convenient characterizations. Integral inequalities have been extensively studied in the literature and their variant forms presented by fractional integrals operators are Hermite-Hadamard, Ostrowski, Lyenger and so forth. The most captivating inequality is the Hermite-Hadamard inequality, which can be used to find lower and upper bounds for fractional integral for some given convex functions.
The following well-known inequality

$$\hbar\Big(\frac{x_1+x_2}{2}\Big) \leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\hbar(z)dz \leq \frac{\hbar(x_1)+\hbar(x_2)}{2} \tag{1. 1}$$

holds for convex functions and is known as the Hermite-Hadamard inequality. This inequality has for quite some time been known as Hadamard's inequality. This inequality of Hermite was found by Mitrinovic, who he had discovered sooner than Hadamard and is referred to now in the writing as Hermite-Hadamard's inequality. The significance of this inequality is because of the way that it is identical to the meaning of convexity under specific conditions. The inequality (1. 1) has been contemplated by a few analysts because of its potential applications, see [19, 20, 21, 22, 23, 24, 25, 26, 27, 28].
Now we demonstrate some basic preliminaries concerning to convexity and preinvexity as follows:

2. Notations and Preliminaries

Now we demonstrate some basic preliminaries concerning to convexity and preinvexity as follows:

Definition 2.1. *A mapping $\hbar : \mathcal{K} \to \mathcal{R}$ is said to be convex, if the inequality*

$$\hbar(\zeta x_1 + (1-\zeta)x_2)) \leq \zeta\hbar(x_1) + (1-\zeta)\hbar(x_2) \tag{2. 2}$$

holds for all $x_1, x_2 \in \mathcal{K}$ and $\zeta \in [0,1]$.

Awan et al. [29] gave the concept of exponentially convex function.

Definition 2.2. *A real-valued mapping $\hbar : \mathcal{K} \to \mathcal{R}$ is said to be exponentially convex function if the inequality*

$$\hbar(\zeta x_1 + (1-\zeta)x_2)) \leq \zeta\frac{\hbar(x_1)}{e^{\vartheta x_1}} + (1-\zeta)\frac{\hbar(x_2)}{e^{\vartheta x_2}} \tag{2. 3}$$

holds for all $x_1, x_2 \in \mathcal{K}$, $\zeta \in [0,1]$ and $\vartheta \in \mathcal{R}$

Exponentially convex functions are introduced by Bernstein [30] in covariance formation then Avriel [31] expounded the concept of exponentially convex function in linear programming. Alirezaei and Mathar [32] explored the application of exponentially convex functions in information theory as well in statistical theory. For more details and information, see [33, 34].

In [35], Craven expounded the term due to their properties discussed as invariance by convexity. In [36], Hanson outfitted the possibility of differentiable invex functions in the context of their accurate global optimum behavior and obtained the significance generalization of convex functions is that of invex functions. Weir and Mond [37] explored the generalizations of convex functions is that of preinvex functions and utilized it to build-up the duality and the sufficient optimality conditions in nonlinear programming. In [38], Noor introduced another version of Hermite-Hadamard inequality for preinvex functions. Mohan and Neogy [39] explored a famous result, which is known as Condition C. Mititelu [40] investigated well-known concept of invex sets as follows:
Now, we will give the definition of preinvex functions with some basic properties. Suppose that $\Omega \subset \mathcal{R}$ be a set, $\hbar : \Omega \to \mathcal{R}$ be a continuous function and $\eta(.,.) : \Omega \times \Omega \to \mathcal{R}$ be a continuous bifunction.

Definition 2.3. [40]*A set $\Omega \subset \mathcal{R}$ is said to be invex set w. r. to the bifunction $\eta(.,.)$, if and only if*

$$x_1 + \zeta\eta(x_2, x_1) \in \Omega, \quad \forall x_1, x_2 \in \Omega, \zeta \in [0, 1]. \tag{2. 4}$$

It is also called η-connected set. Note that, if $\eta(x_2, x_1) = x_2 - x_1$, this implies that every convex set is an invex set, but the converse is not true. We will use the set Ω as an invex set, unless otherwise it is specified.

Definition 2.4. [37] *A mapping $\hbar$ on the invex set Ω is said to be preinvex function w. r. to $\eta(.,.)$, if the inequality*

$$\hbar(x_1 + \zeta\eta(x_2, x_1)) \leq (1 - \zeta)\hbar(x_1) + \zeta\hbar(x_2) \tag{2. 5}$$

holds $\forall x_1, x_1 + \eta(x_2, x_1) \in \Omega$ and $\zeta \in [0, 1]$.

We now introduce the concept of exponentially preinvex, exponentially s-preinvex and exponentially m-preinvex function, respectively.

Definition 2.5. *A real-valued mapping $\hbar$ on the invex set Ω is said to be exponentially preinvex w. r. to $\eta(.,.)$, if the inequality*

$$\hbar(x_1 + \zeta\eta(x_2, x_1)) \leq (1 - \zeta)\frac{\hbar(x_1)}{e^{\vartheta x_1}} + \zeta\frac{\hbar(x_2)}{e^{\vartheta x_2}} \tag{2. 6}$$

holds for all $x_1, x_1 + \eta(x_2, x_1) \in \Omega, \zeta \in [0, 1]$, and $\vartheta \in \mathcal{R}$.

Definition 2.6. *Let $s \in (0, 1]$ and a real-valued mapping $\hbar$ on the invex set Ω is said to be exponentially s-preinvex w. r. to $\eta(.,.)$, if the inequality*

$$\hbar(x_1 + \zeta\eta(x_2, x_1)) \leq (1 - \zeta)^s\frac{\hbar(x_1)}{e^{\vartheta x_1}} + \zeta^s\frac{\hbar(x_2)}{e^{\vartheta x_2}} \tag{2. 7}$$

holds for all $x_1, x_1 + \eta(x_2, x_1) \in \Omega, \zeta \in [0, 1]$, and $\vartheta \in \mathcal{R}$.

Now we extend the concept of exponentially m-preinvexity, by some transformation we will get m-preinvex functions and m-convex functions introduced by Toader [41].

Definition 2.7. *A real-valued mapping $\hbar$ on the invex set Ω is said to be exponentially m-preinvex w. r. to $\eta(.,.)$, if the inequality*

$$\hbar(x_1 + m\zeta\eta(x_2, x_1)) \leq (1-\zeta)^s \frac{\hbar(x_1)}{e^{\vartheta x_1}} + m\zeta^s \frac{\hbar(x_2)}{e^{\vartheta x_2}} \tag{2. 8}$$

holds for all $x_1, x_1 + \eta(x_2, x_1) \in [0, b], \zeta \in [0, 1], m \in [0, 1]$ and $\vartheta \in \mathcal{R}$.

In 1903, Gosta [42] introduced the idea of Mittag-Leffler function $E_\sigma(\zeta)$ as

$$E_\sigma(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{\Gamma(\sigma n + 1)},$$

where $\zeta, \sigma \in \mathbb{C}, \Re(\sigma)$ and $\Gamma(.)$ is the gamma function.
If $\sigma = 1$ then we obtain the direct generalization of the exponential function is Mittag-Leffler function. In the arrangement of fractional differential equations and integral equations, the Mittag-Leffler work emerges normally. Due to its significance Mittag-Leffler work is additionally summed up and stretched out by numerous specialists, we allude the peruser to [5, 6, 33, 43, 44, 45, 47, 48]. As of late in [43], Andric et al. presented generalized Mittag-Leffler work defined as pursues:

Definition 2.8. [43] *Let $\omega, \mu, \sigma, \jmath, \gamma, \nu \in \mathbb{C}, \Re(\mu), \Re(\sigma), \Re(\jmath), \Re(\nu), \Re(\gamma) > 0$ with $q \geq 0, \delta > 0$ and $0 < \kappa \leq \delta + \Re(\mu)$. Let $\hbar \in L_[x_1, x_2]$ and $z \in [x_1, x_2]$. Then the generalized fractional integral operators having Mittag-Leffler function are defined by:*

$$\left(\epsilon_{\mu,\sigma,\jmath,\omega,x_1^+}\hbar\right)(z;q) = \int_{x_1}^{z} E_{\mu,\sigma,\jmath}^{\gamma,\delta,\kappa,\nu}(\omega(z-\zeta)^\mu; q)\hbar(\zeta)d\zeta \tag{2. 9}$$

and

$$\left(\epsilon_{\mu,\sigma,\jmath,\omega,x_2^-}\hbar\right)(z;q) = \int_{z}^{x_2} E_{\mu,\sigma,\jmath}^{\gamma,\delta,\kappa,\nu}(\omega(\zeta-z)^\mu; q)\hbar(\zeta)d\zeta, \tag{2. 10}$$

where $E_{\mu,\sigma,\jmath}^{\gamma,\delta,\kappa,\nu}(.)$ is the Mittag-Leffler function defined as:

$$E_{\mu,\sigma,\jmath}^{\gamma,\delta,\kappa,\nu}(\zeta;q) = \sum_{n=0}^{\infty} \frac{\beta_q(\gamma + n\kappa, \nu - \gamma)}{\beta(\gamma, \nu - \gamma)} \frac{(\nu)_{n\kappa}\zeta^n}{\Gamma(\mu n + \sigma)(\jmath)_{n\delta}},$$

and β_q is the generalized beta function defined as follows:

$$\beta_q(x, y) = \int_0^1 \zeta^{x-1}(1-\zeta)^{y-1} e^{-\frac{q}{\zeta(1-\zeta)}} d\zeta$$

and $(\nu)_{n\kappa}$ is the Pochhammer symbol defined by $(\nu)_{n\kappa} = \frac{\Gamma(\nu+n\kappa)}{\Gamma(\nu)}$.

Remark 2.9. *Definition 2.8 is the generalization of the following fractional integral operators containing Mittag-Leffler function:*
(1) *choosing $q = 0$, we have the fractional integral operators defined by Salim-Faraj in* [33].
(2) *choosing $\jmath = \delta = 1$, we have the fractional integral operators defined by Rahman et*

al. in [6].
(3) *choosing* $q=0$ *and* $\jmath=\delta=1$*, we have the fractional integral operators defined by Srivastava-Tomovski in* [48].
(4) *choosing* $q=0$ *and* $\jmath=\delta=\kappa=1$*, we have the fractional integral operators defined by Prabhakar in* [5].
(5) *choosing* $q=\omega=0$*, we have the right-sided and left-sided Riemann-Liouville fractional integrals.*

Farid et al. [18] obtained as follows:

$$\left(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath,\omega,x_1^+}1\right)(z;q)=(z-x_1)^{\sigma}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma+1,\jmath}(\omega(z-x_1)^{\mu};q) \tag{2. 11}$$

and

$$\left(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath,\omega,x_2^-}1\right)(z;q)=(x_2-z)^{\tau}E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath}(\omega(x_2-z)^{\mu};q). \tag{2. 12}$$

The principal purpose of this paper is to introduce the concept of exponentially (s,m)-preinvex function in section 3, which is the generalizations of s-preinvex, m-preinvex, exponentially preinvex, exponentially s-preinvex and exponentially m-preinvex functions, and explore the estimates of concerning to the Mittag-Leffler functions in their kernels. Our outcomes provide several new generalizations by some variations in parameters.

3. Main Results

In this section, we introduced the concept of exponentially (s,m)-preinvex functions, which is the main motivation of this paper.

Definition 3.1. *Let* $s\in(0,1]$ *and a real valued mapping on invex set* Ω *is said to be exponentially* (s,m)*-preinvex, if the inequality*

$$\hbar(x_1+\zeta\eta(mx_2,x_1))\leq(1-\zeta)^s\frac{\hbar(x_1)}{e^{\vartheta x_1}}+m\zeta^s\frac{\hbar(x_2)}{e^{\vartheta x_2}}$$

holds for all $x_1,x_1+\eta(x_2,x_1)\in\Omega,\ \zeta\in[0,1],m\in[0,1]$ *and* $\vartheta\in\mathcal{R}$.

Remark 3.2. *In Definition 3.1:*
(1) *choosing* $m=1$*, we attain exponentially* s*-preinvex functions.*
(2) *choosing* $\vartheta=0$*, we attain* (s,m)*-preinvex functions.*
(3) *choosing* $\vartheta=0$, $m=1$*, we attain* s*-preinvex functions.*
(4) *choosing* $\vartheta=0,s=1,m=1$*, we attain classical preinvex functions.*
(5) *choosing* $\vartheta=0,s=1,m=1$ *with* $\eta(x_2,x_1)=x_2-x_1$*, we attain classical convex functions.*

Theorem 3.3. *Suppose a real valued function* $\hbar:[x_1,x_1+\eta(x_2,x_1)]\to\mathcal{R}$ *be an exponentially* (s,m)*-preinvex function, then the following fractional integral inequality for*

(2. 9) *and* (2. 10) *holds:*

$$\begin{aligned}&\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath,\omega,x_1^+}\hbar\Big)(z;q)+\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath,\omega,(x_1+\eta(x_2,x_1)^-)}\hbar\Big)(z;q)\\&\leq\frac{(\zeta-x_1)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)}{s+1}\bigg(\frac{\hbar(x_1)}{e^{\vartheta_1x_1}}+m\frac{\hbar(z)}{e^{\vartheta_1z}}\bigg)\\&\quad+\frac{\big(x_1+\eta(x_2,x_1)-z\big)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,(x_1+\eta(x_2,x_1))^-}1\big)(z;q)}{s+1}\\&\quad\times\bigg(\frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}+m\frac{\hbar(z)}{e^{\vartheta_2z}}\bigg),\end{aligned}\tag{3. 13}$$

for all $z\in[x_1,x_1+\eta(x_2,x_1)]$.

Proof. Let $z\in[x_1,x_1+\eta(x_2,x_1)]$. Then for $\zeta\in[x_1,z)$ and $\sigma\geq1$, we have the following inequality:

$$\begin{aligned}&(z-\zeta)^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-\zeta)^{\mu};q)\\&\leq(z-x_1)^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-x_1)^{\mu};q).\end{aligned}\tag{3. 14}$$

Since $\hbar$ is exponentially (s,m)-preinvex, we have

$$\hbar(\zeta)\leq\Big(\frac{z-\zeta}{z-x_1}\Big)^s\frac{\hbar(x_1)}{e^{\vartheta_1x_1}}+m\Big(\frac{\zeta-x_1}{z-x_1}\Big)^s\frac{\hbar(z)}{e^{\vartheta_1z}}.\tag{3. 15}$$

Taking product (3. 14) and (3. 15) and then integrating with respect to ζ from x_1 to z, we have

$$\begin{aligned}&\int_{x_1}^{z}(z-\zeta)^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-\zeta)^{\mu};q)\hbar(\zeta)d\zeta\\&\leq\int_{x_1}^{z}(z-x_1)^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-x_1)^{\mu};q)\\&\quad\times\bigg[\Big(\frac{z-\zeta}{z-x_1}\Big)^s\frac{\hbar(x_1)}{e^{\vartheta_1x_1}}+m\Big(\frac{\zeta-x_1}{z-x_1}\Big)^s\frac{\hbar(z)}{e^{\vartheta_1z}}\bigg]d\zeta.\end{aligned}\tag{3. 16}$$

Using Definition 2.8, we have

$$\begin{aligned}&\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath,\omega,x_1^+}\hbar\Big)(z;q)\\&\leq\frac{(z-x_1)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)}{s+1}\bigg(\frac{\hbar(x_1)}{e^{\vartheta_1x_1}}+m\frac{\hbar(z)}{e^{\vartheta_1z}}\bigg).\end{aligned}\tag{3. 17}$$

Analogously, for $\zeta \in (z, x_1+\eta(x_2,x_1)]$ and $\tau \geq 1$, we have

$$\begin{aligned}&(\zeta-z)^{\tau-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath}(\omega(\zeta-z)^{\mu};q)\\&\leq (x_1+\eta(x_2,x_1)-z)^{\tau-1}\\&\times E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath}\big(\omega(x_1+\eta(x_2,x_1)-z)^{\mu};q\big).\end{aligned} \tag{3. 18}$$

Further, from exponentially (s,m)-convexity of $\hbar$, we have

$$\begin{aligned}\hbar(\zeta)&\leq \Big(\frac{\zeta-z}{x_1+\eta(x_2,x_1)-z}\Big)^s\frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}\\&+m\Big(\frac{x_1+\eta(x_2,x_1)-\zeta}{x_1+\eta(x_2,x_1)-z}\Big)^s\frac{\hbar(z)}{e^{\vartheta_2 z}}.\end{aligned} \tag{3. 19}$$

Taking product (3. 18) and (3. 19) and then integrating with respect to ζ from z to $x_1+\eta(x_2,x_1)$, we have

$$\begin{aligned}&\int_z^{x_1+\eta(x_2,x_1)}(\zeta-z)^{\tau-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath}(\omega(\zeta-z)^{\mu};q)\hbar(\zeta)d\zeta\\&\leq \int_z^{x_1+\eta(x_2,x_1)}(x_1+\eta(x_2,x_1)-z)^{\tau-1}\\&\times E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath}\big(\omega(x_1+\eta(x_2,x_1)-z)^{\mu};q\big)\\&\times\Bigg[\Big(\frac{\zeta-z}{x_1+\eta(x_2,x_1)-z}\Big)^s\frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}\\&+m\Big(\frac{x_1+\eta(x_2,x_1)-\zeta}{x_1+\eta(x_2,x_1)-z}\Big)^s\frac{\hbar(z)}{e^{\vartheta_2 z}}\Bigg]d\zeta.\end{aligned} \tag{3. 20}$$

Again, using Definition 2.8, we get

$$\begin{aligned}&\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}\hbar\Big)(z;q)\\&\leq\frac{\big(x_1+\eta(x_2,x_1)-z\big)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,(x_1+\eta(x_2,x_1))^-}1\big)(z;q)}{s+1}\\&\times\Bigg(\frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}+m\frac{\hbar(z)}{e^{\vartheta_2 z}}\Bigg).\end{aligned} \tag{3. 21}$$

Summing up (3. 18) and (3. 21), we deduce the inequality (3. 13). □

Corollary 3.4. *If $\hbar\in L_\infty[x_1,x_1+\eta(x_2,x_1)]$, then under the assumption of Theorem 3.3, we have*

$$\begin{aligned}&\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath,\omega,x_1^+}\hbar\Big)(z;q)+\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath,\omega,(x_1+\eta(x_2,x_1)^-)}\hbar\Big)(z;q)\\&\leq\frac{\|\hbar\|_\infty}{s+1}\Bigg[(\zeta-x_1)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)\Bigg(\frac{1}{e^{\vartheta_1 x_1}}+m\frac{1}{e^{\vartheta_1 z}}\Bigg)\\&+\big(x_1+\eta(x_2,x_1)-z\big)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,(x_1+\eta(x_2,x_1))^-}1\big)(z;q)\\&\times\Bigg(\frac{1}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}+m\frac{1}{e^{\vartheta_2 z}}\Bigg)\Bigg].\end{aligned} \tag{3. 22}$$

Corollary 3.5. *Choosing $m = 1$ and $\hbar \in L_\infty[x_1, x_1 + \eta(x_2, x_1)]$, then under the assumption of Theorem 3.3, we have*

$$\begin{aligned}
&\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath,\omega,x_1^+}\hbar\Big)(z;q) + \Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath,\omega,(x_1+\eta(x_2,x_1)^-)}\hbar\Big)(z;q)\\
&\leq \frac{\|\hbar\|_\infty}{s+1}\bigg[(\zeta - x_1)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)\bigg(\frac{1}{e^{\vartheta_1 x_1}} + \frac{1}{e^{\vartheta_1 z}}\bigg)\\
&\quad +\big(x_1 + \eta(x_2, x_1) - z\big)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,(x_1+\eta(x_2,x_1))^-}1\big)(z;q)\\
&\quad \times\bigg(\frac{1}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}} + \frac{1}{e^{\vartheta_2 z}}\bigg)\bigg].
\end{aligned}$$

Corollary 3.6. *Choosing $m = s = 1$ and $\hbar \in L_\infty[x_1, x_1 + \eta(x_2, x_1)]$, then under the assumption of Theorem 3.3, we have*

$$\begin{aligned}
&\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath,\omega,x_1^+}\hbar\Big)(z;q) + \Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath,\omega,(x_1+\eta(x_2,x_1)^-)}\hbar\Big)(z;q)\\
&\leq \frac{\|\hbar\|_\infty}{2}\bigg[(\zeta - x_1)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)\bigg(\frac{1}{e^{\vartheta_1 x_1}} + \frac{1}{e^{\vartheta_1 z}}\bigg)\\
&\quad +\big(x_1 + \eta(x_2, x_1) - z\big)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,(x_1+\eta(x_2,x_1))^-}1\big)(z;q)\\
&\quad \times\bigg(\frac{1}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}} + \frac{1}{e^{\vartheta_2 z}}\bigg)\bigg].
\end{aligned}$$

Corollary 3.7. *Choosing $\eta(x_2, x_1) = x_2 - x_1$ and $\hbar \in L_\infty[x_1, x_2]$, then under the assumption of Theorem 3.3, we have*

$$\begin{aligned}
&\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath,\omega,x_1^+}\hbar\Big)(z;q) + \Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath,\omega,x_2^-)}\hbar\Big)(z;q) \qquad (3.\ 23)\\
&\leq \frac{\|\hbar\|_\infty}{s+1}\bigg[(\zeta - x_1)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)\bigg(\frac{1}{e^{\vartheta_1 x_1}} + m\frac{1}{e^{\vartheta_1 z}}\bigg)\\
&\quad +\big(x_2 - z\big)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,x_2^-}1\big)(z;q)\bigg(\frac{1}{e^{\vartheta_2 x_2}} + m\frac{1}{e^{\vartheta_2 z}}\bigg)\bigg].
\end{aligned}$$

Theorem 3.8. *Suppose a real valued function $\hbar : [x_1, x_1+\eta(x_2, x_1)] \to \mathcal{R}$ is differentiable and $|\hbar|'$ is exponentially (s, m) preinvex, then the following fractional integral inequality* *for* (2. 9) *and* (2. 10) *holds:*

$$\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma+1,\jmath,\omega,x_1^+}\hbar\ (z;q) + \ \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}\hbar\ (z;q)$$
$$-\ \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,(x_1+\eta(x_2,x_1))^-}1\ (z;q)$$
$$\hbar(x_1+\eta(x_2,x_1)) - \ \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\ (z;q)\quad \hbar(x_1)$$
$$\leq \frac{((x_1+\eta(x_2,x_1))-z)\ \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,(x_1+\eta(x_2,x_1))^-}1\ (z;q)}{s+1}$$
$$\frac{|\hbar'(x_1+\eta(x_2,x_1))|}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}} + m\frac{|\hbar'(z)|}{e^{\vartheta_2 z}}$$
$$+\frac{(z-x_1)\ \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\ (z;q)}{s+1}\quad \frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}} + m\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}\ , \tag{3. 24}$$

for all $z \in [x_1.x_1+\eta(x_2,x_1)],\ \vartheta_1,\vartheta_2 \in \mathcal{R}$.

Proof. Let $z \in [x_1, x_1+\eta(x_2,x_1)]$ and $\zeta \in [x_1, z)$, by using exponentially (s,m) preinvexity of $|\hbar|'$, we have

$$|\hbar'(\zeta)| \leq \Big(\frac{z-\zeta}{z-x_1}\Big)^s\frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}} + m\Big(\frac{\zeta-x_1}{z-x_1}\Big)^s\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}. \tag{3. 25}$$

Implies (3. 25), we have

$$\hbar'(\zeta) \leq \Big(\frac{z-\zeta}{z-x_1}\Big)^s\frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}} + m\Big(\frac{\zeta-x_1}{z-x_1}\Big)^s\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}. \tag{3. 26}$$

Conducting product between (3. 14) and (3. 26), we get

$$(z-\zeta)^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-\zeta)^\mu;q)\hbar'(\zeta)$$
$$\leq (z-x_1)^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-x_1)^\mu;q)$$
$$\times\quad \Big[\Big(\frac{z-\zeta}{z-x_1}\Big)^s\frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}} + m\Big(\frac{\zeta-x_1}{z-x_1}\Big)^s\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}\Big]. \tag{3. 27}$$

Integrating the above inequality with respect to ζ from x_1 to z, we get

$$\int_{x_1}^{z}(z-\zeta)^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-\zeta)^\mu;q)\hbar'(\zeta)d\zeta$$
$$\leq \int_{x_1}^{z}(z-x_1)^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-x_1)^\mu;q)$$
$$\times\quad \Big[\Big(\frac{z-\zeta}{z-x_1}\Big)^s\frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}} + m\Big(\frac{\zeta-x_1}{z-x_1}\Big)^s\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}\Big]d\zeta$$
$$= \frac{(z-x_1)E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-\zeta)^\mu;q)}{s+1}\Big(\frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}} + m\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}\Big). \tag{3. 28}$$

Now solving left side of (3. 28) gives

$$\int_{x_1}^{z}(z-\zeta)^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-\zeta)^{\mu};q)\hbar'(\zeta)d\zeta$$
$$=\int_{0}^{z-x_1}\theta^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega\theta^{\mu};q)\hbar'(z-\theta)d\theta$$
$$=(z-x_1)^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-x_1)^{\mu};q)$$
$$\hbar(x_1)+\int_{0}^{z-x_1}\theta^{\sigma-2}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath}(\omega\theta^{\mu};q)\hbar(z-\theta)d\theta.$$

Again, substituting $z-\theta=\zeta$ in the second term and solving right hand side of the afore-menioned inequality, we have

$$\int_{0}^{z-x_1}\theta^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath}(\omega\theta^{\mu};q)\hbar'(z-\theta)d\theta$$
$$=(z-x_1)^{\sigma-1}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(z-x_1)^{\mu};q)\hbar(x_1)$$
$$-\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,\omega,x_1^+}\hbar\Big)(z;q).$$

Therefore (3. 28) gets the following form:

$$\Big(\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)\Big)\hbar(x_1)-\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma+1,\jmath,\omega,x_1^+}\hbar\Big)(z;q)$$
$$\leq\frac{(z-x_1)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)}{s+1}\Big(\frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}}+m\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}\Big). \tag{3. 29}$$

Also from (3. 25), we have

$$\hbar'(\zeta)\geq-\bigg(\Big(\frac{z-\zeta}{z-x_1}\Big)^{s}\frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}}+m\Big(\frac{\zeta-x_1}{z-x_1}\Big)^{s}\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}\bigg). \tag{3. 30}$$

Adopting the same proceedure as we have done for (3. 26), we obtain

$$\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma+1,\jmath,\omega,x_1^+}\hbar\Big)(z;q)-\Big(\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)\Big)\hbar(x_1)$$
$$\leq\frac{(z-x_1)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)}{s+1}\Big(\frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}}+m\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}\Big). \tag{3. 31}$$

From (3. 29) and (3. 31), we get

$$\Big|\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma+1,\jmath,\omega,x_1^+}\hbar\Big)(z;q)-\Big(\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)\Big)\hbar(x_1)\Big|$$
$$\leq\frac{(z-x_1)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)}{s+1}\Big(\frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}}+m\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}\Big). \tag{3. 32}$$

Now we let $x \in [x_1, x_1 + \eta(x_2, x_1)]$ and $\zeta \in (z, x_1 + \eta(x_2, x_1)]$. Then by exponentially (s, m)-preinvexity of $|\hbar'|$, we have

$$|\hbar'(\zeta)| \leq \quad \frac{\zeta - z}{x_1 + \eta(x_2, x_1) - z} \quad ^{s} \frac{|\hbar'(x_1 + \eta(x_2, x_1))|}{e^{\vartheta_2 x_2}}$$
$$+m \quad \frac{x_1 + \eta(x_2, x_1) - \zeta}{x_1 + \eta(x_2, x_1) - z} \quad ^{s} \frac{|\hbar'(z)|}{e^{\vartheta_2 z}}. \tag{3. 33}$$

On the same lines as we have done for (3. 14), (3. 26) and (3. 30), we can conclude (3. 17) and (3. 33), we have

$$\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}\hbar \ \ (z;q)$$
$$- \quad \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,(x_1+\eta(x_2,x_1))^-}1 \ \ (z;q) \quad \hbar(x_1 + \eta(x_2, x_1))$$
$$\leq \frac{((x_1 + \eta(x_2, x_1)) - z) \ \ \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,(x_1+\eta(x_2,x_1))^-}1 \ \ (z;q)}{s+1}$$
$$\times \quad \frac{|\hbar'(x_1 + \eta(x_2, x_1))|}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}} + m\frac{|\hbar'(z)|}{e^{\vartheta_2 z}} \quad . \tag{3. 34}$$

From inequalities (3. 32) and (3. 34) via triangular inequality (3. 24) can be concluded. □

The following corollaries are remarkable for Theorem 3.8.

Corollary 3.9. *Choosing* $\eta(x_2, x_1) = x_2 - x_1$*, then under the assumption of Theorem 3.8, we have*

$$\Big|\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma+1,\jmath,\omega,x_1^+}\hbar\Big)(z;q) + \Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_2^-}\hbar\Big)(z;q)$$
$$-\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_2^-}1\big)(z;q)\hbar(x_2) - \big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)\Big)\hbar(x_1)\Big|$$
$$\leq \frac{(x_2 - z)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_2^-}1\big)(z;q)}{s+1}\Big(\frac{|\hbar'(x_2)|}{e^{\vartheta_2 x_2}} + m\frac{|\hbar'(z)|}{e^{\vartheta_2 z}}\Big)$$
$$+\frac{(z - x_1)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)}{s+1}\Big(\frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}} + m\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}\Big),$$

for all $z \in [x_1.x_2], \ \vartheta_1, \vartheta_2 \in \mathcal{R}$*.*

Corollary 3.10. *Choosing* $\eta(x_2, x_1) = x_2 - x_1$*, along with* $s = m = 1$ *then under the assumption of Theorem 3.8, we have*

$$\Big|\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma+1,\jmath,\omega,x_1^+}\hbar\Big)(z;q)+\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_2^-}\hbar\Big)(z;q)$$
$$-\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_2^-}1\big)(z;q)\hbar(x_2)-\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)\Big)\hbar(x_1)\Big|$$
$$\leq\frac{(x_2-z)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_2^-}1\big)(z;q)}{2}\Big(\frac{|\hbar'(x_2)|}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}+\frac{|\hbar'(z)|}{e^{\vartheta_2 z}}\Big)$$
$$+\frac{(z-x_1)\big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,x_1^+}1\big)(z;q)}{2}\Big(\frac{|\hbar'(x_1)|}{e^{\vartheta_1 x_1}}+\frac{|\hbar'(z)|}{e^{\vartheta_1 z}}\Big), \quad (3.\ 35)$$

for all $z\in[x_1,x_2]$, $\vartheta_1,\vartheta_2\in\mathcal{R}$.

Definition 3.11. *Let* $\hbar:[x_1,x_1+\eta(x_2,x_1)]\to\mathcal{R}$ *be a function, we say* $\hbar$ *is exponentially symmetric about* $\frac{2x_1+\eta(x_2,x_1)}{2}$, *if*

$$\frac{\hbar(z)}{e^{\vartheta_1 z}}=\frac{\hbar(2x_1+\eta(x_2,x_1)-z)}{e^{\vartheta_1((2x_1+\eta(x_2,x_1)-z))}},\vartheta_1\in\mathcal{R}.$$

Now we demonstrate the following lemma which will be helpful to produce our coming result.

Lemma 3.12. *Let* $\hbar:[x_1,x_1+\eta(x_2,x_1)]\to\mathcal{R}$ *be an exponentially symmetric, then*

$$\hbar\Big(\frac{2x_1+\eta(x_2,x_1)}{2}\Big)\leq\frac{(1+m)\hbar(z)}{2^s e^{\vartheta_1 z}},\vartheta_1\in\mathcal{R}. \quad (3.\ 36)$$

Proof. Since $\hbar$ is exponentially (s,m)-preinvex, therefore

$$\hbar\Big(\frac{2x_1+\eta(x_2,x_1)}{2}\Big)\leq\frac{\hbar(x_1+\zeta\eta(x_2,x_1))}{2^s e^{\vartheta_1\hbar(x_1+\zeta\eta(x_2,x_1))}}$$
$$+m\frac{\hbar(x_1+(1-\zeta)\eta(x_2,x_1))}{2^s e^{\vartheta_1\hbar(x_1+(1-\zeta)\eta(x_2,x_1))}}. \quad (3.\ 37)$$

Let $z=x_1+\zeta\eta(x_2,x_1)$, where $z\in[x_1,x_1+\eta(x_2,x_1)]$. Then we have $2x_1+\eta(x_2,x_1)=x_1+(1-\zeta)\eta(x_2,x_1)$ and we have

$$\hbar\Big(\frac{2x_1+\eta(x_2,x_1)}{2}\Big)\leq\frac{\hbar(z)}{2^s e^{\vartheta_1 z}}+m\frac{\hbar(2x_1+\eta(x_2,x_1)-z)}{2^s e^{\vartheta_1(2x_1+\eta(x_2,x_1)-z)}}. \quad (3.\ 38)$$

Now using that $\hbar$ is exponentially symmetric we will get (3. 36). □

Theorem 3.13. *Suppose a real valued function* $\hbar:[x_1,x_1+\eta(x_2,x_1)]\to\mathcal{R}$ *is positive, exponentially* (s,m)*-preinvex and expoennetially symmetric about* $\frac{2x_1+\eta(x_2,x_1)}{2}$, *then the* *following fractional integral inequality for* (2. 9) *and* (2. 10) *holds:*

$$
\begin{aligned}
&\frac{2^s}{1+m}\hbar\Big(\frac{2x_1+\eta(x_2,x_1)}{2}\Big)\\
&\times\Big[(e^{\vartheta_1 x_1})\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}1\Big)(x_1;q)\\
&\quad+e^{\vartheta_2(x_1+\eta(x_2,x_1))}\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_1^+}1\Big)(x_1+\eta(x_2,x_1);q)\Big]\\
&\leq\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_1^+}\hbar\Big)(x_1+\eta(x_2,x_1);q)\\
&+\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}\hbar\Big)(x_1;q)\\
&\leq\frac{\eta(x_2,x_1)}{s+1}\Big(\frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}\\
&+m\frac{\hbar(x_1)}{e^{\vartheta_1 x_1}}\Big)\Big[\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,\omega,x_1^-}1\Big)(x_1+\eta(x_2,x_1);q)\\
&\quad+\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}1\Big)(x_1;q)\Big].
\end{aligned}
\tag{3. 39}
$$

Proof. For $z\in[x_1,x_1+\eta(x_2,x_1)]$, we have

$$
\begin{aligned}
&(z-x_1)^\tau E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath}(\omega(z-x_1)^\mu;q)\\
&\leq(\eta(x_2,x_1))^\tau E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath}(\omega(\eta(x_2,x_1)^\mu;q),\tau>0
\end{aligned}
\tag{3. 40}
$$

As $\hbar$ is exponentially (s,m)-preinvex so for $z\in[x_1,x_1+\eta(x_2,x_1)]$, we have

$$
\begin{aligned}
\hbar(z)\leq&\Big(\frac{z-x_1}{\eta(x_2,x_1)}\Big)^s\frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}\\
&+m\Big(\frac{(x_1+\eta(x_2,x_1))-z}{\eta(x_2,x_1)}\Big)^s\frac{\hbar(x_1)}{e^{\vartheta_1 x_1}}.
\end{aligned}
\tag{3. 41}
$$

Conducting product between (3. 40) and (3. 41) and then integrating with respect to the variable z from x_1 and x_2, we get

$$
\begin{aligned}
&\int_{x_1}^{x_2}(z-x_1)^\tau E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath}(\omega(z-x_1)^\mu;q)\hbar(z)dz\\
&\leq(\eta(x_2,x_1))^\tau E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath}(\omega(\eta(x_2,x_1)^\mu;q)\\
&\times\int_{x_1}^{x_2}\Big[\Big(\frac{z-x_1}{\eta(x_2,x_1)}\Big)^s\frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}\\
&+m\Big(\frac{(x_1+\eta(x_2,x_1))-z}{\eta(x_2,x_1)}\Big)^s\frac{\hbar(x_1)}{e^{\vartheta_1 x_1}}\Big]dz.
\end{aligned}
$$

from which we have

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}\hbar\right)(x_1,q) \\
&\leq \frac{(\eta(x_2,x_1))^{\tau} E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath}(\omega(\eta(x_2,x_1))^{\mu};q)}{s+1} \\
&\times\Big(\frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}+m\frac{\hbar(x_1)}{e^{\vartheta_1 x_1}}\Big), \\
&= \frac{\eta(x_2,x_1)}{s+1}\left(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}1\right)(x_1;q) \\
&\times\Big(\frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}+m\frac{\hbar(x_1)}{e^{\vartheta_1 x_1}}\Big).
\end{aligned} \tag{3. 42}$$

Analogously for $z \in [x_1, x_1+\eta(x_2,x_1)]$, we have

$$\begin{aligned}
&(x_1+\eta(x_2,x_1)-z)^{\sigma} E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma+1,\jmath}(\omega(x_1+\eta(x_2,x_1));q) \\
&\leq (\eta(x_2,x_1))^{\sigma} E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma+1,\jmath}(\omega(\eta(x_2,x_1))^{\mu};q), \quad \vartheta_1>0.
\end{aligned} \tag{3. 43}$$

Conducting product (3. 40) and (3. 43) and then integrating with respect to z with respect to x_1 and x_2, we get

$$\begin{aligned}
&\int_{x_1}^{x_2}(x_1+\eta(x_2,x_1)-z)^{\sigma} E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma+1,\jmath}(\omega(x_1+\eta(x_2,x_1))^{\mu};q)\hbar(z)dz \\
&\leq (\eta(x_2,x_1))^{\sigma} E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma+1,\jmath}(\omega(\eta(x_2,x_1))^{\mu};q) \\
&\times \quad \int_{x_1}^{x_2}\Big[\Big(\frac{z-x_1}{\eta(x_2,x_1)}\Big)^{s}\frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}} \\
&+m\Big(\frac{(x_1+\eta(x_2,x_1))-z}{\eta(x_2,x_1)}\Big)^{s}\frac{\hbar(x_1)}{e^{\vartheta_1 x_1}}\Big]dz.
\end{aligned} \tag{3. 44}$$

From which we have

$$\begin{aligned}
&\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath\,\omega,x_1^+}\hbar \;\; (x_1+\eta(x_2,x_1) : q) \\
&\leq \frac{(\eta(x_2,x_1))^{\sigma} E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(\eta(x_2,x_1))^{\mu};q)}{s+1} \\
&\times \;\; \frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}+m\frac{\hbar(x_1)}{e^{\vartheta_1 x_1}} \\
&= \frac{\eta(x_2,x_1)}{s+1} \;\; \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,\omega,x_1^-}1 \;\; (x_1+\eta(x_2,x_1);q) \\
&\times \;\; \frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}+m\frac{\hbar(x_1)}{e^{\vartheta_1 x_1}} \quad .
\end{aligned} \tag{3. 45}$$

Summing up (3. 42) and (3. 45), we attain

$$\begin{aligned}&\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}\hbar\Big)(x_1,q)\\&+\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath\,\omega,x_1^+}\hbar\Big)(x_1+\eta(x_2,x_1):q)\\&\leq\frac{\eta(x_2,x_1)}{s+1}\Big(\frac{\hbar(x_1+\eta(x_2,x_1))}{e^{\vartheta_2(x_1+\eta(x_2,x_1))}}\\&+m\frac{\hbar(x_1)}{e^{\vartheta_1x_1}}\Big)\bigg[\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,\omega,x_1^-}1\Big)(x_1+\eta(x_2,x_1);q)\\&\quad+\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}1\Big)(x_1;q)\bigg].\end{aligned}\tag{3. 46}$$

Taking product (3. 36) with $(z-x_1)^{\tau}E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath}(\omega(z-x_1)^{\mu};q)$ and integrating with respect to the variable z from x_1 and x_2, we have

$$\begin{aligned}&\hbar\Big(\frac{2x_1+\eta(x_2,x_1)}{2}\Big)\int_{x_1}^{x_2}(z-x_1)^{\tau}E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath}(\omega(z-x_1)^{\mu};q)dz\\&\leq\frac{(1+m)}{2^s}\int_{x_1}^{x_2}(z-x_1)^{\tau}E^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath}(\omega(z-x_1)^{\mu};q)\frac{\hbar(z)}{e^{\vartheta_1z}}dz.\end{aligned}\tag{3. 47}$$

Using Definition 2.8, we get

$$\begin{aligned}&\hbar\Big(\frac{2x_1+\eta(x_2,x_1)}{2}\Big)\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}1\Big)(x_1;q)\\&\leq\frac{(1+m)}{2^se^{\vartheta_1x_1}}\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}\hbar\Big)(x_1;q)\end{aligned}\tag{3. 48}$$

Taking product (3. 36) with $(x_1+\eta(x_2,x_1)-z)^{\sigma}E^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma,\jmath}(\omega(x_1+\eta(x_2,x_1))^{\mu};q)$ and integrating with respect to the variable z from x_1 and x_2, we have

$$\begin{aligned}&\hbar\Big(\frac{2x_1+\eta(x_2,x_1)}{2}\Big)\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_1^+}1\Big)(x_1+\eta(x_2,x_1);q)\\&\leq\frac{(1+m)}{2^se^{\vartheta_2(x_1+\eta(x_2,x_1))}}\Big(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_1^+}\hbar\Big)(x_1+\eta(x_2,x_1);q).\end{aligned}\tag{3. 49}$$

Summing up (3. 47) and (3. 48), we get

$$\begin{aligned}&\frac{2^s}{1+m}\hbar\ \ \frac{2x_1+\eta(x_2,x_1)}{2}\\&\times\Big[(e^{\vartheta_1x_1})\ \ \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}1\ \ (x_1;q)\\&\quad+e^{\vartheta_2(x_1+\eta(x_2,x_1)}\ \ \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_1^+}1\ \ (x_1+\eta(x_2,x_1);q)\Big]\\&\leq\ \ \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_1^+}\hbar\ \ (x_1+\eta(x_2,x_1);q)\\&+\ \ \epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,(x_1+\eta(x_2,x_1))^-}\hbar\ \ (x_1;q)\end{aligned}\tag{3. 50}$$

By combining (3. 46) and (3. 50), we get inequality (3. 39). □

Corollary 3.14. *Choosing $\eta(x_2, x_1) = x_2 - x_1$, then under the assumption of Theorem 3.13, we have*

$$\begin{aligned}&\frac{2^s}{1+m}\hbar\Big(\frac{x_1+x_2}{2}\Big)\Big[e^{\vartheta_1 x_1}\left(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_2^-}1\right)(x_1;q)\\&\quad+e^{\vartheta_2 x_2}\left(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_1^+}1\right)(x_2);q)\Big]\\&\le\left(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_1^+}\hbar\right)(x_2);q)+\left(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau+1,\jmath,\omega,x_2^-}\hbar\right)(x_1;q)\\&\le\frac{(x_2-x_1)}{s+1}\Big(\frac{\hbar(x_2)}{e^{\vartheta_2 x_2}}+m\frac{\hbar(x_1)}{e^{\vartheta_1 x_1}}\Big)\\&\times\left[\left(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\sigma-1,\jmath,\omega,x_1^+}1\right)(x_2;q)+\left(\epsilon^{\gamma,\delta,\kappa,\nu}_{\mu,\tau-1,\jmath,\omega,x_2^-}1\right)(x_1;q)\right]\end{aligned} \tag{3. 51}$$

4. CONCLUSION

In this study, we have introduced the more general form of fractional integral inequalities by choosing the appropriate and suitable choice of parameters can be attained. For instance, choosing $q = 0$, and $\jmath = \delta = 1$ in Definition 2.8 our results reduces to the results for fractional integral operators defined by Salim and Faraj [33] and Rahman et al. [6], respectively. Moreover, for $q = 0$ and $\jmath = \delta = 1$, our resutlts reduces to fractional integral operators defined by Shukla and Prajapati in [47], choosing $q = 0$ and $\jmath = \delta = \kappa = 1$, for fractional integral operators defined by Prabhakar in [5], under some certain conditions defined by exponentially (s, m)-preinvex functions. By choosing $q = \omega = 0$ fractional integral inequalities for Riemann-Liouville fractional integrals. All the results hold for s-preinvex, m-preinvex, exponentially preinvex, exponentially s-preinvex as well as convex functions. The results can be used to study some new and know results having application in scientific and technological fields.

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6. AUTHORS CONTRIBUTIONS

Both the authors worked jointly. They all read and approved the final manuscript.

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