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Offset Approximation of Rational Trigonometric Bézier Curves

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Abstract.: Offset curves are one of the crucial curves, but the presence of square root function in the representation is main hindrance towards their applications in CAD/CAM. The presented technique is based on offset approximation using rational trigonometric Bézier curves. The idea is to construct a new control polygon parallel to original one. The two end points of the offset control polygon have been taken as exact offset end points, while the middle control points and weights have been computed using definition of parallel curves. As a result, offsets of rational and non-rational trigonometric Bézier curves have been approximated by rational cubic trigonometric Bézier curve. An error between exact and approximated offset curves have also been computed to show the efficacy of the method.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Offset curves, Approximation, Bézier curves

1. Introduction

While working on shape generation in computer aided design (CAD) and computer aided manufacturing (CAM), a designer often needs to draw a curve which is parallel to a base curve at a certain distance. These curves are referred to as offset curves [1]. Applications of offset curves include numerical control machining [2], shoe, textile, automotive industry and font designing. Offset curves are also used in robot path planning [3], surface generation [11], animation used in movies, pocket machining ([1], [4]), ship designing and tolerance analysis [5]. Some applications of offset curves include manufacturing of railways track [12], conveyor belts, road maps and path of rides [3] (see Figure 1 for practical usage of offset curves).

Among the famous and user friendly curve designing techniques, Bézier curves are most popular, both in rational and non rational form, that provide the user an innate control on

FIGURE 1. Applications of offset curves

the generation of curve. The mathematical representation of these curves is compact and elegant. Also the involvement of shape parameters [12] makes them more flexible for shape generation. Since, Bézier curves involve easy computations and geometrically are highly stable, so are widely used in the fields ranging from CAGD to generic object shape descriptors, geology [10] and even economy.
On the whole, the offset of a rational Bézier curve does not preserve its rational structure. This is mainly due to the square root function present in the denominator of their formula. Due to this factor, offset curves achieve a high algebraic degree and hence, cannot be applied in NC-machining. For example, the degree of offset of cubic Bézier curve is ten. This drawback has motivated the researchers to find the various approximation methods for generating offset curves, including, offset approximation of cubic splines by using curvature and tangents at the end points [6], approximation of offset curves using control knots [7]. The introduction of approximation methods has increased the use of offset curves in industry where a designer can draw offset curves at desired distance as per need and requirement. This study presents offset approximation of trigonometric Bézier curves. Offset curves of rational and non rational Bézier curves of degree two and three have been approximated by rational cubic trigonometric Bézier curves. Trigonometric Bézier curves are devised in comparison with the traditional Bézier curves. These curves share the same geometric properties as of traditional Bézier curves and additionally, offer better designing powers.

2. Notations and Preliminaries

2.1. **Offset Curves.** Offset curves are generally parallel curves that are drawn at a fixed distance from a given curve. The fixed distance d is known as offset distance [1]. Suppose the given parametric curve is

$$S(\zeta) = (S_x(\zeta), S_y(\zeta)). \tag{2. 1}$$

Then its offset curve $S_d(\zeta)$ at a distance d is defined as:

$$S_d(\zeta) = S(\zeta) \pm d\mathbf{N}(\zeta), \tag{2. 2}$$

$$\mathbf{N}(\zeta) = \pm \frac{(S_y^{'}(\zeta), S_x^{'}(\zeta))}{\sqrt{S_x^{'}(\zeta)^2 + S_y^{'}(\zeta)^2}}. \tag{2. 3}$$

$S_x^{'}$ and $S_y^{'}$ represents derivatives with respect to ζ, of x and y coordinates respectively. Where as $\pm$ indicates signed normal vector.

2.2. **Rational Trigonometric Bézier curves.** A rational trigonometric Bézier curve with control points $Q_j = (x_j, y_j), j = 0, 1, 2, 3, ..., n$, is defined as [7]:

$$S(\zeta) = \big(S_x(\zeta), S_y(\zeta)\big) = \left(\frac{x(\zeta)}{h(\zeta)}, \frac{y(\zeta)}{h(\zeta)}\right) = \frac{\sum_{j=0}^{n} \lambda_j b_j^n(\zeta) Q_j}{\sum_{j=0}^{n} \lambda_j b_j^n(\zeta)}, \qquad \zeta \in [0,1] \quad (2.\ 4)$$

$$x(\zeta) = \sum_{j=0}^{n} \lambda_j b_j^n(\zeta) x_j,$$

$$y(\zeta) = \sum_{j=0}^{n} \lambda_j b_j^n(\zeta) y_j,$$

$$h(\zeta) = \sum_{j=0}^{n} \lambda_j b_j^n(\zeta),$$

where $b_j^n(t)$ for $\zeta \in [0,1]$, are trigonometric basis functions defined in [8], [7] and $\lambda_j > 0$ are the weights.

2.2.1. *Rational Quadratic Trigonometric Bézier curves.* For $n = 2$ the rational quadratic trigonometric Bézier curve is defined as:

$$S(\zeta) = \frac{\sum_{j=0}^{2} \lambda_j b_j^2(\zeta) Q_j}{\sum_{j=0}^{2} \lambda_j b_j^2(\zeta)}, \qquad \zeta \in [0,1] \quad (2.\ 5)$$

The basis functions $b_j^2(\zeta)$ with $r_1, r_2 \in [-1, 1]$ as shape parameters, are defined as [7].

$$\begin{aligned} b_0^2(\zeta) &= (1 - \sin\frac{\pi}{2}\zeta)(1 - r_1 \sin\frac{\pi}{2}\zeta) \\ b_1^2(\zeta) &= -1 + \sin\frac{\pi}{2}\zeta\big(1 + r_1(1 - \sin\frac{\pi}{2}\zeta)\big) + \cos\frac{\pi}{2}\zeta\big(1 + r_2(1 - \cos\frac{\pi}{2}\zeta)\big) \\ b_2^2(\zeta) &= (1 - \cos\frac{\pi}{2}\zeta)(1 - r_2 \cos\frac{\pi}{2}\zeta) \end{aligned} \quad (2.\ 6)$$

2.2.2. *Rational cubic trigonometric Bézier curves.* For $n = 3$ the rational cubic trigonometric Bézier curve is defined as([8]):

$$S(\zeta) = \frac{\sum_{j=0}^{3} \lambda_j b_j^3(\zeta) Q_j}{\sum_{j=0}^{3} \lambda_j b_j^3(\zeta)}, \qquad \zeta \in [0,1] \quad (2.\ 7)$$

The basis functions $b_j^3(\zeta)$ with $r_1, r_2 \in [-1,1]$ as shape parameters, are defined as [8].

$$\begin{aligned}
b_0^3(\zeta) &= (1-\sin\frac{\pi}{2}\zeta)\big((1-\sin\frac{\pi}{2}\zeta)+r_1\sin\frac{\pi}{2}\zeta\cos\frac{\pi}{2}\zeta\big)\\
b_1^3(\zeta) &= \sin\frac{\pi}{2}\zeta(1-\sin\frac{\pi}{2}\zeta)(2-r_1\cos\frac{\pi}{2}\zeta)\\
b_2^3(\zeta) &= \cos\frac{\pi}{2}\zeta(1-\cos\frac{\pi}{2}\zeta)(2-r_2\sin\frac{\pi}{2}\zeta)\\
b_3^3(\zeta) &= (1-\cos\frac{\pi}{2}\zeta)\big((1-\cos\frac{\pi}{2}\zeta)+r_2\sin\frac{\pi}{2}\zeta\cos\frac{\pi}{2}\zeta\big).
\end{aligned} \tag{2. 8}$$

Theorem 2.3. [8] *for $n = 2,3$ the quadratic and cubic trigonometric basis functions defined in equation 2. 6 and 2. 8 satisfy the following properties which can be observed in figure 2.*

(i) Non negativity: $b_j^n(\zeta) \geq 0$, $j = 0,1,2,...,n$

(ii) Partition of unity: $\sum_{j=0}^n b_j^n(\zeta) = 1$.

(iii) Monotonicity: For the specified values of shape parameters r_1 and r_2, the basis function $b_0^n(\zeta)$ decreases and $b_n^n(\zeta)$ increases monotonically.

(iv) Symmetry: Trigonometric basis functions are symmetric with respect to the parameters i.e;

$$b_j^n(\zeta)(\zeta, r_1, r_2) = b_{n-j}^n(\zeta)(1-\zeta, r_2, r_1), \quad j = 0,1,2,...,n$$

The properties of basis functions defined above give rise to promising properties of the generating curve, shown respectively in the figure 2, are as follows:

(i) End Point Property: One can witness end points values of the curve as first and last control points, $S(0) = Q_0, S(1) = Q_n$. Whereas first order derivate of curve at the end points represents the tangents at those points.

(ii) Symmetry: The symmetry in basis functions is responsible for the symmetry in the curve. $S(\zeta, Q_0, Q_1, Q_2, , Q_n) = S(1-\zeta, Q_n, Q_{n-1}, Q_{n-2}, , Q_0)$

(iii) Geometric Invariance: Under the effect of translation and rotation the curve is invariant due to partition of unity of basis.

(iv) Convex Hull Property: The curve will always remain in the convex hull of its describing control polygon for non-negative weights.

3. Offset Approximation of the Trigonometric Bézier Curve

3.1. **Offset Approximation of Rational Quadratic Trigonometric Bézier Curves.** In this section, the offset curve $S_d(\zeta)$ of the curve defined in equation 2. 2 , is approximated by rational cubic trigonometric Bézier curve. The approximating curve $U(\zeta)$, with control points $R_k = (x_k, y_k)$ and non-negative weights ν_k is defined as:

$$\begin{aligned}
U(\zeta) = \big(U_x(\zeta), U_y(\zeta)\big) &= \left(\frac{x(\zeta)}{h(\zeta)}, \frac{y(\zeta)}{h(\zeta)}\right)\\
&= \frac{\sum_{k=0}^3 \nu_k b_k^3(\zeta) R_k}{\sum_{k=0}^3 \nu_k b_k^3(\zeta)}, \qquad \zeta \in [0,1]
\end{aligned} \tag{3. 9}$$

Where the end control points R_0 and R_3 can be given as:

$$R_0 = Q_0 + d\mathbf{N}(0), \tag{3. 10}$$

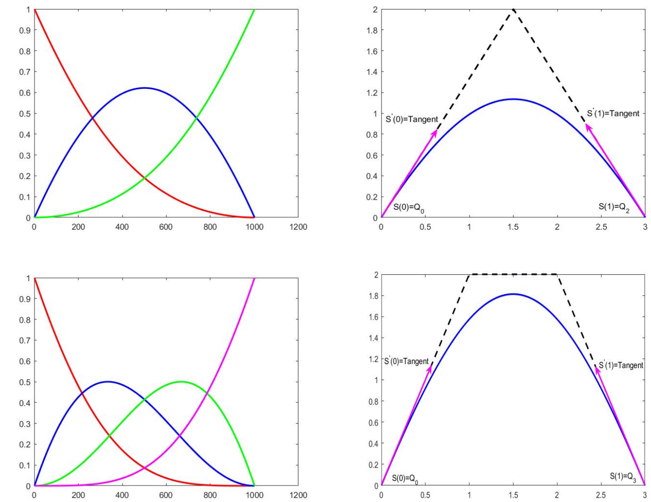


FIGURE 2. Quadratic and Cubic Trigonometric basis along with their curves

$$R_3 = Q_2 + d\mathbf{N}(1) \tag{3. 11}$$

and by the definition of parallel tangents at the end points, the intermediate control points R_1 and R_2 can be written as:

$$R_1 = R_0 + k_1 \frac{\nu_0 \lambda_1}{\nu_1 \lambda_0}(Q_1 - Q_0), \tag{3. 12}$$

$$R_2 = R_3 + k_2 \frac{\nu_3 \lambda_1}{\nu_2 \lambda_2}(Q_1 - Q_2), \tag{3. 13}$$

where $k_i,\ i = 1, 2$ are unknowns.
To find the unknowns k_i, take (2m+1) points A_i on the offset curve $S_d(\zeta)$, with parameter values $\zeta_i = \frac{i}{2m}$, where $i = 0, 1, 2, ..., 2m$ and $m = max(3, 2)$. Inserting these A_i points into the approximated offset curve $U(\zeta)$ gives,

$$A_i = U(\zeta_i) + \xi_i, \tag{3. 14}$$

$$\xi_i = A_i - U(\zeta_i), \tag{3. 15}$$

where ξ_i are error vectors. The minimization of this error vector will lead us to finest approximation of the offset curves. To carry out the task, Least square approximation method is used. According to which the square of the norm of the error vector $\xi = \Sigma_{i=0}^{2m}(\xi_i)^2$ that is:

$$\xi = \Sigma_{i=0}^{2m}(A_i - U(\zeta_i))^2, \tag{3. 16}$$

$$\xi = \sum_{i=0}^{2m} \Bigg((A_i - R_0)\big(b_0^3(\zeta_i) + \nu_1 b_1^3(\zeta_i)\big) + (A_i - R_3)\big(b_3^3(\zeta_i) + \nu_2 b_2^3(\zeta_i)\big) - k_1 \frac{\lambda_1}{\lambda_0}(Q_1 - Q_0) b_1^3(\zeta_i) - k_2 \frac{\lambda_1}{\lambda_2}(Q_1 - Q_2) b_2^3(\zeta_i) \Bigg)^2. \tag{3. 17}$$

should be as small as possible.
To attain the objective of Least Square approximation method the partial derivative of ξ with respect to four unknown parameters k_1, k_2, ν_1 and ν_2 must be vanished. Mathematically

$$\begin{aligned} \partial\xi/\partial k_1 &= 0, \\ \partial\xi/\partial k_2 &= 0 \\ \partial\xi/\partial \nu_1 &= 0, \\ \partial\xi/\partial \nu_2 &= 0 \end{aligned} \tag{3. 18}$$

These four equations lead to a linear system in four unknowns as the system is a combination of nonzero weights, control points and basis. Solving these four equations gives unknown values of k_1, k_2, ν_1 and ν_2 and consequently obtain the intermediate control points R_1 and R_2 of the approximated offset curve.

3.1.1. *Special Case: offset Approximation of Non Rational Quadratic Trigonometric Bézier Curves.* The non-rational curve is obtained by taking all weights equal to unity in equation 2. 4 . $U(\zeta)$ with control points R_k and weights ν_k; is the approximant of the offset curve $S_d(\zeta)$ of the curve defined in equation 2. 7 . The end control points of the approximant curve R_0 and R_3 are computed from equations 3. 10 and 3. 11 . The definition of parallel tangents at the end points gives, middle control points obtained from 3. 12 and 3. 13 . The approximated curve $U(\zeta)$ and the points A_i are related as in 3. 14 and the error vector in this case is given as,

$$\xi = \sum_{i=0}^{2m} \Bigg((A_i - R_0)\big(b_0^3(\zeta_i) + \nu_1 b_1^3(\zeta_i)\big) + (A_i - R_3)\big(b_3^3(\zeta_i) + \nu_2 b_2^3(\zeta_i)\big) - k_1 (Q_1 - Q_0) b_1^3(\zeta_i) - k_2 (Q_1 - Q_2) b_2^3(\zeta_i) \Bigg)^2. \tag{3. 19}$$

This equation lead to the computation of unknown control points using Least Square approximation.

3.2. **Offset Approximation of Rational Cubic Trigonometric Bézier Curves.** In this section offset approximation of rational cubic trigonometric Bézier curve is discussed. Offset curve $S_d(\zeta)$ is approximated by the curve $U(\zeta)$ defined in equation 2. 7 . Since the difference between the end points of the curve and its offset is d. Therefore, the end control points R_0 and R_3 of approximated offset $U(\zeta)$ of rational cubic trigonometric Bézier curve are calculated as:

$$R_0 = Q_0 + d\mathbf{N}(0), \tag{3. 20}$$

$$R_3 = Q_3 + d\mathbf{N}(1). \tag{3. 21}$$

Furthermore, applying geometric continuity conditions specifically tangent continuity at the end points gives [13],

$$R_1 = R_0 + k_1 \frac{\nu_0 \lambda_1}{\nu_1 \lambda_0}(Q_1 - Q_0), \tag{3. 22}$$

$$R_2 = R_3 + k_2 \frac{\nu_3 \lambda_2}{\nu_2 \lambda_3}(Q_2 - Q_3), \tag{3. 23}$$

where k_i, $i = 1, 2$ are unknowns. To determine the unknowns k_i, take (2m+1) points A_i on the offset curve $S_d(t)$ with parameter values $\zeta_i = \frac{i}{2m}$, where $i = 0, 1, 2, ..., 2m$ and $m = 3$. After setting these A_i points into the approximated offset curve $U(\zeta)$ and taking the minimum vector ξ as,

$$\begin{aligned} \xi = \sum_{i=0}^{2m} \Bigg((A_i - R_0)\Big(b_0^3(\zeta_i) + \nu_1 b_1^3(\zeta_i)\Big) + (A_i - R_3)\Big(b_3^3(\zeta_i) + \nu_2 b_2^3(\zeta_i)\Big) \\ - k_1 \frac{\lambda_1}{\lambda_0}(Q_1 - Q_0) b_1^3(\zeta_i) - k_2 \frac{\lambda_2}{\lambda_3}(Q_2 - Q_3) b_2^3(\zeta_i)\Bigg)^2. \end{aligned} \tag{3. 24}$$

This error vector approximate the intermediate control points R_1 and R_2 of the approximated curve are evaluated.

3.2.1. *Special case: Offset Approximation of the non-rational cubic Trigonometric Bézier Curves.* One can easily obtain the non-rational cubic trigonometric Bézier curve by taking all weights equal to unity. $U(\zeta)$ defined in equation 2. 7 with control points R_k and weights ν_k; is the offset approximation of $S_d(t)$, offset of the non-rational cubic trigonometric Bézier curve. The end control points are calculated from equation 3. 20 and 3. 21 . Since the given curve and its offset are parallel to each other therefore, both curves have parallel tangents at the end points, and obtained the mid control points from equations 3. 22 and 3. 23 . The minimum vector in this case have the form

$$\begin{aligned} \xi = \sum_{i=0}^{2m} \Bigg((A_i - R_0)\Big(b_0^3(\zeta_i) + \nu_1 b_1^3(\zeta_i)\Big) + (A_i - R_3)\Big(b_3^3(\zeta_i) + \nu_2 b_2^3(\zeta_i)\Big) \\ - k_1 (Q_1 - Q_0) b_1^3(\zeta_i) - k_2 (Q_2 - Q_3) b_2^3(\zeta_i)\Bigg)^2. \end{aligned} \tag{3. 25}$$

Algorithm 1: Offset approximation of trigonometric Bézier curves

All four techniques are summed up in the form of algorithm for computation of approximated offset curve of $S_d(\zeta)$, by a rational cubic trigonometric Bézier curve:

Input: Choose number k of points A_i, in general $k = 2\mu$ ($\mu = max(3, n) = 3$), (where n=2,3 is the degree of given curve) and error ϵ_o.

Calculate $A_i = S_d(\zeta_i)$ at $\zeta_i = i/k$, $i = 0, 1, 2, ..., k$;

Calculate k_1, k_2, ν_1, and ν_2 by least square method;

Output: The approximated offset curve $U(\zeta)$.

Find the maximum deviation $\epsilon = max|A_i - S_d(\zeta_i)|$;

if $\epsilon \leqslant \epsilon_o$ **then**

  STOP (the result is defined by 2. 7)

end

else

  split the given curve into two segments;

  Goto 1;

  Compute approximated offset curve of $S_d(\zeta)$ for each segment.

end

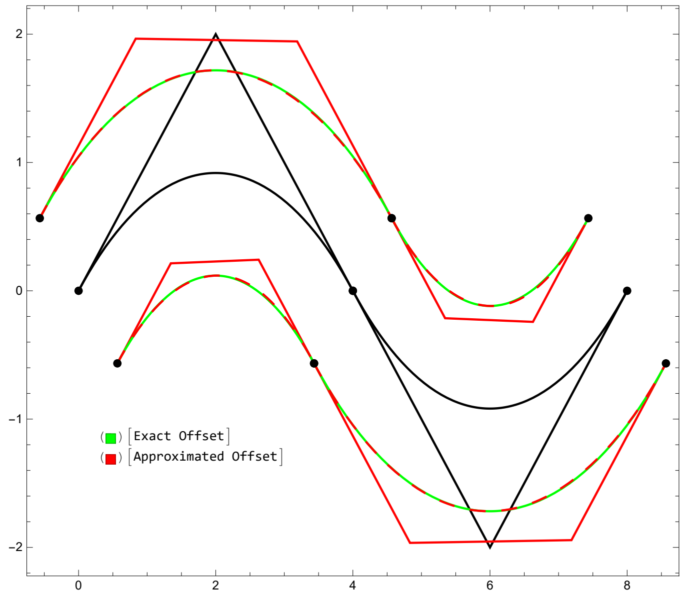
4. Applications

4.1. **Example 1.** S like curves and their offsets have vast applications in railway tracks and road path designing. Figure 3(b) represents the approximated offset curves drawn at $d = \pm 0.8$. The original curve consists of two segments; joined together by geometric continuity conditions and offset approximation of each segment is done separately (see figure 3(a)). Exact offset obtained by formulation is drawn in solid green color while approximated offset is drawn in dashed red color.

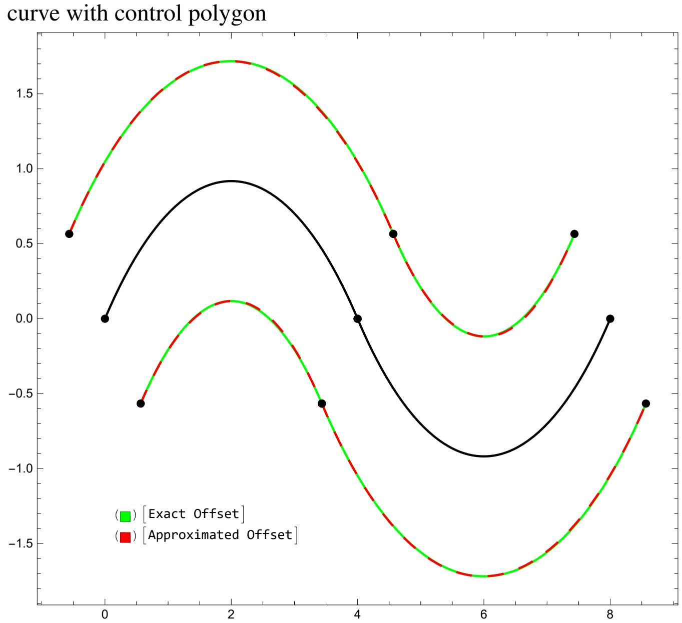
4.2. **Example 2.** In the figure 4(a) the base curve consists of five segments of non-rational quadratic curve with shape parameters 0.2 and offset curve of each segment is approximated separately using rational cubic curve. Approximated offset curve had drawn at offset distance -0.05 as shown in figure 4(b).

4.3. **Example 3.** In this example base curve is rational cubic curve and its approximated offset is also a rational cubic curve (see figure 5(a)). The complete curve is composed of four segments connected by C^2 continuity and offset of each segment approximated separately as shown in figure 5(b). Offset curve is drawn at offset distance $d = 0.1$. The solid black colored curve represent approximated offset, while magenta dashed colored curve shows formulated offset curve

4.4. **Example 4.** The figure 6(a) has been representing a helix curve generated by non-rational cubic curve and its approximated offset is a rational cubic curve. The curve has been composed of seven segments and offset of each segment approximated separately as shown in figure 6(b). Each approximated offset segment had drawn at a distance equal to 0.1.

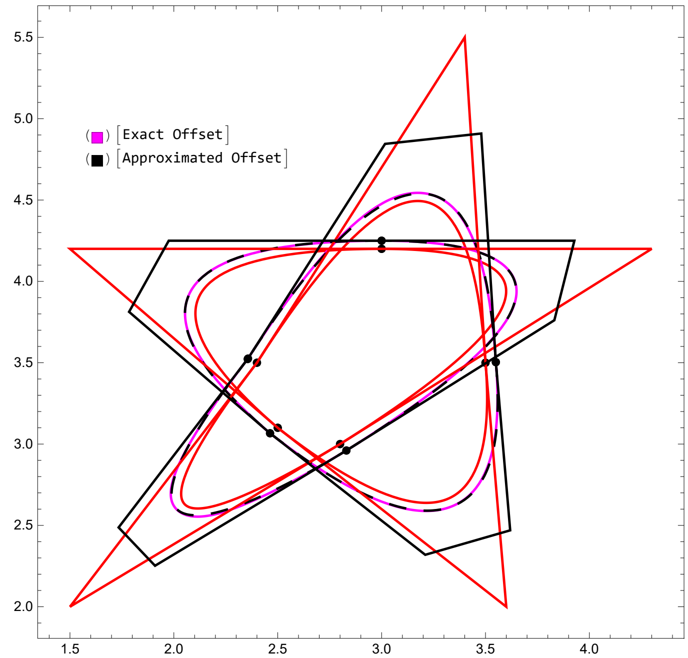


(A) Approximated offset of rational quadratic trigonometric Bézier curve with control polygon

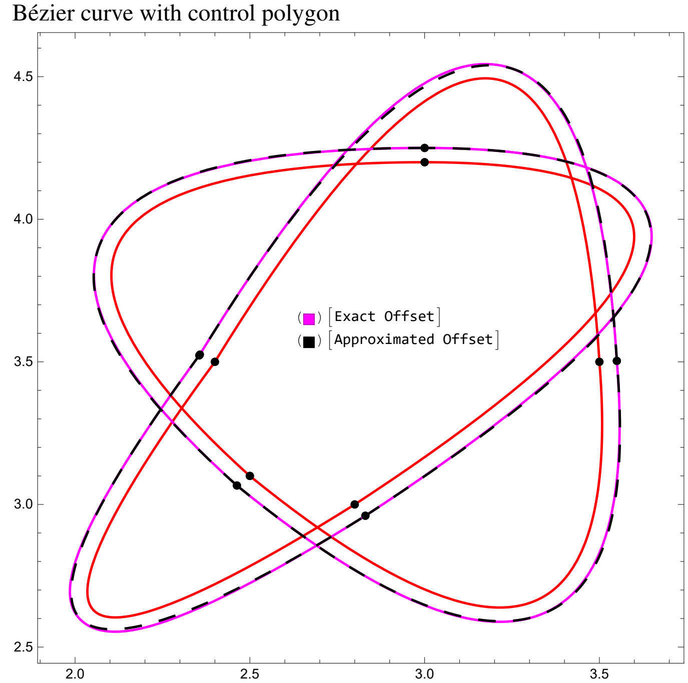


(B) Approximated offset of rational quadratic trigonometric Bézier curve without control polygon

FIGURE 3. Offset approximation of rational quadratic trigonometric Bézier curve

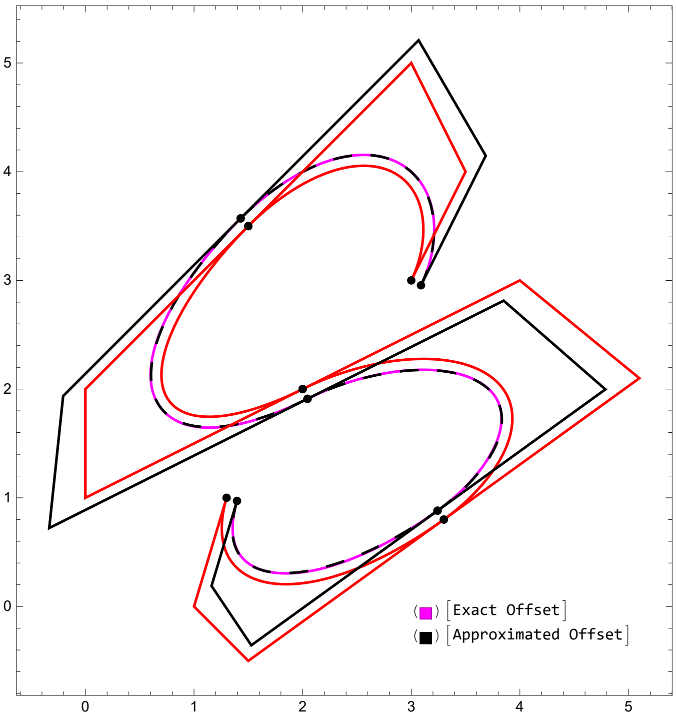


(A) Approximated offset of non-rational quadratic trigonometric Bézier curve with control polygon

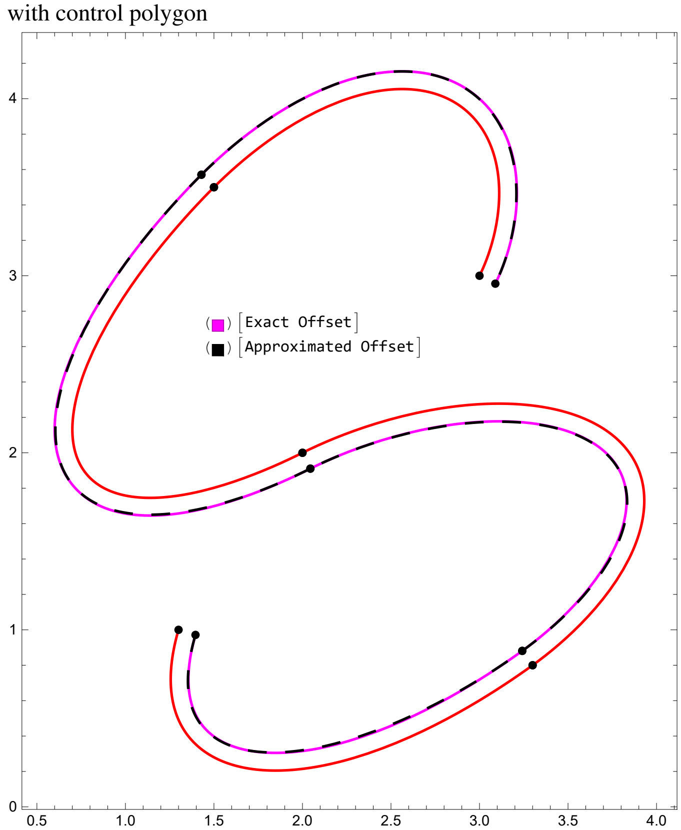


(B) Approximated offset of non-rational quadratic trigonometric Bézier curve without control polygon

FIGURE 4. Offset approximation of non-rational quadratic trigonometric Bézier curve

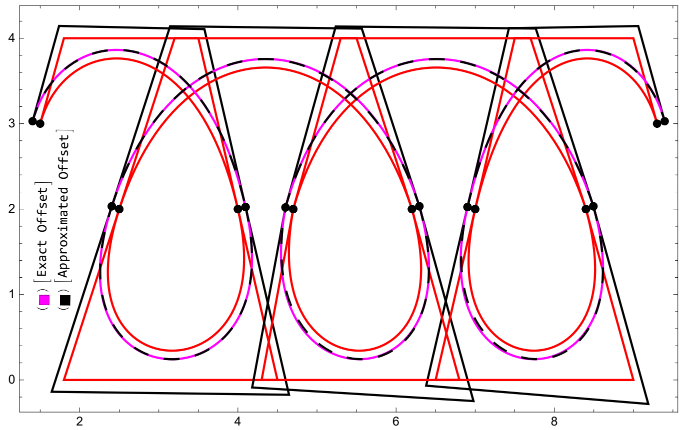


(A) Approximated offset of rational cubic trigonometric Bézier curve with control polygon

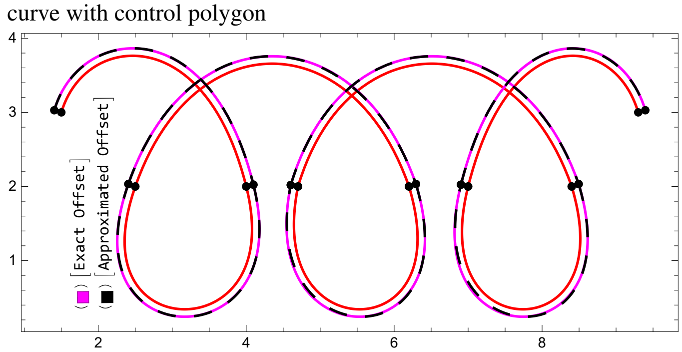


(B) Approximated offset of rational cubic trigonometric Bézier curve without control polygon

FIGURE 5. Offset approximation of rational cubic trigonometric Bézier curve



(A) Approximated offset of non-rational cubic trigonometric Bézier curve with control polygon



(B) Approximated offset of non-rational cubic trigonometric Bézier curve without control polygon

FIGURE 6. Offset approximation of non-rational cubic trigonometric Bézier curve

4.5. **Example 5.** This example discuss the approximation of offset of a circle generated by a non-rational quadratic trigonometric Bézier curve. Whereas the approximated offset is a rational trigonometric Bézier curve of order three. The tolerance is $\epsilon_o = (0.005, 0.005)$ 7. The error in this case is zero which shows the efficiency of the method.

4.6. **Example 6.** In this example approximated offset of a curve has been drawn at a distance where it include a cusp (figure 8(a)) and tolerance is set as $\epsilon_o = (0.05, 0.05)$. To overcome this singularity the curve is segmented into parts and approximated their offsets separately which are joined by C^2 continuity condition (figure 8(b),8(c)).

5. CONCLUSION

The presented research work is concerned with the offset approximation based on trigonometric Bézier curves. The blending functions of these curves involve two shape parameters

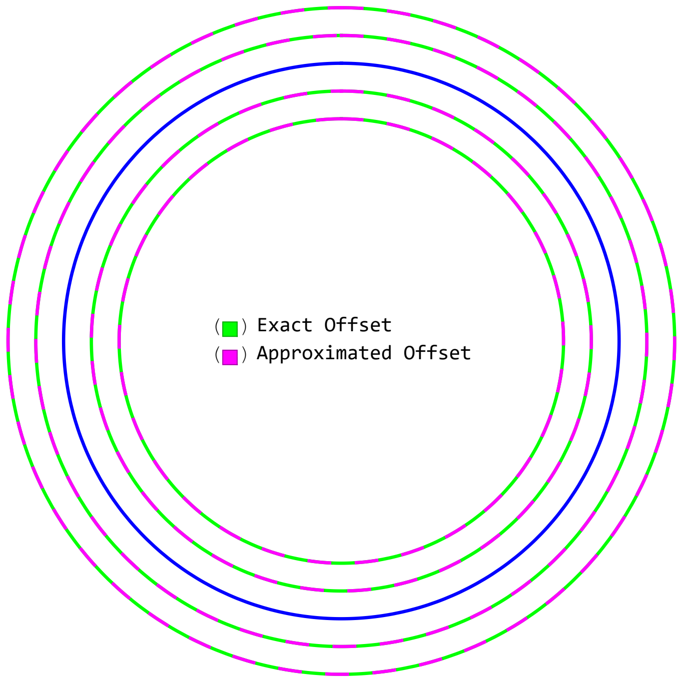
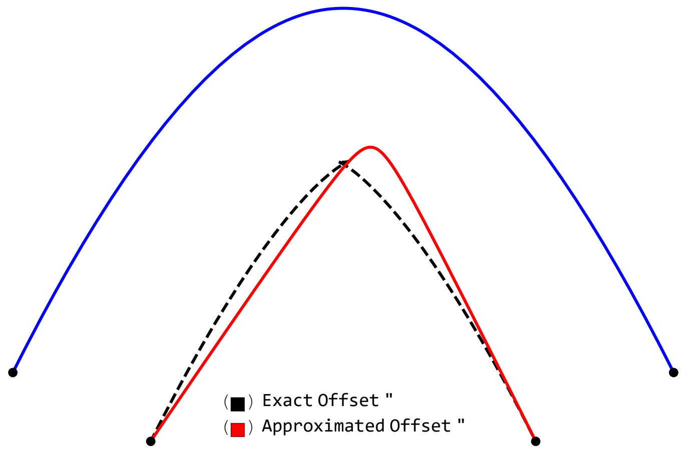


FIGURE 7. Offset approximation of a circle

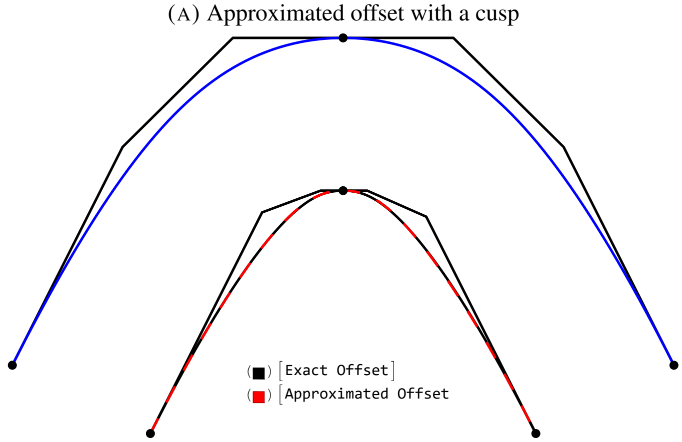
TABLE 1. Data for examples

Example	Offset distance	Degree of given curve	Degree of approximated curve	Max Error
	d	$\xi(\zeta)$	$U(\zeta)$	$\epsilon = max\|A_i - S_d(\zeta_i)\|$
Example 1	± 0.8	2	3	0.0001
Example 2	-0.05	2	3	0.0004
Example 3	0.1	3	3	0.0001
Example 4	0.1	3	3	0.0016
Example 5	$\pm 0.1, \pm 0.2$	2	3	zero
Example 6	-0.7	3	3	0.0001

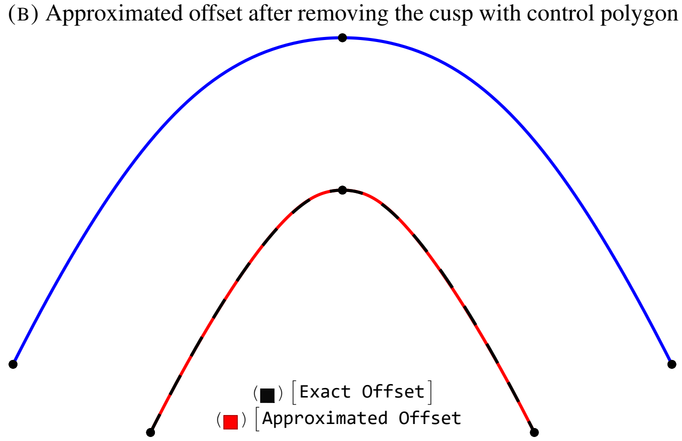
which enhance the degree of freedom in the construction of free form curves. The offset curves of rational and non-rational trigonometric (both quadratic and cubic) Bézier curves have been approximated by constructing a new control polygon. The approximated offset curve lies in the convex hull of this control polygon. The technique involves a rational (non-rational) trigonometric Bézier curve of degree three for the approximation of the offset curves and can be used in industry. The presented technique have imposing results and in case of circle the error is zero. The work can be extended further by considering high degree curves and the space curves.



(A) Approximated offset with a cusp



(B) Approximated offset after removing the cusp with control polygon



(C) Approximated offset after removal of cusp without control polygon

FIGURE 8. Removal of cusp in offsetting

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