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Analytical Solution of Nonlinear Nonhomogeneous Space and Time Fractional Physical Models by Improved Adomian Decomposition Method

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Abstract.: The paper aims to obtain exact analytical solution of nonlinear nonhomogeneous space-time fractional order partial differential equations in Gas dynamics model, Advection model, Wave model and Klein-Gordon model by improved Adomian decomposition method coupled with fractional Taylor expansion series.The solution of these equations are in series form may have rapid convergence to a closed-form solution. The effectiveness and sharpness of this method is shown by obtaining the exact solution of these equations with suitable initial conditions(ICs). With the help of this method, it is possible to investigate nature of solutions when we vary order of the fractional derivative. Behaviour of the solution of these equations are represented by graphs using MATHEMATICA software.

AMS (MOS) Subject Classification Codes: 26A33; 33E12; 34A08; 35R11
Key Words: Improved Adomian decomposition method; Fractional Taylor expansion series; Gas dynamics model; Advection model; Wave model; Klein-Gordon model.

1. Introduction

The study of nonlinear partial differential equations(PDEs) is a well developed research area. There are physical models governed by nonlinear fractional partial differential equations(FPDEs) in various sciences such as sciences of Mathematics, sciences of Physics, sciences of Chemistry, and sciences of Biology as well as in technologies [8, 12, 13, 21, 28, 30]. Several researchers have focused on the study of physical models directed by FPDEs. The difficulty of getting the exact solution of equations in such models is an important and attractive area of research. Not all nonlinear equations in physical models have an exact solution, therefore, many researchers have developed various methods of solving nonlinear FPDEs. Therefore, few methods like integral transform method [22, 33], homotopy perturbation method[14, 15], new iterative method [35, 4, 7], variational iteration method [16], iterative Laplace transform method [36, 37], differential transform method [27] etc. are developed and applied extensively. The Adomian decomposition method (ADM)[1, 2] is a powerful weapon to determine solution of fractional differential equations. A.M.Wazwaz in [39, 41] has applied ADM to solve a variety of problems in differential equations. Chitalkar-Dhaigude and Bhadgaonkar in [3] have shown that the ADM is more convenient than the Charpit's method to solve first-order nonlinear PDEs. Shawagfeh [38], D.Gejji and Jafari [17] has occupied ADM for determining solution of nonlinear fractional differential equations. Also, Dhaigude and Birajdar [5, 6] extended the discrete ADM for obtaining the computative solution of a nonlinear system of FPDEs. Kaya [18, 19] applied ADM for determining solution of specific types of nonlinear PDEs. O.González-Gaxiola and R. Bernal-Jaquez [11] present a model for tumour growth under medical treatment represented by a nonlinear partial differential equation that is solved by using ADM. R.Santanu Saha [34] presents the analytical solutions of the fractional diffusion equations by ADM. A modification of the classical ADM is proposed within the current study based on the recent publications by T. Mustafa [23, 24, 25] successfully generating fast convergent ADM series solutions with as small Adomian polynomials as possible in the solution series.

Rach et al.[32] developed a new modification of the ADM for the resolution of ordinary differential equations(ODEs) by applying the Taylor expansion series for a nonhomogeneous term. N.Khodabakshi et al.[20], discussed the basic method of the ADM and also extend the proposed method in [32] to solve time-fractional ODEs. Sparked from these ideas, we coupled the ADM with fractional Taylor expansion series for obtaining almost analytical solution of a FPDEs. Fractional Taylor expansion series used in this method is represent as differential transform in the differential transform method [27]. The applicability of improved ADM is shown by some physically important nonlinear models and have a great results.

The paper is structured in this way: in Section (2) few basic results about fractional calculus and related properties are given which are used in this paper, while in Section (3) we clarify the steps of the improved ADM for solving nonlinear nonhomogeneous space and time fractional order PDEs by applying the Taylor expansion series for nonhomogeneous function. The effectiveness and sharpness of the method is shown by obtaining solution of equations in physical models like Gas dynamics model, Advection model, Wave model and Klein-Gordon model in Section (4). Section (5) is conclusion.

2. Preliminaries

In this section, basic definitions on fractional calculus are discussed which are useful for further discussion.

Definition 2.1. [30] *Let $f \in \mathcal{C}_\alpha$ and $\alpha \geq -1$, then Riemann-Liouville fractional integral operator(RLFIO) of $w(x,t)$ with respect to t of order α is indicated by $I_t^\alpha w(x,t)$ and is explained as*

$$I_t^\alpha w(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} w(x,\tau) d\tau, \quad t > 0, \alpha > 0. \tag{2. 1}$$

Definition 2.2. [30] *Let $m-1<\alpha<m$, $t\in R$ and $t>0$. The Caputo fractional derivative operator(CFDO) for the function $f\in H^1([a,b],\mathbb{R}_+)$ with order $\alpha\geq 0$ is explained as*

$$D_t^\alpha w(x,t)=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m w}{\partial\tau^m}d\tau,\\ \dfrac{\partial^m w}{\partial t^m}, \qquad \alpha=m\in\mathbb{N}.\end{cases} \tag{2. 2}$$

We have following properties of RLFIO and CFDO

$$D_t^\alpha t^\mu=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{(\mu-\alpha)}, \tag{2. 3}$$

$$I_t^\alpha t^\mu=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}t^{(\mu+\alpha)}, \quad \alpha>0,\ \mu>-1. \tag{2. 4}$$

Note that the relation between RLFIO and CFDO is given by:

$$I_t^\alpha D_t^\alpha w(x,t)=w(x,0)-\sum_{k=0}^{m-1}w^{(k)}(x,0)\frac{t^k}{k!}, \quad m-1<\alpha\leq m. \tag{2. 5}$$

Mittage-Leffler function(MLF) : The MLF for one parameter and two parameter is explained as follows

$$E_\alpha(t)=\sum_{n=0}^\infty\frac{t^n}{\Gamma(\alpha n+1)}, \quad (\alpha\in\mathbf{C}, Re(\alpha)>0),$$

$$E_{\alpha,\beta}(t)=\sum_{n=0}^\infty\frac{t^n}{\Gamma(\alpha n+\beta)}, \quad (\alpha,\beta\in\mathbf{C}, Re(\alpha,\beta)>0).$$

When we apply CFDO on MLF we get

$$D_t^\alpha E_\alpha(at^\alpha)=aE_\alpha(at^\alpha), \tag{2. 6}$$

where a is constant.

3. Analysis of Adomian Decomposition Method

In this section, we proposed improved ADM for solving various physical models governed by different types of nonlinear, nonhomogeneous space and time FPDEs.

- **Type I:Nonlinear, Nonhomogeneous Space and Time FPDEs**

Consider the IVP for nonlinear, nonhomogeneous space and time FPDE of order $0<\alpha\leq 1$

$$D_t^\alpha w(x,t)+D_x^\alpha w(x,t)+f(x,t,w(x,t))=g(x,t), \tag{3. 1}$$

$$w(x,0)=h(x) \tag{3. 2}$$

or equivalently

$$Lw(x,t)=g(x,t)-N(w(x,t)) \tag{3. 3}$$

$$w(x,0)=h(x) \tag{3. 4}$$

where $w(x,t)$ is unrecognized function which we want to determined, t is time variable, x is the space coordinate, $N(w(x,t)) = f(x,t,w(x,t))$ is nonlinear data and $g(x,t)$ is nonhomogeneous function.
Now, applying the RLFIO I_t^α on both side of equation(3. 1) and use the IC (3. 2), we attain:

$$w(x,t) = w(x,0) + I_t^\alpha\big[g(x,t) - D_x^\alpha w(x,t) - N(w(x,t))\big]. \tag{3. 5}$$

The unrecognized function $w(x,t)$ can be expressed as an infinite series of the form

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) \tag{3. 6}$$

and $N(w(x,t))$ can be decayed as

$$N(w(x,t)) = \sum_{n=0}^{\infty} A_n \tag{3. 7}$$

where

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\big[N\sum_{i=0}^{n}\lambda^i w_i\big]_{\lambda=0}, \qquad n = 0,1,2,3,\dots \tag{3. 8}$$

are the nonlinear Adomian polynomials. Suppose that $g(x,t)$ is analytic.Its fractional Taylor expansion series [9, 26, 27] is:

$$g(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h)x^{k\alpha}t^{h\alpha}, \tag{3. 9}$$

where

$$G_{\alpha,\alpha}(k,h) = \frac{1}{\Gamma(k\alpha+1)\Gamma(h\alpha+1)}(D^\alpha)_x^k(D^\alpha)_t^h g(x,t)\big|_{x=t=0}$$

$$\text{and} \quad (D^\alpha)_x^k = D_x^\alpha D_x^\alpha ... D_x^\alpha, \quad k \quad times.$$

By using (3. 6), (3. 7) and (3. 9) in (3. 5) we attain

$$\begin{aligned}\sum_{n=0}^{\infty} w(x,t) = h(x) + I_t^\alpha\Bigg[&\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h)x^{k\alpha}t^{h\alpha} \\ &- D_x^\alpha\sum_{n=0}^{\infty} w_n(x,t) - \sum_{n=0}^{\infty} A_n(x,t)\Bigg],\end{aligned} \tag{3. 10}$$

$$\begin{aligned}\sum_{n=0}^{\infty} w(x,t) = h(x) + I_t^\alpha\Bigg[&\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} \\ &+ \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} + \cdots\Bigg] \\ &- I_t^\alpha\Bigg[D_x^\alpha\sum_{n=0}^{\infty} w_n(x,t) + \sum_{n=0}^{\infty} A_n(x,t)\Bigg].\end{aligned} \tag{3. 11}$$

Taking term by term comparison on both side of equation (3. 11), we set recursion scheme like:

$$
\begin{aligned}
w_0(x,t) &= h(x),\\
w_1(x,t) &= I_t^{\alpha}\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} - D_x^{\alpha}w_0 - A_0\Big],\\
w_2(x,t) &= I_t^{\alpha}\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} - D_x^{\alpha}w_1 - A_1\Big],\\
w_3(x,t) &= I_t^{\alpha}\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,2)x^{k\alpha}t^{2\alpha} - D_x^{\alpha}w_2 - A_2\Big],
\end{aligned}
$$

and so forth. Then the solution $w(x,t)$ of IVP (3. 1) $-$ (3. 2) is

$$\phi_{m+1} = \sum_{n=0}^{m} w_n(x,t) \tag{3. 12}$$

which gives

$$\lim_{m\to\infty} \phi_{m+1} = w(x,t). \tag{3. 13}$$

- **Type II:Nonlinear, Nonhomogeneous Space and Time FPDEs**

We consider the IVP for nonlinear, nonhomogeneous space and time FPDE of order $1 < \alpha \leq 2$

$$D_t^{\alpha}w(x,t) + D_x^{\alpha}w(x,t) + f(x,t,w(x,t)) = g(x,t), \tag{3. 14}$$

$$w(x,0) = h(x) \quad \text{and} \quad w_t(x,0) = 0 \tag{3. 15}$$

or equivalently

$$Lw(x,t) = g(x,t) - N(w(x,t)) \tag{3. 16}$$

$$w(x,0) = h(x) \quad \text{and} \quad w_t(x,0) = 0 \tag{3. 17}$$

where $w(x,t)$ is unrecognized function which we want to determined, t is time variable, x is the space coordinate, $N(w(x,t)) = f(x,t,w(x,t))$ is nonlinear term and $g(x,t)$ is nonhomogeneous function. Suppose that $g(x,t)$ is analytic.
Now, applying the RLFIO I_t^{α} on both side of equation(3. 14) and use ICs (3. 15), we attain:

$$w(x,t) = w(x,0) + w_t(x,0) + I_t^{\alpha}\big[g(x,t) - D_x^{\alpha}w(x,t) - N(w(x,t))\big]. \tag{3. 18}$$

By using (3. 6), (3. 7) and (3. 9) in (3. 18), we attain

$$\sum_{n=0}^{\infty} w(x,t) = h(x) + I_t^{\alpha}\Big[\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h)x^{k\alpha}t^{h\alpha} - D_x^{\alpha}\sum_{n=0}^{\infty} w_n(x,t) - \sum_{n=0}^{\infty} A_n(x,t)\Big], \tag{3. 19}$$

$$\sum_{n=0}^{\infty} w(x,t) = h(x) + I_t^{\alpha}\Bigg[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} + \cdots\Bigg] - I_t^{\alpha}\Bigg[D_x^{\alpha}\sum_{n=0}^{\infty} w_n(x,t) + \sum_{n=0}^{\infty} A_n(x,t)\Bigg]. \tag{3. 20}$$

Taking term by term comparison on both side of equation (3. 19), we set recursion scheme like:

$$\begin{aligned}
w_0(x,t) &= h(x),\\
w_1(x,t) &= I_t^{\alpha}\Bigg[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} - D_x^{\alpha}u_0 - A_0\Bigg],\\
w_2(x,t) &= I_t^{\alpha}\Bigg[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} - D_x^{\alpha}u_1 - A_1\Bigg],\\
w_3(x,t) &= I_t^{\alpha}\Bigg[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,2)x^{k\alpha}t^{2\alpha} - D_x^{\alpha}u_2 - A_2\Bigg],
\end{aligned}$$

and so forth. Then the solution $w(x,t)$ of IVP (3. 14)-(3. 15) is

$$\phi_{m+1} = \sum_{n=0}^{m} w_n(x,t) \tag{3. 21}$$

which gives

$$\lim_{m\to\infty} \phi_{m+1} = w(x,t). \tag{3. 22}$$

4. Application to Some Physical Models

The effectiveness and sharpness of the improved Adomian decomposition method can be demonstrated by applying it to some physical models in space and time fractional nonlinear nonhomogeneous PDEs.

Gas Dynamics Model: Gas dynamics equations includes in mathematic. They are depend on the physical law like law of conservation of mass, momentum,energy etc. The several types of gas dynamics equations in physics have been studied in[10, 31].

Example 4.1. *Consider the gas dynamics equation of order* $0 < \alpha \leq 1$

$$D_t^{\alpha}w(x,t) + wD_x^{\alpha}w - w(1-w) = -E_{\alpha}(-x^{\alpha})E_{\alpha}(t^{\alpha}), \tag{4. 1}$$

with IC

$$w(x,0) = 1 - E_{\alpha}(-x^{\alpha}). \tag{4. 2}$$

Solution: Applying I_t^{α} on both side of equation (4. 1) and use IC (4. 2), we attain

$$w(x,t) = 1 - E_{\alpha}(-x^{\alpha}) - I_t^{\alpha}\big[wD_x^{\alpha}w - w + w^2 + E_{\alpha}(-x^{\alpha})E_{\alpha}(t^{\alpha})\big]. \tag{4. 3}$$

Here

$$g(x,t) = E_{\alpha}(-x^{\alpha})E_{\alpha}(t^{\alpha}) \quad and \quad N(w(x,t)) = wD_x^{\alpha}w + w^2$$

by using (3. 6), (3. 7) and (3. 9) in (4. 3) we have

$$\begin{aligned}\sum_{n=0}^{\infty} w_n(x,t) &= 1 - E_\alpha(-x^\alpha) - I_t^\alpha\Big[\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} w_n + \sum_{n=0}^{\infty} B_n \\ &+ \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h) x^{k\alpha} t^{h\alpha}\Big], \\ &= 1 - E_\alpha(-x^\alpha) - I_t^\alpha\Big[\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} w_n + \sum_{n=0}^{\infty} B_n \\ &+ \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0) x^{k\alpha} t^{0\alpha} + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1) x^{k\alpha} t^{\alpha} + \cdots\Big]. \end{aligned} \tag{4. 4}$$

Here first few coefficients of $G_{\alpha,\alpha}(k,h)$ are given in Table1.

TABLE 1. The components of $G_{\alpha,\alpha}(k,h)$

$G_{\alpha,\alpha}(k,h)$	$G_{\alpha,\alpha}(k,0)$	$G_{\alpha,\alpha}(k,1)$	$G_{\alpha,\alpha}(k,2)$	$\cdots$
$G_{\alpha,\alpha}(0,h)$	1	$\frac{1}{\Gamma(\alpha+1)}$	$\frac{1}{\Gamma(2\alpha+1)}$	$\cdots$
$G_{\alpha,\alpha}(1,h)$	$\frac{-1}{\Gamma(\alpha+1)}$	$\frac{-1}{[\Gamma(\alpha+1)]^2}$	$\frac{-1}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}$	$\cdots$
$G_{\alpha,\alpha}(2,h)$	$\frac{1}{\Gamma(2\alpha+1)}$	$\frac{1}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}$	$\frac{1}{[\Gamma(2\alpha+1)]^2}$	$\cdots$
$G_{\alpha,\alpha}(3,h)$	$\frac{-1}{\Gamma(3\alpha+1)}$	$\frac{-1}{\Gamma(3\alpha+1)\Gamma(\alpha+1)}$	$\frac{-1}{\Gamma(3\alpha+1)\Gamma(2\alpha+1)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Taking term by term comparison on both side of equation(4. 4), we set recursion scheme like :

$$\begin{aligned}
w_0(x,t) &= 1 - E_\alpha(-x^\alpha),\\
w_1(x,t) &= -I_t^\alpha\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} + A_0 - w_0 + B_0\Big],\\
&= -I_t^\alpha\Big[E_\alpha(-x^\alpha) + w_0 D_x^\alpha w_0 - w_0 + w_0^2\Big],\\
&= -E_\alpha(-x^\alpha)\frac{t^\alpha}{\Gamma(\alpha+1)},\\
w_2(x,t) &= -I_t^\alpha\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} + A_1 - w_1 + B_1\Big],\\
&= -I_t^\alpha\Big[E_\alpha(-x^\alpha)\frac{t^\alpha}{\Gamma(\alpha+1)} + w_0 D_x^\alpha w_1 + w_1 D_x^\alpha w_0 - w_1 + 2w_0w_1\Big],\\
&= -E_\alpha(-x^\alpha)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
w_3(x,t) &= -I_t^\alpha\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,2)x^{k\alpha}t^{2\alpha} + A_2 - w_2 + B_2\Big],\\
&= -I_t^\alpha\Big[E_\alpha(-x^\alpha)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + w_2 D_x^\alpha w_0 + w_0 D_x^\alpha w_2\\
&\quad + w_1 D_x^\alpha w_1 - w_2 + 2w_0 D w_2 + w_1^2\Big],\\
&= -E_\alpha(-x^\alpha)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
w_4(x,t) &= -I_t^\alpha\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,3)x^{k\alpha}t^{3\alpha} + A_3 - w_3 + B_3\Big],\\
&= -I_t^\alpha\Big[E_\alpha(-x^\alpha)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + w_3 D_x^\alpha w_0 + w_0 D_x^\alpha w_3\\
&\quad + w_1 D_x^\alpha w_2 + w_2 D_x^\alpha w_1 - w_3 + 2w_0w_3 + 2w_1w_2\Big],\\
&= -E_\alpha(-x^\alpha)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}
\end{aligned}$$

and so on.Then analytical solution of IVP (4. 1)-(4. 2) is

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = 1 - E_\alpha(-x^\alpha)E_\alpha(t^\alpha). \tag{4. 5}$$

If $\alpha = 1$ then IVP (4. 1)-(4. 2) is

$$w_t + ww_x - w(1-w) = -e^{-x+t}, \tag{4. 6}$$

with IC

$$w(x,0) = 1 - e^{-x}. \tag{4. 7}$$

the exact solution of given IVP is

$$w(x,t) = 1 - e^{-x+t}. \tag{4. 8}$$

This result represents the exact solution of the IVP (4. 6)-(4. 7).

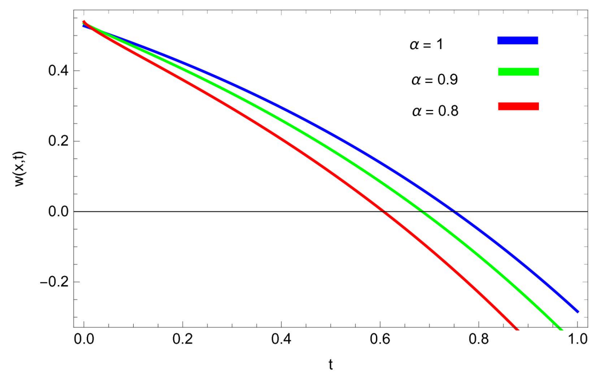


FIGURE 1. 2D Graphical representation of solution (4. 5) of IVP (4. 1)-(4. 2) for different values of α such as $\alpha = 1, 0.9, 0.8$ when $x = 0.50$.

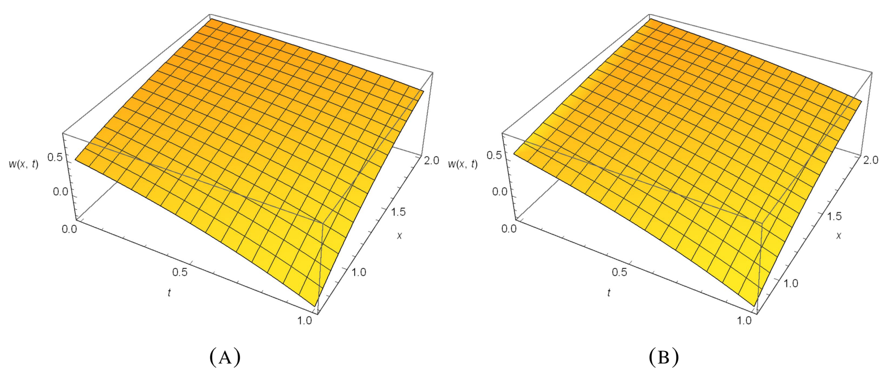
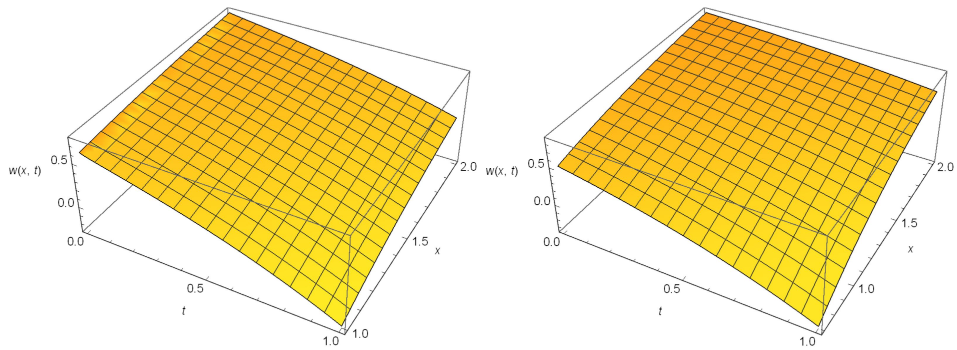


FIGURE 2. 3D Graphical representation of solution (4. 5) of IVP (4. 1)-(4. 2) when $\alpha = 1, 0.9$ with respect to time

Remark 4.1. *: Figure* (1) *is the graphical behaviour of improved ADM solution* (4. 5) *for different values of* α *such as* $\alpha = 1, 0.9, 0.8$ *and exact solution* (4. 8) *when* $x = 0.50$*. Figure* $2a$, $2b$, $3a$, $3b$ *shows the surface of the 4 terms of the improved ADM solution* (4. 5) *for values of* $\alpha = 1, 0.9, 0.8$ *and surface of exact solution* (4. 8)*. It is clear from Figure* (1) *and Figure* (2a) *to*(3b)*, in the limit while* $\alpha \to 1$, (4. 5) *approaches to the exact*



(A) (B)

FIGURE 3. 3D Graphical representation of solution (4. 5) of IVP (4. 1)-(4. 2) when $\alpha = 0.8$ and exact solution (4. 8) with respect to time

solution (4. 8). *Therefore, the improved ADM is an effective and sharp method which can be handled to detect exact analytical solution of fractional-order Gas dynamics equation.*

Advection model: This equation moves towards the direction of a conserved scalar field which is transported by a recognized velocity vector range. It is obtained by applying the scalar field's conservation law, coupled with Gauss's theorem, and have the insignificant limit. A nonlinear advection problem express the Earth's bow shock affiliated with the solar wind as well as traffic flow on a highway.

Example 4.2. *Consider the advection equation of order* $0 < \alpha \leq 1$

$$D_t^{\alpha} w(x,t) + w D_x^{\alpha} w = 1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{4. 9}$$

with IC

$$w(x,0) = \frac{x^{\alpha}}{\Gamma(\alpha+1)}. \tag{4. 10}$$

Solution: Applying I_t^{α} on both side of equation (4. 9) and use initial condition (4. 10), we attain

$$w(x,t) = \frac{x^{\alpha}}{\Gamma(\alpha+1)} + I_t^{\alpha}\left[- w D_x^{\alpha} w + 1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]. \tag{4. 11}$$

Here

$$g(x,t) = 1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \quad \textit{and} \quad N(w(x,t)) = w D_x^{\alpha} w.$$

By using (3. 6), (3. 7) and (3. 9) in (4. 11), we attain

$$\begin{aligned}\sum_{n=0}^{\infty} w_n(x,t) &= \frac{x^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha\Big[\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h)x^{k\alpha}t^{h\alpha} - \sum_{n=0}^{\infty} A_n(x,t)\Big],\\ &= \frac{x^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha}\\ &+ \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} + \cdots\Big] - I_t^\alpha \sum_{n=0}^{\infty} A_n(x,t). \qquad (4.\ 12)\end{aligned}$$

Here first few coefficients of $G_{\alpha,\alpha}(k,h)$ are given in Table 2.

Taking term by term comparison on both side of equation(4. 12), We set recursion scheme

TABLE 2. The components of $G_{\alpha,\alpha}(k,h)$

$G_{\alpha,\alpha}(k,h)$	$G_{\alpha,\alpha}(k,0)$	$G_{\alpha,\alpha}(k,1)$	$G_{\alpha,\alpha}(k,2)$	$\cdots$
$G_{\alpha,\alpha}(0,h)$	1	$\frac{1}{\Gamma(\alpha+1)}$	0	$\cdots$
$G_{\alpha,\alpha}(1,h)$	$\frac{1}{\Gamma(\alpha+1)}$	0	0	$\cdots$
$G_{\alpha,\alpha}(2,h)$	0	0	0	$\cdots$
$G_{\alpha,\alpha}(3,h)$	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

like:

$$\begin{aligned}w_0(x,t) &= \frac{x^\alpha}{\Gamma(\alpha+1)},\\ w_1(x,t) &= I_t^\alpha\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} - A_0\Big],\\ &= I_t^\alpha\Big[1 + \frac{x^\alpha}{\Gamma(\alpha+1)} - w_0 D_x^\alpha w_0\Big] = \frac{t^\alpha}{\Gamma(\alpha+1)},\\ w_2(x,t) &= I_t^\alpha\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} - A_1\Big],\\ &= I_t^\alpha\Big[\frac{t^\alpha}{\Gamma(\alpha+1)} - w_0 D_x^\alpha w_1 - w_1 D_x^\alpha w_0\Big] = 0,\\ w_3(x,t) &= I_t^\alpha\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,2)x^{k\alpha}t^{2\alpha} - A_2\Big],\\ &= I_t^\alpha\Big[0 - w_0 D_x^\alpha w_2 - w_2 D_x^\alpha w_0 - w_1 D_x^\alpha w_1\Big] = 0.\end{aligned}$$

and all remaining terms are zero. The analytical solution of IVP (4. 9)-(4. 10) is

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{t^\alpha}{\Gamma(\alpha+1)}. \qquad (4.\ 13)$$

If $\alpha = 1$ then IVP (4. 9)-(4. 10) is

$$w_t + ww_x = 1 + x + t, \tag{4. 14}$$

with IC

$$w(x, 0) = x. \tag{4. 15}$$

the exact solution of given IVP is

$$w(x, t) = x + t. \tag{4. 16}$$

This result represents the exact solution of the IVP (4. 14)-(4. 15) as presented in [39].

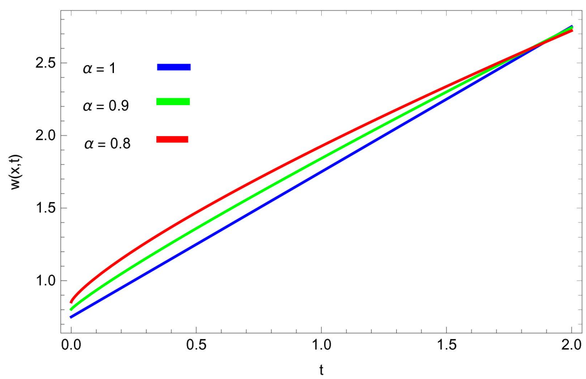


FIGURE 4. 2D Graphical representation of solution (4. 13) of IVP (4. 9)-(4. 10) for different values of α such as $\alpha = 1, 0.9, 0.8$ when $x = 0.50$.

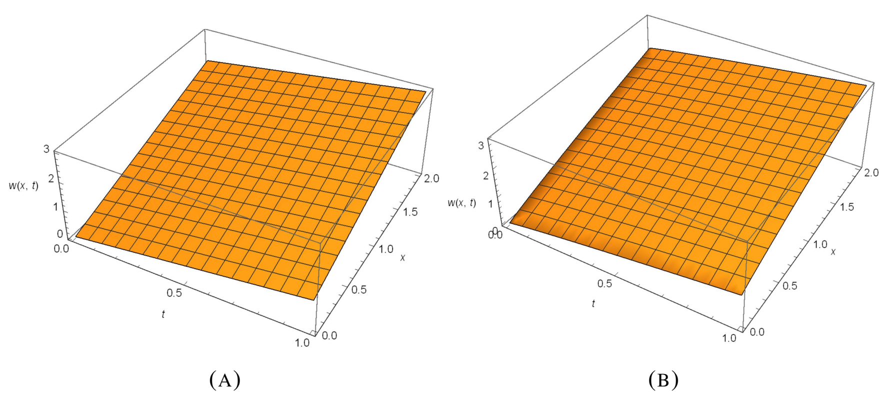
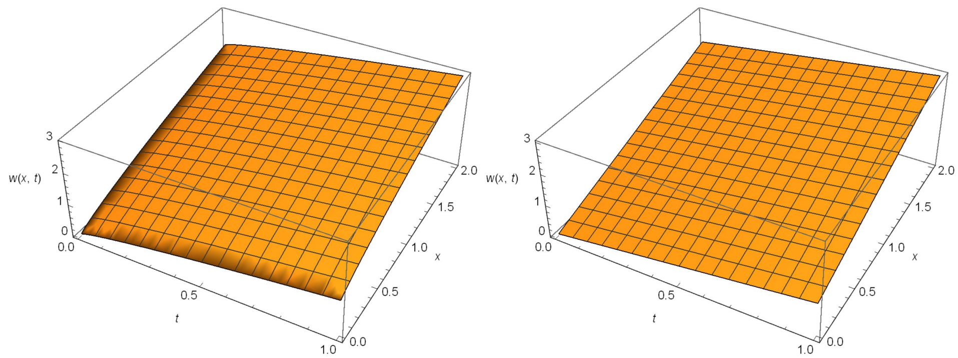


FIGURE 5. 3D Graphical representation of solution (4. 13) of IVP (4. 9)-(4. 10) when $\alpha = 1, 0.9$ with respect to time



(A) (B)

FIGURE 6. 3D Graphical representation of solution (4. 13) of IVP (4. 9)-(4. 10) when $\alpha = 0.8$ and exact solution (4. 16) with respect to time

Remark 4.2. *: Figure* (4) *is the graphical behaviour of improved ADM solution* (4. 13) *for different values of* α *such as* $\alpha = 1, 0.9, 0.8$ *and exact solution* (4. 16) *when* $x = 0.75$. *Figure* $5a$, $5b$, $6a$, $6b$ *shows the surface of the 4 terms of the improved ADM solution* (4. 13) *for values of* $\alpha = 1, 0.9, 0.8$ *and surface of exact solution* (4. 16). *It is clear from Figure* (4) *and Figure* (5a) *to* (6b) *that, when the limit* $\alpha \to 1$, *the solution* (4. 13) *approaches to the exact solution* (4. 16). *Therefore, the improved ADM is an effective and sharp method which can be handled to detect exact analytical solution of the fractional-order Advection equation.*

Now we discuss examples based on type II.

Wave Model: Various second-order PDEs are applied to express wave propagations. These equations are also utilized in geometrical differentiations, in several areas of chemical engineering, hydro technologies and gas dynamics see [40] for details. The solution $w(x, t)$ is the amplitude of wave at position x and time t.

Example 4.3. *Consider the wave equation of order* $1 < \alpha \leq 2$

$$D_t^\alpha w(x,t) = wD_x^\alpha w + 1 - \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{4. 17}$$

with ICs

$$w(x,0) = \frac{x^\alpha}{\Gamma(\alpha+1)} \quad and \quad w_t(x,0) = 0. \tag{4. 18}$$

Solution: Applying I_t^α on both side of equation (4. 17) and use initial condition (4. 18), we attain

$$w(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha\left[wD_x^\alpha w + 1 - \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{t^\alpha}{\Gamma(\alpha+1)}\right]. \tag{4. 19}$$

Here

$$g(x,t) = 1 - \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{t^\alpha}{\Gamma(\alpha+1)}$$

and

$$N(w(x,t)) = wD_x^\alpha w.$$

By using (3. 6), (3. 7) and (3. 9) in (4. 19), we attain

$$\begin{aligned}\sum_{n=0}^{\infty} w_n(x,t) &= \frac{x^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha\Bigg[\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h)x^{k\alpha}t^{h\alpha} + \sum_{n=0}^{\infty} A_n(x,t)\Bigg],\\ &= \frac{x^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha\Bigg[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha}\\ &+ \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,2)x + \cdots\Bigg] + I_t^\alpha \sum_{n=0}^{\infty} A_n(x,t).\end{aligned} \tag{4. 20}$$

Here first few coefficients of $G_{\alpha,\alpha}(k,h)$ are given in Table 3.

Taking term by term comparison on both side of equation (4. 20), We set recursion scheme

TABLE 3. The components of $G_{\alpha,\alpha}(k,h)$

$G_{\alpha,\alpha}(k,h)$	$G_{\alpha,\alpha}(k,0)$	$G_{\alpha,\alpha}(k,1)$	$G_{\alpha,\alpha}(k,2)$	$\cdots$
$G_{\alpha,\alpha}(0,h)$	1	$\frac{-1}{\Gamma(\alpha+1)}$	0	$\cdots$
$G_{\alpha,\alpha}(1,h)$	$\frac{-1}{\Gamma(\alpha+1)}$	0	0	$\cdots$
$G_{\alpha,\alpha}(2,h)$	0	0	0	$\cdots$
$G_{\alpha,\alpha}(3,h)$	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

like:

$$\begin{aligned}w_0(x,t) &= \frac{x^\alpha}{\Gamma(\alpha+1)},\\ w_1(x,t) &= I_t^\alpha\Bigg[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} + A_0\Bigg],\\ &= I_t^\alpha\Bigg[1 - \frac{x^\alpha}{\Gamma(\alpha+1)} + w_0D_x^\alpha w_0\Bigg] = \frac{t^\alpha}{\Gamma(\alpha+1)},\\ w_2(x,t) &= I_t^\alpha\Bigg[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} + A_1\Bigg],\\ &= I_t^\alpha\Bigg[-\frac{t^\alpha}{\Gamma(\alpha+1)} + w_0D_x^\alpha w_1 + w_1D_x^\alpha w_0\Bigg] = 0,\\ w_3(x,t) &= I_t^\alpha\Bigg[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,2)x^{k\alpha}t^{2\alpha} + A_2\Bigg],\\ &= I_t^\alpha\Bigg[0 + w_0D_x^\alpha w_2 + w_2D_x^\alpha w_0 + w_1D_x^\alpha w_1\Bigg] = 0.\end{aligned}$$

and all remaining terms are zero.Then analytical solution of IVP (4. 17)-(4. 18) is

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}. \tag{4. 21}$$

If $\alpha = 2$ then IVP (4. 17)-(4. 18) is

$$w_{tt} = w w_{xx} + 1 - \frac{x^2+t^2}{2}, \tag{4. 22}$$

with ICs

$$w(x,0) = \frac{x^2}{2} \quad and \quad w_t(x,0) = 0 \tag{4. 23}$$

the exact solution of given IVP is

$$w(x,t) = \frac{x^2+t^2}{2} \tag{4. 24}$$

This result represents the exact solution of the IVP (4. 22)-(4. 23) as presented in [39].

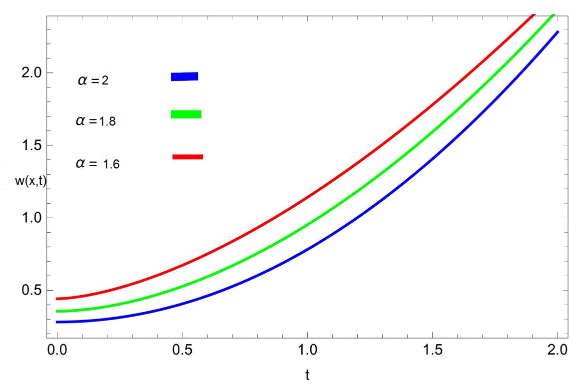
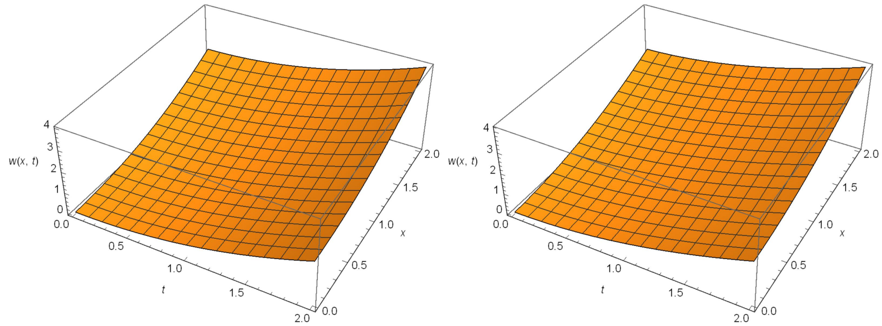


FIGURE 7. 2D Graphical representation of solution (4. 21) of IVP (4. 17)-(4. 18) for different values of α such as $\alpha = 2, 1.8, 1.6$ when $x = 0.75$.

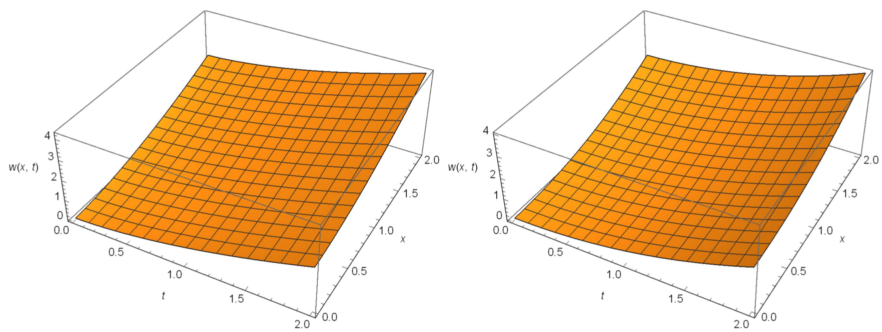
Remark 4.3. *: Figure* (7) *is the graphical behaviour of improved ADM solution* (4. 21) *for different values of* α *such as* $\alpha = 2, 1.8, 1.6$ *and exact solution* (4. 24) *when* $x = 0.75$. *Figure* (8a), (8b), (9a), (9b) *shows the surface of the 4 terms of the improved ADM solution* (4. 21) *for values of* $\alpha = 2, 1.8, 1.6$ *and surface of exact solution* (4. 24). *It is clear from Figure* (7) *and Figures* (8a) *to* (9b)*, in the limit while* $\alpha \to 2$*, the solution* (4. 21) *approaches to the exact solution* (4. 24)*. Therefore, the improved ADM is an effective and sharp method which can be handled to detect exact analytical solution of the fractional-order wave equation.*

- **Klein-Gordon Model**:



(A) (B)

FIGURE 8. 3D Graphical representation of solution (4. 21) of IVP (4. 17)-(4. 18) when $\alpha = 2, 1.8$ with respect to time



(A) (B)

FIGURE 9. 3D Graphical representation of solution (4. 21) of IVP (4. 17)-(4. 18) when $\alpha = 1.6$ and exact solution (4. 24) with respect to time

Quantum field theory includes various mathematical models and the Klein-Gordon model [29] is one of the most popular model. In general, this equation occurs in relativistic physics which is applied to express dispersive wave phenomena. It also occurs in nonlinear optics and plasma physics. It is well known that wave models are occurs in various wave aspects like radio waves, beam waves, water waves etc.

Example 4.4. *Consider the Klein-Gordon equation of order* $1 < \alpha \leq 2$

$$D_t^{\alpha} w(x,t) = D_x^{\alpha} w + w^2 + \frac{x^{\alpha} - t^{\alpha}}{\Gamma(\alpha+1)} - \frac{(xt)^{2\alpha}}{[\Gamma(\alpha+1)]^4}, \tag{4. 25}$$

with ICs

$$w(x,0) = 0 \quad and \quad w_t(x,0) = 0. \tag{4. 26}$$

Solution: Applying I_t^α on both side of equation (4. 25) and use ICs (4. 26), we attain

$$w(x,t) = I_t^\alpha\Big[D_x^\alpha w + w^2 + \frac{x^\alpha - t^\alpha}{\Gamma(\alpha+1)} - \frac{(xt)^{2\alpha}}{[\Gamma(\alpha+1)])^4}\Big]. \tag{4. 27}$$

Here

$$g(x,t) = \frac{x^\alpha - t^\alpha}{\Gamma(\alpha+1)} - \frac{(xt)^{2\alpha}}{[\Gamma(\alpha+1)])^4} \quad and \quad N(w(x,t)) = w^2.$$

By using (3. 6), (3. 7) and (3. 9) in (4. 27), we attain

$$\begin{aligned}\sum_{n=0}^{\infty} w_n(x,t) &= 0 + I_t^\alpha\Big[D_x^\alpha \sum_{n=0}^{\infty} w_n(x,t) + \sum_{n=0}^{\infty} A_n(x,t) + \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h)x^{k\alpha}t^{h\alpha}\Big],\\ &= I_t^\alpha\Big[D_x^\alpha \sum_{n=0}^{\infty} w_n(x,t) + \sum_{n=0}^{\infty} A_n(x,t) + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} + \cdots\Big].\end{aligned} \tag{4. 28}$$

Here first few coefficients of $G_{\alpha,\alpha}(k,h)$ are given in Table (4):
Taking term by term comparison on both side of equation (4. 28), We set recursion scheme

TABLE 4. The components of $G_{\alpha,\alpha}(k,h)$

$G_{\alpha,\alpha}(k,h)$	$G_{\alpha,\alpha}(k,0)$	$G_{\alpha,\alpha}(k,1)$	$G_{\alpha,\alpha}(k,2)$	$\cdots$
$G_{\alpha,\alpha}(0,h)$	0	$\frac{-1}{\Gamma(\alpha+1)}$	0	$\cdots$
$G_{\alpha,\alpha}(1,h)$	$\frac{1}{\Gamma(\alpha+1)}$	0	0	$\cdots$
$G_{\alpha,\alpha}(2,h)$	0	0	$\frac{-1}{[\Gamma(\alpha+1)]^4}$	$\cdots$
$G_{\alpha,\alpha}(3,h)$	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

like:

$$\begin{aligned} w_0(x,t) &= 0,\\ w_1(x,t) &= I_t^\alpha\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} + D_x^\alpha w_0 + A_0\Big],\\ &= I_t^\alpha\Big[\frac{x^\alpha}{\Gamma(\alpha+1)} + 0 + w_0^2\Big] = \frac{x^\alpha t^\alpha}{[\Gamma(\alpha+1)]^2},\end{aligned}$$

$$w_2(x,t) = I_t^{\alpha}\Bigg[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} + D_x^{\alpha}w_1 + A_1\Bigg],$$
$$= I_t^{\alpha}\Bigg[-\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2w_0w_1\Bigg] = 0,$$
$$w_3(x,t) = I_t^{\alpha}\Bigg[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,2)x^{k\alpha}t^{2\alpha} + D_x^{\alpha}w_2 + A_2\Bigg],$$
$$= I_t^{\alpha}\Bigg[-\frac{x^{2\alpha}t^{2\alpha}}{[\Gamma(\alpha+1)]^4} + 0 + 2w_0w_2 + w_1^2\Bigg] = 0.$$

Note that all remaining terms are zero.Then analytical solution of IVP (4. 25)-(4. 26) is

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = \frac{x^{\alpha}t^{\alpha}}{[\Gamma(\alpha+1)]^2}. \tag{4. 29}$$

If $\alpha = 2$ then IVP (4. 25)-(4. 26) is

$$w_{tt} = w_{xx} + w^2 + \frac{x^2-t^2}{2} - \frac{(xt)^4}{16}, \tag{4. 30}$$

with ICs

$$w(x,0) = 0 \quad and \quad w_t(x,0) = 0 \tag{4. 31}$$

the exact solution of given IVP is

$$w(x,t) = \frac{x^2t^2}{4} \tag{4. 32}$$

This result represents the exact solution of the IVP (4. 30)-(4. 31) as presented in [39].

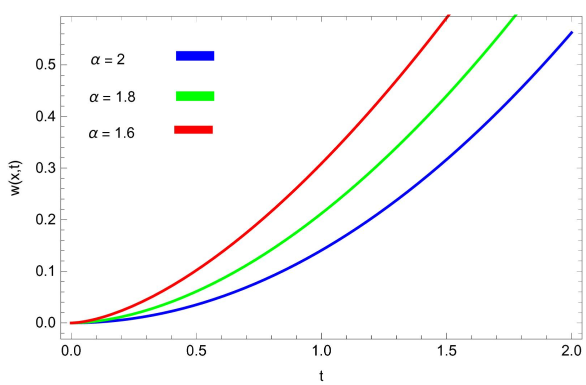
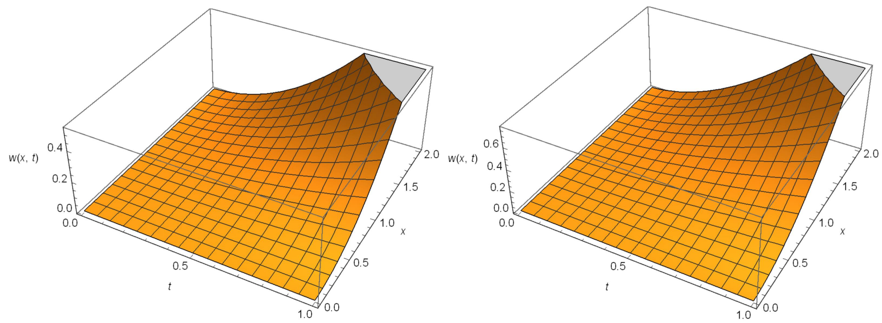


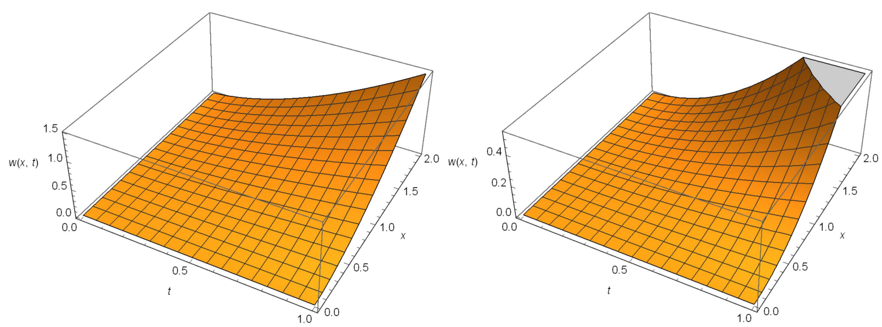
FIGURE 10. 2D Graphical representation of solution (4. 29) of IVP (4. 25)-(4. 26) for different values of α such as $\alpha = 2, 1.8, 1.6$ when $x = 0.75$.

Remark 4.4. *: Figure* (10) *is the graphical behaviour of improved ADM solution* (4. 29) *for different values of* α *such as* $\alpha = 2, 1.8, 1.6$ *and exact solution* (4. 24) *when* $x = 0.75$. *Figure* (11a), (11b), (12a), (12b) *shows the surface of the 4 terms of the improved ADM*



(A) (B)

FIGURE 11. 3D Graphical representation of solution (4. 29) of IVP (4. 25)-(4. 26) when $\alpha = 2, 1.8$ with respect to time



(A) (B)

FIGURE 12. 3D Graphical representation of solution (4. 29) of IVP (4. 25)-(4. 26) when $\alpha = 1.6$ and exact solution (4. 32) with respect to time

solution (4. 21) *for values of* $\alpha = 2, 1.8, 1.6$ *and surface of exact solution* (4. 32). *It is clear from Figure* (10) *and Figures* (11a) *to* (12b) *that, when the limit* $\alpha \to 2$*, the solution* (4. 29) *approaches to the exact solution* (4. 32). *Therefore,the improved ADM is an effective and sharp method which can be handled to detect exact analytical solution of the space and time fractional-order Klein-Gordon equation.*

5. CONCLUSION

The various space-time fractional models such as Gas dynamics model, Advection model, Wave model, and Klein-Gordon model are studied successfully by virtue of improved Adomian decomposition method developed for nonlinear nonhomogeneous space and time fractional PDEs by using fractional Taylor expansion series to nonhomogeneous functions.

The solution of these models are in series form may have rapid convergence to a closed-form solution. It is a more convenient way to solve such types physical models with the help of improved ADM than ADM.

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Authors Contributions:The first author gave the idea of main result. Numerical calculations, data explanations and graphical representation are done by second author. Both the authors gave equal contribution in the reading, writing of this paper and approved final manuscript.

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