Punjab University Journal of Mathematics (2023),55(11-12),471-482
https://doi.org/10.52280/pujm.2023.55(11-12))05

New Exact Solutions of a Second Grade MHD Flow Through Porous Media Using Travelling Wave Method

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Received: 01 April, 2023 / Accepted: 26 January, 2024 / Published online: 29 February, 2024

Abstract. MHD flow equations are non-linear partial differential equations. In this essay,we have considered the flow of an incompressible fluid through a porous medium and have employed a strategy based on the traveling wave solution. By utilizing wave parameters, the non-linear equations are transformed into ordinary differential equations. In this approach, all the fluid parameters, initially functions of (x, y, t), are now represented by a single variable 'z' through the transformation z = bx + ay + ct. Finally, expressions for the pressure function, magnetic field, vorticity, velocity components, and other parameters are derived for two different cases, and comparisons are drawn with previously known results.

AMS (MOS) Subject Classification Codes: 76W05; 76B47; 76D05; 76S05; 35G50
Key Words: Travelling wave, porous media, Second grade fluid, vorticity, Aligned flow, Heat transfer

1. INTRODUCTION

Finding an exact solution to the partial differential equations of fluid dynamics is crucial in nonlinear physics [21]. Exact solutions are not only important as solutions for fundamental flows, but they also serve as accuracy checks for exponential, numerical, and asymmetric methods. The numerous applications of non-Newtonian fluids in industrial processes such as nuclear reactors, blood flow, food processing, exotic lubricant suspension solutions, etc., make understanding their behavior [18], [2] significantly important. Due to their nonlinear nature, the Navier-Stokes equations are extremely challenging to solve. Only a small number of researchers have been able to transform these nonlinear equations into ordinary differential equations using various well-known approaches, including similarity transformation [28],hodograph transformation [27],magnetograph transformation [12],traveling wave solution [15], lapalce transform [4] and the Martine's method [6]. In order to verify numerical solutions and aid in the stability analysis of solutions, it is crucial to find the exact solution, if one exists, for these nonlinear equations. Higher-order equations of motion exist in non-Newtonian second-order fluids than in the Navier-Stokes equations [30]. The second-grade fluid has already been investigated and is still being studied by researchers worldwide. Various techniques have been employed to solve the second-grade fluid, including the inverse method [17], by assuming certain conditions on the stream function [5], perturbation method [7],the Lie group approach [31] and others.

One of the key methods for addressing the issue is the inverse technique [22]. However, despite this, the results are not general answers to the original equations because they were reached by assuming a particular shape for the vorticity or stream function. The perturbation approach is a well-known method in science and engineering that aids in our understanding of numerous non-linear issues. However, the Lie group technique [8] requires extremely time-consuming calculations, and algebraic programming frameworks find it challenging to incorporate arbitrary functions when they appear in Lie algebra.

One of the crucial method in addressing non-linear issues is the traveling wave solution approach [11]. Many fields, including plasma physics, mathematical biology, chemical kinetics, fiber optics, etc., use traveling waves. Specific example of findings that Khan et al. [16] achieved after studying micro-polar fluid and using the traveling wave technique to solve the issue is provided by Shahzad et al. [26]. J. E. Dunn and R. L. Fosdick [9] as well as Kashif Ali Abro [1],J. E. Dunn and K. R. Rajagopal [10] conducted a thermodynamic analysis of second-grade fluid. They utilized the concept of extremum of stored energy to identify restrictions on the response function for stress and stored energy in incompressible fluids of differential type. In recent years, a number of other researchers, including M. Aldhabani [3], have used the traveling wave method to determine precise solutions to the equations governing the motion of MHD and the flow of heat through porous media.

Umer Rehman et al. [24] described a mathematical formulation on the coating of a thin film for a compressible isothermal MHD viscous-plastic fluid flowing across a narrow gap between two rotating rolls. They obtained a relation that explains the relationship between MHD wave and instability through analytical calculations. Saeed Ur Rehman and Jose Luis Diaz Palecia [23] modeled fluid flow with a p-Laplacian operator. They provided an analytical assessment of weak solution together with a numerical validation analysis for a 1-D fluid in MHD flowing in porous media.

In another research, Jose Luis Diaz Palecia et al. [19] conducted a study to provide an analysis of the solution to a 1-D Eyring-Powell fluid in MHD with general initial conditions.Mohammad Ishaq et al. [13] investigated two dimensional nanofluid film flow of Eyring Powell Fluid with variable heat transmission in the existence of uniform magnetic field (MHD) on an unsteady porous stretching sheet. They studied The influence of the unsteady parameter (A) over thin film analytically for different values. Extending this work, again, Jose Luis Diaz Palecia et al. [20] applied the traveling wave solution to a fluid flowing in porous media with non-linear diffusion, in which the viscosity term is formulated with an Eyring-Powell law, together with non-homogeneous diffusion. Recently, Najeeb et al. [15] applied the traveling wave solution method and studied heat transfer in a second-grade incompressible fluid with aligned flow, by converting the non-linear partial differential equation into a solvable ordinary differential equation.

For the present paper, the adopted method is as follows. In Section 4, we present the governing equations of the problem. In Section 5, the traveling wave solution is introduced, and the coupled partial differential equation is converted into an ordinary partial differential equation. In Section 6, solutions for the various fluid variables are obtained under two conditions by solving the derived differential equation.

2. NOMENCLATURE

$\vec{f}$ = Body force per unit stress,
μ =Constant viscosity,
ρ =Density,
σ = Electrical conductivity,
$\vec{H}$ = Magnetic field intensity,
H_1 and H_2 = Magnetic field components,

μ^* =Magnetic permeability,
P =Pressure,
ϕ^* = Porosity,
k = Permeability of the medium,
α_1 and α_2 the constant normal stress moduli,
$\vec{A}_1$ and $\vec{A}_2$ the 1st two Rivlin-Ericson tensor
$\vec{V}$ = Velocity vector,
u and v = Velocity components.

3. FLOW DEVELOPMENT

The fundamental equations that control the aligned motion of an unsteady, plane, incompressible, electrically conducting second-grade fluid through porous media, when a magnetic field is present, are as follows [25]:

$$\frac{\partial \rho}{\partial t} + div\left(\rho \vec{V}\right) = 0, \tag{3. 1}$$

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right] = \vec{\nabla}\cdot\left(\vec{A}\right) + \rho\vec{f} + \mu^*\left(curl\vec{H}\right)\times\vec{H} - \frac{\phi^*}{k}\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\vec{V}, \tag{3. 2}$$

$$\frac{\partial H}{\partial t} = curl\left(\vec{V}\times\vec{H}\right) - \frac{1}{\mu^*\sigma}\left(curl\vec{H}\right)\times\vec{H}, \tag{3. 3}$$

$$div\vec{H} = 0. \tag{3. 4}$$

The stress constitutive equation is, [29]

$$\vec{A} = -PI + \mu\vec{A}_1 + \alpha_1\vec{A}_2 + \alpha_2\vec{A}_1^2, \tag{3. 5}$$

where,

$$\vec{A}_1 = \left(\vec{\nabla}\vec{V}\right) + \left(\vec{\nabla}\vec{V}\right)^T, \tag{3. 6}$$

$$\vec{A}_2 = \frac{\partial\vec{A}_1}{\partial t} + \left(\vec{\nabla}\vec{A}_1\right)\vec{V} + \left(\vec{\nabla}\vec{V}\right)^T + \vec{A}_1\left(\vec{\nabla}\vec{V}\right). \tag{3. 7}$$

Dunn and Fosdick [26] discovered that the material constant must satisfy $\mu \geq 0,\ \ \alpha_1 \geq 0,\ \alpha_1 + \alpha_2 = 0.$, in order for the second grade fluid to be compatible with thermodynamics and to meet the requirement that its Helmholtz free energy be a minimum. According to Fosdick and Rajugopal [9], the second-grade fluid exhibits anamolous behaviour for $\alpha_1 \leq 0$. Hence, it is reasonable to suppose $\mu \geq 0$ and $\alpha_1 \geq 0$.

Here we shall consider the two dimensional MHD flow in which the body force is minimal or negligible. So we must have $\vec{V} = u\hat{i} + v\hat{j}$, [where $u = u(x,y,t)$ and $v = v(x,y,t)$] ,
$\vec{H} = H_1\hat{i} + H_2\hat{j}$, [where $H_1 = H_1(x,\ y,\ t)$ and $H_2 = H_2(x,\ y,\ t)$] , $\vec{f} = 0$ and P = P (x, y, t).

In view of the above, system of equations governing the flow is replaced by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3. 8}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{-\partial P}{\rho\,\partial x}+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\alpha\begin{bmatrix}\frac{\partial^3 u}{\partial t\partial^2 x}+\frac{\partial^3 u}{\partial t\partial^2 y}+u\frac{\partial^3 u}{\partial^3 x}+13\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial^2 x}+\\ v\frac{\partial^3 u}{\partial y\partial^2 x}+4\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial^2 x}+2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y\partial x}+3\frac{\partial^2 v}{\partial^2 x}\frac{\partial u}{\partial y}\\ +4\frac{\partial^3 u}{\partial^2 y\partial x}+3\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x}+v\frac{\partial^3 u}{\partial^3 y}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial^2 y}\end{bmatrix}+$$

$$\beta(8\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}+2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x}+2\frac{\partial^2 v}{\partial x^2}\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y\partial x})-\eta H_2\left(\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}\right)-\frac{\phi^*}{k\rho}\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)u, \tag{3. 9}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial y}=\frac{-\partial P}{\rho\,\partial y}+\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\alpha\begin{bmatrix}\frac{\partial^3 v}{\partial t\partial^2 x}+\frac{\partial^3 v}{\partial t\partial^2 y}+u\frac{\partial^3 v}{\partial^3 x}-\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial^2 x}-4\frac{\partial^3 u}{\partial y\partial^2 x}\\ +13\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y\partial x}-v\frac{\partial^3 u}{\partial^3 x}-3\frac{\partial^2 u}{\partial x^2}\frac{\partial v}{\partial x}+3\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial y^2}\\ -v\frac{\partial^3 u}{\partial^2 y\partial x}-2\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y}+4\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\end{bmatrix}+$$

$$\beta(8\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}-2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2}-2\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y})-\eta H_1\left(\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}\right)-\frac{\phi^*}{k\rho}\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)v, \tag{3. 10}$$

$$\frac{\partial^2 H_2}{\partial t\partial x}-\frac{\partial^2 H_1}{\partial t\partial y}=\frac{1}{\mu^*\sigma}\left[\frac{\partial^3 H_2}{\partial x^3}+\frac{\partial^3 H_2}{\partial^2 y\partial x}-\frac{\partial^3 H_1}{\partial^2 x\partial y}-\frac{\partial^3 H_1}{\partial y^3}\right]+v\frac{\partial^2 H_1}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}H_1+\frac{\partial^2 v}{\partial y^2}H_1- \\ v\frac{\partial^2 H_1}{\partial y^2}-u\frac{\partial^2 H_2}{\partial x^2}-\frac{\partial^2 u}{\partial x^2}H_2-u\frac{\partial^2 H_2}{\partial y^2}-\frac{\partial^2 u}{\partial y^2}H_2, \tag{3. 11}$$

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0, \tag{3. 12}$$

where,

$$\upsilon=\frac{\mu}{\rho},\ \eta=\frac{\mu^*}{\rho},\ \alpha=\frac{\alpha_1}{\rho},\ \beta=\frac{\alpha_2}{\rho},\ \phi=\frac{\phi^*}{k}. \tag{3. 13}$$

4. SUMMARY OF THE METHOD

The approach being considered can be summarised up as follows:
Regarding the specified coupled partial differential equation system

$$C\left(\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y}\right)=0, \tag{4. 14}$$

$$L_1\left(u,\ v,\ \frac{\partial P}{\partial x},\ \frac{\partial u}{\partial t},\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial^2 u}{\partial^2 x}.......\frac{\partial v}{\partial t},\ \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial^2 v}{\partial x^2},.......H_1,\ \frac{\partial H_1}{\partial y},\ \frac{\partial H_2}{\partial x}\right)=0, \tag{4. 15}$$

$$L_2\left(u,\ v,\ \frac{\partial P}{\partial y},\ \frac{\partial u}{\partial t},\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial^2 u}{\partial x^2}.......\frac{\partial v}{\partial t},\ \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial^2 v}{\partial x^2},.......H_1,\ \frac{\partial H_1}{\partial y},\ \frac{\partial H_2}{\partial x}\right)=0, \quad (4.\ 16)$$

$$D\left(u,\ v,\ \frac{\partial^2 u}{\partial x^2},\ \frac{\partial^2 u}{\partial y^2},.........\frac{\partial^2 v}{\partial x^2},\ \frac{\partial^2 v}{\partial y^2},.........H_1,\ H_2.........\frac{\partial H_1}{\partial x},........\frac{\partial H_2}{\partial x},.......\right)=0, \quad (4.\ 17)$$

$$S\left(\frac{\partial H_1}{\partial x},\ \frac{\partial H_2}{\partial y}\right)=0, \quad (4.\ 18)$$

We seek the following travelling wave solution,

$$u(x,y,t)=u(z),\ v(x,y,t)=v(z),\ P(x,y,t)=P(z),\ H_1(x,y,t)=L(z),\ H_2(x,y,t)=J(z), \quad (4.\ 19)$$

where, $z=a_1x+a_2y+a_3t$, then system (4.14) - (4.18) can be reduced to a system of ordinary differential equation

$$C\left(\frac{\partial u}{\partial z},\ \frac{\partial v}{\partial z}\right)=0, \quad (4.\ 20)$$

$$L_1\left(u,\ v,\ \frac{\partial P}{\partial z},\ \frac{\partial u}{\partial z},\ \frac{\partial^2 u}{\partial z^2},\ \frac{\partial^3 u}{\partial z^3},.........\frac{\partial v}{\partial z},\ \frac{\partial^2 v}{\partial z^2},\ \frac{\partial^3 v}{\partial z^3}......L,\ \frac{\partial L}{\partial z}.........\frac{\partial J}{\partial z}\right)=0, \quad (4.\ 21)$$

$$L_2\left(u,\ v,\ \frac{\partial P}{\partial z},\ \frac{\partial u}{\partial z},\ \frac{\partial^2 u}{\partial z^2},\ \frac{\partial^3 u}{\partial z^3},.........\frac{\partial v}{\partial z},\ \frac{\partial^2 v}{\partial z^2},\ \frac{\partial^3 v}{\partial z^3}......L,\ \frac{\partial L}{\partial z}.........\frac{\partial J}{\partial z}\right)=0, \quad (4.\ 22)$$

$$D=\left(u,\ v,\ \frac{\partial^2 u}{\partial z^2},\ \frac{\partial^2 v}{\partial z^2},\ L, J, \frac{\partial L}{\partial z},.......\frac{\partial J}{\partial z}\right)=0, \quad (4.\ 23)$$

$$S\left(\frac{\partial L}{\partial z},\ \frac{\partial J}{\partial z}\right)=0, \quad (4.\ 24)$$

Where a_1, a_2, a_3 are constants. We use the equation of continuity (4.20) to remove P from equations (4.21) and (4.22) in order to determine the solution of the system of ordinary differential equations. The resultant equation is simple to integrate for the variables u and v. The values of u and v can be used to calculate the other variable.

5. SOLUTION

On substituting the representation of the solution into (3.9)-(3.13), we get

$$a_1\frac{\partial u}{\partial z}+a_2\frac{\partial v}{\partial z}=0, \quad (5.\ 25)$$

$$(a_3 + a_1 u + a_2 v)\frac{\partial u}{\partial z} = \frac{-a_1}{\rho}\frac{\partial P}{\partial z} + \nu\left({a_2}^2 + {a_1}^2\right)\frac{\partial^2 u}{\partial z^2} + \alpha\left[\begin{array}{l} a_3\left({a_2}^2 + {a_1}^2\right)\frac{\partial^3 u}{\partial z^3} + {a_1}^3 u\frac{\partial^3 u}{\partial z^3} + 13{a_1}^3\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} + \\ a_2{a_1}^2 v\frac{\partial^2 u}{\partial z^2} + 4{a_1}^3\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2} + 2a_2{a_1}^2\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial z^2} + \\ 3a_2{a_1}^2\frac{\partial^2 v}{\partial z^2}\frac{\partial u}{\partial z} + {a_2}^2 a_1 u\frac{\partial^2 u}{\partial z^2} + {a_2}^3 v\frac{\partial^2 u}{\partial z^2} + \\ 4{a_2}^2 a_1\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} \end{array}\right]$$

$$\begin{aligned} &+ \beta\left[8{a_1}^3\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} + 2{a_1}^3\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2} + 2{a_2}^2 a_1\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} + 2a_2{a_1}^2\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial z^2} + 2{a_1}^2\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial z^2}\right] \\ &- \eta J\left(a_1\frac{\partial J}{\partial z} - a_2\frac{\partial L}{\partial z}\right) - \phi\upsilon u - \phi\alpha a_3\frac{\partial u}{\partial z}, \end{aligned} \tag{5. 26}$$

$$(a_3 + a_1 u + a_2 v)\frac{\partial v}{\partial z} = \frac{-a_1}{\rho}\frac{\partial P}{\partial z} + \nu\left({a_2}^2 + {a_1}^2\right)\frac{\partial^2 v}{\partial z^2} + \alpha\left[\begin{array}{l} a_3\left({a_2}^2 + {a_1}^2\right)\frac{\partial^3 u}{\partial z^3} + {a_1}^3 u\frac{\partial^3 v}{\partial z^3} - {a_1}^3\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial z^2} - \\ a_2{a_1}^2 u\frac{\partial^2 u}{\partial z^2} + 11{a_1}^3\frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial z^2} - {a_1}^3 v\frac{\partial^2 u}{\partial z^2} - \\ 3{a_1}^3\frac{\partial^2 u}{\partial z^2}\frac{\partial v}{\partial z} + 3{a_2}^2 a_1\frac{\partial^2 u}{\partial z^2}\frac{\partial v}{\partial z} - {a_2}^3 a_1 v\frac{\partial^2 u}{\partial z^2} + \\ 4{a_2}^2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} \end{array}\right]$$

$$\begin{aligned} &+ \beta\left[8a_2{a_1}^3\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} - 2{a_1}^3\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial z^2} + 2{a_2}^2 a_1\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} + 4a_2 a_1\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial z^2} - 2a_2{a_1}^2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right] \\ &- \eta J\left(a_1\frac{\partial J}{\partial z} - a_2\frac{\partial L}{\partial z}\right) - \phi\upsilon v - \phi\alpha a_3\frac{\partial v}{\partial z}, \end{aligned} \tag{5. 27}$$

$$a_3\left(a_1\frac{\partial^2 J}{\partial z^2} - a_2\frac{\partial^2 L}{\partial z^2}\right) = \left({a_2}^2 + {a_1}^2\right)\left[\frac{1}{\mu^*\sigma}\left(a_1\frac{\partial J}{\partial z} - a_2\frac{\partial^2 L}{\partial z^2}\right) + v\frac{\partial^2 L}{\partial z^2} + L\frac{\partial^2 v}{\partial z^2} - u\frac{\partial^2 J}{\partial z^2} - J\frac{\partial^2 u}{\partial z^2}\right], \tag{5. 28}$$

$$a_1\frac{\partial L}{\partial z} + a_2\frac{\partial J}{\partial z} = 0, \tag{5. 29}$$

On Integrating the equation (5.25) and (5.29), we get

$$a_1 u + a_2 v = m_1, \tag{5. 30}$$

$$a_1 L + a_2 J = m_2. \tag{5. 31}$$

From equation (5.26) and (5.27), eliminating pressure and making use of equation (5.30) and (5.31), we get

$$\alpha\left({a_2}^2+{a_1}^2\right)(a_3+m_1)\frac{\partial^3 u}{\partial z^3}+\upsilon\left({a_2}^2+{a_1}^2\right)\frac{\partial^2 u}{\partial z^2}-(a_3+m_1)\frac{\partial u}{\partial z}-\phi\upsilon\left({a_2}^2+{a_1}^2\right)\frac{u}{a_2}+\phi\upsilon\frac{a_1m_1}{a_2}$$
$$-\frac{\phi\alpha a_3}{a_2}\left({a_2}^2+{a_1}^2\right)\frac{\partial u}{\partial z}=0. \tag{5. 32}$$

Now there are two possible cases
Case I, $a_3=-m_1$, **Case II**, $a_3\neq -m_1$

Let's discuss the two cases one by one

Case I, $a_3=-m_1$, In this case equation (32) becomes,

$$\frac{\partial^2 u}{\partial z^2}-\frac{\phi\alpha a_3}{a_2\upsilon}\frac{\partial u}{\partial z}-\frac{\phi}{a}u=\frac{\phi a_1m_1}{a_2\left({a_2}^2+{a_1}^2\right)}, \tag{5. 33}$$

whose solution is given as

$$u=m_3e^{k_1z}+m_4e^{k_2z}-\frac{a_1m_1}{\left({a_2}^2+{a_1}^2\right)}, \tag{5. 34}$$

here,

$$k_1=\frac{A+\sqrt{A^2+4B^2}}{2},\;\; k_2=\frac{A-\sqrt{A^2+4B^2}}{2},\;\; A=\frac{\phi\alpha a_3}{a_2\upsilon},\;\; B=\frac{\phi}{a_2}$$

.

And, $v=\frac{m_1}{a_2}-\frac{a_1}{a_2}u$,

$$v=\frac{m_1}{a_2}-\frac{a_1}{a_2}\left[m_3e^{k_1z}+m_4e^{k_2z}-\frac{a_1m_1}{{a_2}^2+{a_1}^2}\right]. \tag{5. 35}$$

Substituting equation (5.34) and (5.29) in equation (5.28), we get

$$\frac{\partial^3 L}{\partial z^3}=0, \tag{5. 36}$$

which on integration gives

$$L=m_5z^2+m_6z+m_7. \tag{5. 37}$$

Using equation (5.37) in equation (5.31), we get

$$J=-\frac{a_1}{a_2}\left[m_5z^2+m_6z+m_7\right]. \tag{5. 38}$$

The pressure function for this case is,

$$P=m_8\left(m_3k_1e^{k_1z}+m_4e^{k_2z}\right)+m_9\left(m_3k_1^2e^{k_1z}+m_4k_2^2e^{k_2z}\right)-m_{10}e^{2k_1z}-m_{11}e^{(k_1+k_2)z}-m_{12}e^{2k_2z}-$$
$$m_{14}\left(m_5z^2+m_6z+m_7\right)^2-\phi\upsilon\left(\frac{m_3e^{k_1z}}{k_1}+\frac{m_4}{k_2}e^{k_2z}\right)-\phi\alpha m_1\left(m_3e^{k_1z}+m_4e^{k_2z}\right)-\frac{a_1m_1}{{a_2}^2+{a_1}^2}z-m_{15} \tag{5. 39}$$

where,

$$m_8=\frac{\rho}{b}\left(\upsilon\left({a_2}^2+{a_1}^2\right){a_2}^2m_1+\frac{{a_2}^2{a_1}^2}{{a_2}^2+{a_1}^2}m_1\right),\ m_9=\frac{\rho}{a_1}\beta m_1{a_2}^2,\ m_{10}=\frac{\rho}{a_2}\beta\begin{bmatrix}\frac{2{a_1}^5}{{a_2}^2}+2{a_2}^2a_1+\\ 2{a_1}^3+\frac{2{a_1}^3}{a_2}\end{bmatrix}\frac{m_3^2k_1}{2}$$

$$m_{11}=\frac{\beta\rho m_3m_4k_1k_2}{a_1}\left[\left(\frac{2{a_1}^5}{{a_2}^2}+2{a_2}^2a_1+2{a_1}^3+\frac{2{a_1}^3}{a_2}\right)\right],\ m_{12}=\frac{\beta m_4^2k_2^2}{2a_1},\ m_{14}=\frac{\eta\rho}{2{a_2}^2}\left({a_2}^2+{a_1}^2\right)$$

$m_{15}=\frac{\rho}{a_2}\left(\frac{\phi\alpha m_1^2a_1}{{a_2}^2+{a_1}^2}\right)$.
$Also,$

$$H_1(z)=m_5z^2+m_6z+m_7, \tag{5. 40}$$

$$H_2\left(z\right)=\frac{-a_1}{a_2}\left(m_5z^2+m_6z+7\right). \tag{5. 41}$$

Case II, $a_3\neq -m_1$

Then equation (5.32) takes the form

$$Eu'''+Fu''-Gu'-Nu=-Q, \tag{5. 42}$$

where, E=$\alpha\left({a_2}^2+{a_1}^2\right)\left(a_3+m_1\right),\ F=\upsilon\left({a_2}^2+{a_1}^2\right),\ G=\left[a_3+m_1+\frac{\phi\alpha a_3}{a_2}\left({a_2}^2+{a_1}^2\right)\right]$
$N=\frac{\phi\upsilon}{a_2}\left({a_2}^2+{a_1}^2\right),\ Q=\frac{\phi\upsilon a_1m_1}{a_2}$.

Solution of this equation (5.42) is,

$$u=m_{16}e^{(\alpha^*-\beta^*)z}+m_{17}e^{\left(\alpha^*W^2-\beta^*W\right)z}+m_{18}e^{\left(\alpha^*W-\beta^*W^2\right)z}-\frac{Q}{N} \tag{5. 43}$$

where, $\alpha^*=\frac{-F^2}{9E^2\beta^*}-\frac{G}{3\beta^*F},\beta^*=\left[\left(\frac{F^3}{9E^3}+\frac{G}{6E}+\frac{N}{2E}\right)+\sqrt{\left(\frac{F^2}{9E^2}-\frac{2F^2}{9E^2}-\frac{G}{3F}\right)^3+\left(\frac{F^3}{18E^2}+\frac{G}{6E}+\frac{N}{2E}\right)^2}\right]^{\frac{1}{3}}$,
$and, W=\frac{-1}{2}+i\frac{\sqrt{3}}{2}$, m_{16}, m_{17} and m_{18} are constant.

Also,

$$v=\frac{m_1}{a_2}-\frac{a_1}{a_2}\left[m_{16}e^{(\alpha^*-\beta^*)z}+m_{17}e^{\left(\alpha^*W^2-\beta^*W\right)z}+m_{18}e^{\left(\alpha^*W-\beta^*W^2\right)z}-\frac{Q}{N}\right]. \tag{5. 44}$$

Substituting equation (5.43), (5.44) and (5.29) in equation (5.28), we get

$$\frac{\partial^3L}{\partial z^3}+m_{19}\frac{\partial^2L}{\partial z^2}=0, \tag{5. 45}$$

where,

$$m_{19} = \mu^* \sigma a_2 \left(\frac{\left({a_2}^2 + {a_1}^2\right) \frac{m_1}{a_2} - \frac{Q}{N} + \frac{a_3}{a_2} \left({a_2}^2 + {a_1}^2\right)}{\left({a_2}^2 + {a_1}^2\right)^2} \right)$$

which gives the solution,

$$L = m_{20} e^{m_{19} z} + m_{21} z + m_{22}. \tag{5. 46}$$

And using (5.46) in (5.31), we get

$$J = -\frac{a_1}{a_2} \left(m_{20} e^{m_{19} z} + m_{21} z + m_{22} \right), \tag{5. 47}$$

where, $m_{20}, m_{21,}$ and m_{22} are constants.

The pressure function for this case is

$$\begin{aligned} P = {} & m_{23} e^{(\alpha^* - \beta^*)z} + m_{24} e^{\left(\alpha^* W^2 - \beta^* W\right)z} + m_{25} e^{\left(\alpha^* W - \beta^* W^2\right)z} \\ & + m_{26} e^{2(\alpha^* - \beta^*)z} m_{27} e^{2\left(\alpha^* W^2 - \beta^* W\right)z} + m_{28} e^{2\left(\alpha^* W - \beta^* W^2\right)z} + \\ & m_{29} e^{\left(\alpha^* - \beta^* + \alpha^* W^2 - \beta^* W\right)z} + m_{30} e^{\left(\alpha^* - \beta^* + \alpha^* W - \beta^* W^2\right)z} + \\ & m_{31} e^{\left(\alpha^* W - \beta^* W^2 + \alpha^* W^2 - \beta^* W\right)z} - m_{32} [m_{20} e^{m_{19} z} + m_{21} z + m_{22}]^2 - m_{33} z \end{aligned} , \tag{5. 48}$$

here,

$$m_{23} = \frac{\rho}{a_1} \begin{bmatrix} \upsilon \left({a_2}^2 + {a_1}^2\right) - \beta m_1 {a_2}^2 m_{16} \left(\alpha^* - \beta^*\right) - \\ \beta \left(a_3 \left({a_2}^2 + {a_1}^2\right) + {a_1}^2 m_1\right) m_{17} \left(\alpha^* W^2 - \beta^* W\right)^2 \end{bmatrix} + \frac{\phi \upsilon m_{17}}{\alpha^* - \beta^*} - \phi \alpha a_3 m_{16}$$

,

$$m_{24} = \frac{\rho}{a_1} \begin{bmatrix} \upsilon \left({a_2}^2 + {a_1}^2\right) - \beta m_1 {a_2}^2 m_{17} \left(\alpha^* W^2 - \beta^* W\right) - \\ \beta \left(a_3 \left({a_2}^2 + {a_1}^2\right) + {a_1}^2 m_1\right) m_{17} \left(\alpha^* W^2 - \beta^* W\right)^2 \end{bmatrix} + \frac{\phi \upsilon m_{17}}{(\alpha^* W - \beta^* W^2)} - \phi \alpha a_3 m_{17}$$

,

$$m_{25} = \frac{\rho}{a_1} \begin{bmatrix} \upsilon \left({a_2}^2 + {a_1}^2\right) - \beta m_1 {a_2}^2 m_{18} \left(\alpha^* W - \beta^* W^2\right) - \\ \beta \left(a_3 \left({a_2}^2 + {a_1}^2\right) + {a_1}^2 m_1\right) m_{18} \left(\alpha^* W^2 - \beta^* W^2\right)^2 \end{bmatrix} + \frac{\phi \upsilon m_{18}}{(\alpha^* W - \beta^* W^2)} - \phi \alpha a_3 m_{18}$$

,

$$m_{26} = \frac{-\beta \rho}{a_1} \left[\frac{2{a_1}^5}{{a_2}^2} + 2{a_2}^2 a_1 + 2{a_1}^3 + \frac{2{a_1}^3}{a_2} \right] \frac{m_{16}^2 (\alpha^* - \beta^*)^2}{2}$$

,

$$m_{27} = \frac{-\beta \rho}{a_1} \left[\frac{2{a_1}^5}{{a_2}^2} + 2{a_2}^2 a_1 + 2{a_1}^3 + \frac{2{a_1}^3}{a_2} \right] \frac{m_{17}^2 \left(\alpha^* W^2 - \beta^* W\right)^2}{2}$$

,

$$m_{28}=\frac{-\beta\rho}{a_1}\left[\frac{2{a_1}^5}{{a_2}^2}+2{a_2}^2a_1+2{a_1}^3+\frac{2{a_1}^3}{a_2}\right]\frac{m_{18}^2\left(\alpha^*W-\beta^*W^2\right)^2}{2}$$

,

$$m_{29}=\frac{-\beta\rho}{a_1}\left[\frac{2{a_1}^5}{{a_2}^2}+2{a_2}^2a_1+2{a_1}^3+\frac{2{a_1}^3}{a_2}\right]\frac{m_{16}m_{17}\left(\alpha^*-\beta^*\right)\left(\alpha^*W^2-\beta^*W\right)^2}{\left(\alpha^*-\beta^*+a^*W^2-\beta^*W\right)}$$

$$+\frac{m_{16}m_{17}(\alpha^*-\beta^*)^2\left(\alpha^*W^2-\beta^*W\right)^2}{\left(\alpha^*-\beta^*+\alpha^*W^2-\beta^*W\right)}$$

$$m_{30}=\frac{-\beta\rho}{a_1}\left[\frac{2{a_1}^5}{{a_2}^2}+2{a_2}^2a_1+2{a_1}^3+\frac{2{a_1}^3}{a_2}\right]\frac{m_{16}m_{18}\left(\alpha^*-\beta^*\right)\left(\alpha^*W-\beta^*W^2\right)^2}{\left(\alpha^*-\beta^*+\alpha^*W-\beta^*W^2\right)}$$

$$+\frac{m_{16}m_{18}(\alpha^*-\beta^*)^2\left(\alpha^*W-\beta^*W^2\right)^2}{\left(\alpha^*-\beta^*+\alpha^*W-\beta^*W^2\right)}$$

$$m_{31}=\frac{\rho}{a_1}\left[\frac{2{a_1}^5}{{a_2}^2}+2{a_2}^2a_1+2{a_1}^3+\frac{2{a_1}^3}{a_2}\right]\frac{m_{17}m_{18}\left(\alpha^*W^2-\beta^*W\right)\left(\alpha^*W-\beta^*W^2\right)}{\left(a^*W^2-\beta^*W+\alpha^*W-\beta^*W^2\right)}$$

,

$$m_{32}=\frac{\rho}{a_1}\left[\frac{\eta a_1}{{a_2}^2}\left({a_2}^2+{a_1}^2\right)\right],\ \ m_{33}=\frac{\phi Qv\rho}{Na_1}$$

and,

$$H_1\left(z\right)=m_{20}e^{m_{19}z}+m_{21}z+m_{22}, \tag{5. 49}$$

$$H_2\left(z\right)=-\frac{a_1}{a_2}\left(m_{20}e^{m_{19}z}+m_{21}z+m_{22}\right). \tag{5. 50}$$

6. RESULT AND DISCUSSION

In the present work, the method of obtaining traveling wave solutions has been employed for the second-grade fluid flow through porous media, leading to the determination of exact solutions for the variables of interest. By transforming the variables (x, y, t) into a single variable, z, we simplified the coupled partial differential equations into a much simpler linear differential equation. This simplified form facilitates easy integration to find the solution. Through the elimination of pressure from equations (5.26) and (5.27) and the utilization of equations (5.30) and (5.31), a second and third-order linear differential equation in u was obtained. Subsequently, two possible cases arose for the variable c. By separately considering these two cases, Case I, $C=-m_1$, Case II, $C\neq-m_1$, we were able to find the desired solutions with ease.

7. CONCLUSION

The purpose of this study was to determine the exact solutions of second-grade fluids flowing through porous media with electrically conducting, incompressible MHD oriented flow. To linearize the partial differential equation,for this, we employed the travelling wave solution method. The approach was implemented in a straightforward manner without imposing restrictive assumptions or convoluted calculations.It has been noted that the answer $C = -m_1$, includes both exponential and polynomial terms. But that for $C \neq -m_1$, the solution is of the exponential kind. By inserting $\phi = \frac{\phi^*}{\rho} = 0$, we obtain the same solution as Najeeb et al. [14]. We can verify this as follows:

For non-porous media, $\phi^* = 0$.

Putting this in equation (5.32), we get,
$\alpha\left(a^2+b^2\right)(C+m_1)\, u'''+\upsilon\left(a^2+b^2\right)u''-(C+m_1)\, u'=0,$

which gives the result,
$u''=0$, for $C=-m_1$ (case-I), and

$Eu'''+Fu''-Gu'=0$, for $C \neq -m_1$, (case-II)

which is the same as obtained by Najeeb et al. [14].

8. ACKNOWLEDGMENTS

I would like to thank my supervisor, Dr. Manoj Kumar, for the patient guidance, encouragement and advice he has provided throughout my time as his student. I have been extremely lucky to have a supervisor who cared so much about my work, and who responded to my questions and queries so promptly. I would also like to thank Dr. Sayanatan Sil and Dr. Mantu Prajapati who helped me in my supervisors absence.

CONFLICT OF INTEREST
The authors agree with the contents of the manuscript, and there is no conflict of interest among the authors.

AUTHOR'S CONTRIBUTIONS
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

FUNDING
The author(s) received no financial support for the research, authorship and/or publication of this article.

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