Punjab University Journal of Mathematics (2023),55(9-10),397-421
https://doi.org/10.52280/pujm.2023.55(9-10)05

Generalized ρ-dependent polynomials of topological indices of the identity graph for the ring Z_ρ

Rukhshanda Anjum
Department of Mathematics and Statistics,
University of Lahore, Lahore, Pakistan,
Email: rukhshanda.anjum@math.uol.edu.pk

Muhammad Umar Mirza
Department of Mathematics and Statistics,
University of Lahore, Lahore, Pakistan,
Email: umarmirza096@gmail.com

Naila Niaz
Department of Mathematics and Statistics,
University of Lahore, Lahore, Pakistan,
Email: nailau744@gmail.com

Received: 19 June, 2023 / Accepted: 21 November, 2023 / Published online: 23 December, 2023

Abstract. In this article, we present the generalized ρ dependent polynomials for the calculations of eccentricity, distance, total distance, and degree-based topological indices of the identity graph of Z_ρ. This is a thorough work in which we present many topological indices and co-indices as ρ dependent polynomials. The polynomials presented here can play a key role in the further development of the theory of topological indies for commutative rings. This paper presents a brand new approach to generalizing the topological indices because instead of the traditional way. A set-theoretic method is introduced here that can be very helpful and game-changing in the field of algebraic graph theory first of all sets of vertices for the identity graph of the commutative ring Z_ρ are partitioned into various sets, which makes it easier to generalize the degrees, distances, and eccentricities of this graph. This paper presents various results that make it easier to generalize the topological indices of the identity graph of Z_ρ.

AMS (MOS) Subject Classification Codes: 05C90; 05C92
Key Words: Topological indices, Identity graph, Co-indices, Algebraic graph theory

1. Introduction

The history of algebra dates back to ancient times, with the oldest surviving example being an Egyptian papyrus from around 1500 BCE, which explains how to calculate volumes using chords and arc length. The term ”algebra” was first recorded around 1000 BC, and it comes from the title of the book written by Abu Ja’far Muhammad Ibn Musa al-Khwarizmi called ”Kitab al-jabr wa’l muqabala”, which was later translated into different European languages under various titles, including Liber abaci, Algebra, and others.

Ring theory is a mathematical branch that studies rings, and it was first mentioned by the Italian mathematician, Gerolamo Cardano in 1391, who proposed a method for reducing fractions. Although it was an interesting development in mathematics, ring theory didn’t receive much attention as an independent topic until around 2000 AD.

Modern algebra’s foundational idea of rings has its roots in the late 19th-century efforts to demonstrate Fermat’s final theorem. Ring research was started by Richard Dedekind, and the word ’ring’ was first used by David Hilbert [58]. Rings are two-bit binary algebraic systems that have strong ties to groups, particularly Abelian groups [36]. Algebraic number theory and algebraic geometry depend heavily on commutative rings, whereas non-commutative geometry and quantum groups use non-commutative rings [4]. The investigation of unique rings, such as rings of power series, rings of polynomials, and Boolean rings, as well as the creation of numerous theorems, like Wedderburn’s theorem on finite division rings, have all resulted from the study of rings [15]. The development of modern algebra has been aided by the history of ring theory, which has applications in many branches of physics and mathematics [28].

Topological indices were initially developed in the field of chemistry to understand the representations of various chemical structures, but they have now expanded beyond just chemical graph theory. For instance Bielak and Broniszewska [14] presented the lower bounds of the eccentric distance sum index of connected graph and cacti, while [42] calculated the same index for the bridge graph. Additionally, [18] computed the same index for the thorn graph into polynomial form. Studies on the topological indices of tree graphs are also available. [49] conducted a study on the first Zagreb index of the tree graph, calculating the upper bounds of the first Zagreb index of the tree graph, while [26] works on the eccentric distance sum of a tree. The eccentric distance sum index and eccentric connectivity index of unicyclic graphs are calculated in [59] and [43], respectively. Moreover, [38], [20], [35], and [60] presented topological indices of bipartite graphs, composite graphs, windmill graphs, and Sierpinski graphs. The topological indices of some graph operations can also be calculated, and a study is done on this in [23].

Graphs are the best tools to understand any type of relationship between the elements of a set. Algebraic graph theory enables us to better visualize the elements of groups or rings. Different algebraic structures are used to develop new graphs such as commuting graph of quaternion and dihedral groups [53] and the non-commuting graphs of quaternion and dihedral groups are discussed in [50]. In [34], the different groups and subgroups are presented as graphs with examples, with many useful results presented especially for the identity graph of groups. [22] discussed the conjugate graph of the group, while [10] studied the subgroup graph of groups. [41], [52], [6], [7], [31], and [30] conducted studies on the non-commuting graph of quasi-dihedral groups, dihedral groups, and other finite groups, in general. Beside that [1] and [2] calculated the topological indices of the subgroup graph of the symmetric group and dihedral groups.

Although a lot of work has been done on the graph of groups, not much research has been conducted on graphs of rings. In [3], the eccentric connectivity index of a finite commutative ring is discussed. Topological co-indices are not widely researched; thus, this article presents topological co-indices along with the topological indices of the identity graph of the ring Z_ρ where ρ is any prime number. Set theory is used to simplify the topological indices of the identity graph of Z_ρ into Z_ρ-dependent polynomials. As for larger prime numbers, the identity graphs of Z_ρ become very large, and traditional methods for calculating topological indices become complicated. However, using the polynomials presented in this article, topological indices for the identity graph of Z_ρ for very large values of Z_ρ can be calculated easily.

This article discusses the eccentric connectivity index of a finite commutative ring, along with topological co-indices and indices of the identity graph of Z_ρ using set theory to simplify the expressions into Z_ρ-dependent polynomials. This method is useful for larger prime numbers where traditional methods become too complicated. While much work has been done on the graph of groups, not much work has been done on graphs of rings, making this article a valuable contribution to the field.

Additionally, the proposed method can be extended to other types of rings and graphs, providing a new approach to the study of topological indices in algebraic structures. This can open up new avenues for research in this field and potentially lead to new applications in other areas such as computer science and physics.

The study of topological indices in algebraic structures such as rings and groups is a rich and growing field with many potential applications. The development of new techniques and methods for calculating these indices, as well as the exploration of new types of rings and graphs, will continue to expand our understanding of the connections between algebra and graph theory.

Exploring the physical and structural aspects of graphs, [33] employs topological indices and diverse graph operations on simple connected graphs, revealing insights into the hyper-Zagreb coindex of derived graphs. Delving into graph labeling, [44] demonstrates C3-supermagic labeling for triangular book-snake graphs and extends findings to establish Cm-supermagicness for polygonal book-snake graphs. Complementing the exploration of book-snake graphs, [13] establishes C3-supermagic labeling and extends this analysis to confer Cm-supermagicness upon polygonal book-snake graphs. Addressing degree-based topological indices, [5] formulates indices for grape seed Proanthocyanidin networks, offering a nuanced understanding of their structural characteristics.

2. Definitions.

In this section, we present some definitions that are crucial for the further development of this article. The identity graph of the ring Z_ρ as defined in [32] is denoted by $Id(Z_\rho)$ is defined as the graph with vertex set equals to the set of units in Z_ρ and two different vertices σ and ς are adjacent if $\sigma\varsigma = 1$ and every vertex of $Id(Z_\rho)$ is adjacent to the multiplicative identity of Z_ρ. $Id(Z_5)$,$Id(Z_7)$ and $Id(Z_{13})$ are given in the Figure 1 and Figure 2. the size of Id(Z_ρ) is defined as the number of edges in $Id(Z_\rho)$ and is denoted by $\psi(Id(Z_\rho))$. The order of the graph Id(Z_ρ) is defined as the number of vertices in Id(Z_ρ)

and is denoted by $\varphi(Id(Z_\rho))$. The degree of a vertex of $\sigma \varepsilon V(Id(Z_\rho))$ is defined by the number of edges adjacent to σ and is denoted by $\varpi(\sigma)$. The distance between $\sigma, \varsigma \in V(Id(Z_\rho))$ is defined as the number of vertices in the shortest path connecting them and we denote it as $\Upsilon(\sigma, \varsigma)$ In this article (σ, ς) denotes the distinct unordered pair of vertices of $Id(Z_\rho)$ i.e. $\sigma \neq \varsigma$ and (σ, ς)=$(\varsigma.\sigma)$ unless specified otherwise. Also, we define β to be the set of all nonadjacent pairs of vertices i.e $\beta = \{(\sigma, \varsigma) : (\sigma, \varsigma) \notin E(Id(Z_\rho))\}$. The set γ is defined as the set of pairs of nonadjacent vertices in which one vertex is $p-1$ or $\gamma = \{(\rho-1, \varsigma) : (\rho-1, \varsigma) \notin E(Id(Z_\rho))\}$. The cardinality of a set A is defined as the number of elements in A and is denoted by $|A|$. In this article, the set μ is defined as the set of pairs of vertices with a distance equal to 1. $\mu = \{(\sigma, \varsigma) : \Upsilon(\sigma, \varsigma) = 1\}$. The Ω set is defined as $\Omega = \{(1, \varsigma) : \varsigma \varepsilon V(Id(Z_\rho)) \wedge \varsigma \neq p-1\}$. The η set is defined as $\eta = E(Id(Z_\rho)) - \Omega - (1, \rho-1)$.

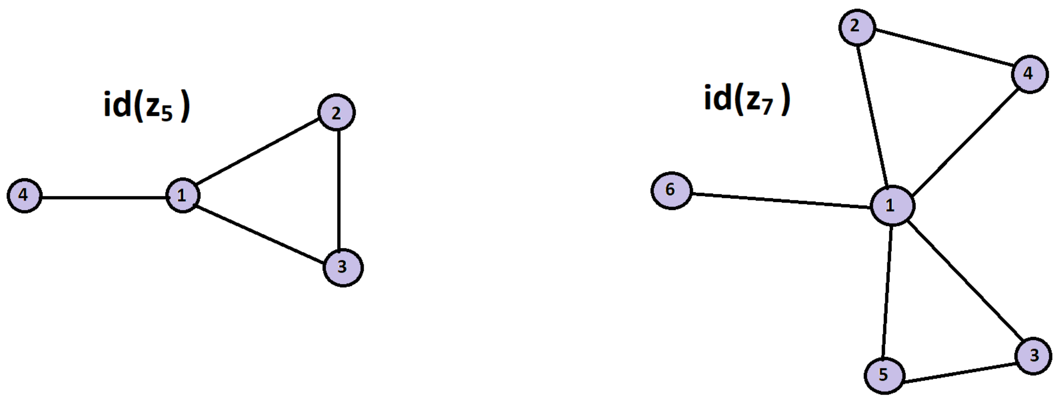

FIGURE 1. *Identity Graph of Z_5 and Z_7*

3. MAIN RESULTS

In this section first, we partitioned the vertex and edge set of $Id(Z_\rho)$ into subsets with certain properties of distances, eccentricities, degrees, and total distances. We calculated the cardinality of each of those sets and used it to generalize the ρ dependent polynomials for topological indices of $Id(Z_\rho)$.

3.1. **Some sets defined on identity graph of Z_ρ.** In this article, we define β as the set of all nonadjacent pairs of vertices i.e $\beta = \{(\sigma, \varsigma) : (\sigma, \varsigma) \notin E(Id(Z_\rho))\}$ and the set γ is defined as the set of pairs of nonadjacent vertices in which one vertex is $\rho - 1$ i.e $\gamma = \{(\rho-1, \varsigma) : (\rho-1, \varsigma) \notin E(Id(Z_\rho))\}$. The Ω set is defined as $\Omega = \{(1, \varsigma) : \varsigma \varepsilon V(Id(Z_\rho)) \wedge \varsigma \neq \rho - 1\}$.The η set is defined as $\eta = E(Id(Z_\rho)) - \Omega - (1, \rho-1)$. The set A, B and C are defined as $A = \{\rho - 1\}, B = \{1\}$ and $C = V(G) - (A \bigcup B)$. The degree of a vertex is denoted by ϖ.

Theorem 3.2. *The size of* $(Id(z_p)) = \frac{3p-7}{2}$.

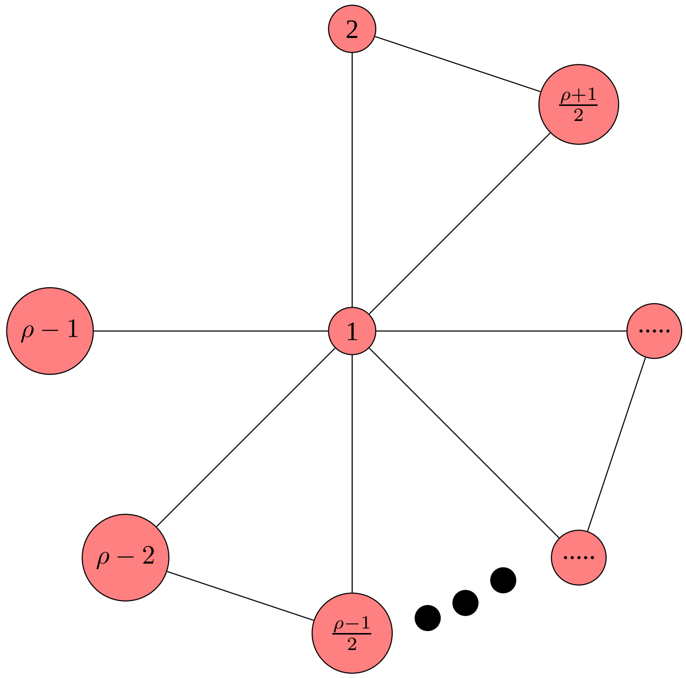

FIGURE 2. *Generalized Identity Graph of Z_ρ*

Proof. As we can see that number of vertices in $(Id(z_p)) = p-1$. $\varpi(1) = p-2, \varpi(p-1) = 1$. it can be observed that $\varpi(\kappa) = 2 \ \forall \ \kappa \in V(Id(z_p)) \wedge \kappa \notin \{1, p-1\}$. The number of vertices of degree 2 are (p-1)-2=p-3. Hence by hand shaking lemma

$$\begin{aligned}\psi(Id(z_p)) &= \frac{1+p-2+(p-3)2}{2}\\ &= \frac{3p-7}{2}.\end{aligned}$$

□

Theorem 3.3. *Let $\beta = \{(\sigma,\varsigma) : (\sigma,\varsigma) \notin E(Id(Z_\rho))\}$ and $\gamma = \{(\rho-1,\varsigma) : (\rho-1,\varsigma) \notin E(Id(Z_\rho))\}$ then*

(1) $|\beta| = \frac{(\rho-3)^2}{2}$.
(2) $|\gamma| = \rho - 3$.
(3) $|\beta - \gamma| = \frac{(\rho-5)(\rho-3)}{2}$.
(4) *if* $(\sigma,\varsigma) \in \gamma$ *then* $\varpi(\sigma) = 1 \wedge \varpi(\varsigma) = 2$.
(5) *if* $(\sigma,\varsigma) \in \beta-\gamma$ *then* $\varpi(\sigma) = \varpi(\varsigma) = 2$.

Proof. **(1)** As we can see that $\psi(Id(Z_\rho)) = \rho-1$, the size of complete graph of order $\rho-1$ is $\frac{(\rho-1)(\rho-2)}{2}$. According to theorem 3.2, the size of $Id(Z_\rho)$ is $\frac{3\rho-7}{2}$, so

$$\begin{aligned}|\beta| &= \frac{(\rho-1)(\rho-2)}{2} - \frac{3\rho-7}{2}\\ &= \frac{(\rho-3)^2}{2}.\end{aligned}$$

(2) As $\varphi(Id(Z_\rho)) = \rho-1$ and the vertex $\rho-1$ is adjacent to only 1. The number of non-adjacent vertices to $\rho-1$ are $\rho-1-2 = \rho-3$, hence $|\gamma| = \rho-3$.

(3) As $\gamma \subset \beta$ so $|\beta-\gamma| = |\beta| - |\gamma|$, hence,

$$\begin{aligned}|\beta-\gamma| &= (\frac{(\rho-3)^2}{2}) - (\rho-3)\\ &= \frac{(\rho-5)(\rho-3)}{2}.\end{aligned}$$

(4) Let $(\sigma,\varsigma) \in \gamma$ and assume that $\sigma = \rho-1$ then $\varsigma\, neq \rho - 1 \wedge \varsigma \neq 1$, hence $\varpi(\sigma) = 1 \wedge \varpi(\varsigma) = 2$.

(5) Let $(\sigma,\varsigma) \in \beta-\gamma \Rightarrow (\sigma,\varsigma) \in \beta \wedge (\sigma,\varsigma) \notin \gamma$. $(\sigma,\varsigma) \in \beta \Rightarrow \sigma \neq 1 \neq \varsigma$ because 1 is adjacent to every vertex of $Id(Z_\rho)$. $(\sigma,\varsigma) \notin \gamma \Rightarrow \sigma \neq \rho-1 \neq \varsigma$. Since every vertex other than 1 and $\rho-1$ have degree two so $\varpi(\sigma) = \varpi(\varsigma) = 2$. □

Theorem 3.4. *Let* $\Omega = \{(1,\varsigma) : \varsigma \varepsilon V(Id(Z_\rho)) \wedge \varsigma \neq \rho-1\}$ *and* $\eta = E(Id(Z_\rho)) - \Omega - (1,\rho-1)$.
(1) $|\Omega| = \rho-3$.
(2) $\forall\ (1,\varsigma) \in \Omega\ \ \varpi(\varsigma) = 2$.
(3) $|\eta| = \frac{\rho-3}{2}$.
(4) $\forall\ (\sigma,\varsigma) \in \eta\ \ \varpi(\sigma) = \varpi(\varsigma) = 2$.

Proof. **(1)** $(1,\varsigma) \in \Omega\ \ \Rightarrow \varsigma \neq 1 \wedge \varsigma \neq \rho-1$. Since 1 is adjacent to all the vertices of $Id(Z_\rho)$ and there are $\rho-1$ vertices in Z_ρ, so $|\Omega| = \rho-3$.

(2) $(1,\varsigma) \in \Omega\ \ \Rightarrow \varsigma \neq 1 \wedge \varsigma \neq \rho-1$. All the vertices of $Id(Z_\rho)$ other than 1 and $\rho-1$ have degree two, so $\varpi(\varsigma) = 2$.

(3) As η, Ω, and $\{(1,\rho-1)\}$ form a partition of $E(Id(Z_\rho))$, hence

$$|E(Id(Z_\rho))| = |\eta| + |\beta| + \{(1,\rho-1)\}.$$

$$\begin{aligned}|\eta| &= |E(Id(Z_\rho))| - |\Omega| - \{(1,\rho-1)\}\\ &= \frac{3\rho-7}{2} - (\rho-3) - 1\\ &= \frac{\rho-3}{2}.\end{aligned}$$

(4) $(\sigma,\varsigma) \in \eta\ \ \Rightarrow \sigma\varsigma \neq 1 \wedge \sigma\varsigma \neq \rho-1$. All the vertices of $Id(Z_\rho)$ other than 1 and $\rho-1$ have degree two, so $\varpi(\sigma) = \varpi(\varsigma) = 2$. □

Theorem 3.5. *Let* $A = \{\rho - 1\}$, $B = \{1\}$, *and* $C = V(Id(Z_\rho)) - (A \cup B)$. *Then,*
(1) $\forall\ v \in A, \varpi(v) = 1.$
(2) $\forall\ v \in B, \varpi(v) = \rho - 2.$
(3) $\forall\ v \in C, \varpi(v) = 2.$
(4) $|A| = 1$, $|B| = 1$, $|C| = \rho - 3.$
(5) $\forall\ v \in A, e(v) = 2.$
(6) $\forall\ v \in B, e(v) = 1.$
(7) $\forall\ v \in C, e(v) = 2.$
(8) $\forall\ v \in A, D(v) = 2\rho - 5.$
(9) $\forall\ v \in B, D(v) = \rho - 2.$
(10) $\forall\ v \in C, D(v) = 2\rho - 6.$

Proof. **(1)** For all prime ρ, $\rho - 1$ is self inverse in Z_ρ hence, in $V(Id(Z_\rho))$, $\varpi(\rho - 1) = 1$.

(2) Since there are $\rho - 1$ vertices in $Id(Z_\rho)$, 1 is adjacent to every vertex of $Id(Z_\rho)$ other than itself. Hence, $\varpi(1) = \rho - 2$.

(3) Since for all $v \in C$, $v \neq 1$ and $v \neq \rho - 1$. Then, v is adjacent to 1 and adjacent to its inverse. Since Z_ρ is a ring, there exists an inverse. Also, $v \neq \rho - 1$, so v is not self-inverse. Hence, $\varpi(v) = 2$.

(4) There is only one element in A and B. C contains all the vertices of $Id(Z_\rho)$ other than 1 and $\rho - 1$. The number of vertices of $Id(Z_\rho)$ other than 1 and $\rho - 1$ is $\rho - 1$, hence $|C| = \rho - 3$.

(5, 6, 7) Since every vertex is adjacent to 1, the eccentricity of all vertices other than 1 is 2, and the eccentricity of 1 is 1.

(8)

$$D(\rho - 1) = d(\rho - 1, 1) + d(\rho - 1, 2) + d(\rho - 1, 3) + \ldots + d(\rho - 1, \rho - 2).$$

Since $\rho - 1$ is adjacent to only 1, its distance from 1 is 1. Every vertex is adjacent to 1, so the distance of $\rho - 1$ from every other vertex other than 1 is 2. Hence,

$$\begin{aligned} D(\rho - 1) &= 1 + 2(\rho - 3) \\ &= 2\rho - 5. \end{aligned}$$

(9)

$$D(1) = d(1, 2) + d(1, 3) + \ldots + d(1, \rho - 1)$$

Since 1 is adjacent to every vertex of $Id(Z_\rho)$, its distance from every vertex of $Id(Z_\rho)$ is equal to 1. There are $\rho - 2$ vertices in $Id(Z_\rho)$ other than 1, hence,

$$D(1) = (\rho - 2).$$

(10) Let $v \in C$, then $d(v, v^{-1}) = 1$ and $d(v, 1) = 1$. For all $\sigma \in V(Id(Z_\rho))$, $v^{-1} \neq \sigma \neq 1$. Then $d(v, \sigma) = 2$, so

$$\begin{aligned} D(v) &= 2 + 2(\rho - 4) \\ &= 2\rho - 8 + 2 \\ &= 2\rho - 6. \end{aligned}$$

$\square$

3.6. **Mixed topological indices based on the distance between vertices and degrees of vertices.** In this section, we dig into a complete examination of graph indices for the graph $Id(Z_\rho)$. We continue with the definition of the Wiener index, denoted as $W(Id(Z_\rho))$ [21], which measures the summation of distances between pairs of vertices, represented by $\Upsilon(\sigma,\varsigma)$. Specifically,

$$W(Id(Z_\rho)) = \sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))} \Upsilon(\sigma,\varsigma). \quad (3.\ 1)$$

The mentioned reference [21] offers insights into several graph operations, including the generalized hierarchical product, T-th subdivision, and Mycielski's construction, all of which are pertinent to this index. Furthermore, we study the hyper Wiener index, written as $WW(Id(Z_\rho))$ [41], which extends the Wiener index by including both distances and their squares. Specifically,

$$WW(Id(Z_\rho)) = \frac{1}{2}\sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))} [\Upsilon(\sigma,\varsigma) + (\Upsilon(\sigma,\varsigma))^2]. \quad (3.\ 2)$$

In reference [21], non-commuting graphs for generalized quaternion groups are explored.

Moving on, the old Harary index, referred to as $H_{old}(Id(Z_\rho))$ [63], emerges as a measure that takes into consideration the reciprocal of the squared distances between vertex pairs. Specifically,

$$H_{old}(Id(Z_\rho)) = \frac{1}{2}\sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))} \frac{1}{\Upsilon(\sigma,\varsigma)^2}. \quad (3.\ 3)$$

In reference [63], upper bounds for triangle-free and quadrangle-free graphs are computed. Meanwhile, the Harary index, defined as $H(Id(Z_\rho))$, captures the reciprocal of distances between vertices. Specifically,

$$H(Id(Z_\rho)) = \frac{1}{2}\sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))} \frac{1}{\Upsilon(\sigma,\varsigma)}. \quad (3.\ 4)$$

In reference [56], they focused on the upper and lower boundaries of the Harary index.

Our exploration continues with the degree distance index, also known as the Schultz index $DD(Id(Z_\rho))$ [54], which combines vertex degrees and distances between vertex pairings. Specifically,

$$DD(Id(Z_\rho)) = \sum_{\sigma\neq\varsigma}[\varpi(\sigma)+\varpi(\varsigma)]\Upsilon(\sigma,\varsigma). \quad (3.\ 5)$$

The Gutman index $Gut(Id(Z_\rho))$ [11] is a metric that considers the product of vertex degrees and the related distances. Specifically,

$$Gut(Id(Z_\rho)) = \sum_{\sigma\neq\varsigma}[\varpi(\sigma)\varpi(\varsigma)]\Upsilon(\sigma,\varsigma). \quad (3.\ 6)$$

Lastly, we analyze the reciprocal degree distance index, frequently referred to as the additively weighted Harary index $H_\alpha(Id(Z_\rho))$ [8], and the multiplicatively weighted Harary index $H_M(Id(Z_\rho))$, discussed in references [19], [57], and [9]. These indices utilise weighted reciprocals of distances and are instrumental in distinguishing distinctive structural elements within the graph.

This section presents a complete review of numerous graph indices, each aimed to expose different facets of the graph $Id(Z_\rho)$. These indexes, based in precise mathematical definitions and backed by applicable references, offer useful insights for graph analysis and applications.

TABLE 1. Mixed topological indices based on the distance between vertices and degrees of vertices

Name of the Index	Formula for index
Wiener index [21]	$W(Id(Z_\rho)) = \sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))} \Upsilon(\sigma,\varsigma)$
Old Harary index [63]	$H_{old}(Id(Z_\rho)) = \frac{1}{2}\sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))} \frac{1}{\Upsilon(\sigma,\varsigma)^2}$
Hyper Wiener index [41]	$WW(Id(Z_\rho)) = \frac{1}{2}\sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))}[\Upsilon(\sigma,\varsigma) + (\Upsilon(\sigma,\varsigma))^2]$
The Harary index [56],[17]	$H(Id(Z_\rho)) = \frac{1}{2}\sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))} \frac{1}{\Upsilon(\sigma,\varsigma)}$
The degree distance index or Schultz index [54]	$DD(Id(Z_\rho)) = \sum_{\sigma\neq\varsigma}[\varpi(\sigma) + \varpi(\varsigma)]\Upsilon(\sigma,\varsigma)$
The Gutman index [11]	$Gut(Id(Z_\rho)) = \sum_{\sigma\neq\varsigma}[\varpi(\sigma)\varpi(\varsigma)]\Upsilon(\sigma,\varsigma)$
The reciprocal degree distance index or additively weighted Harary index[8]	$H_\alpha(Id(Z_\rho)) = \sum_{\sigma\neq\varsigma} \frac{[\varpi(\sigma)+\varpi(\varsigma)]}{\Upsilon(\sigma,\varsigma)}$
The multiplicatively weighted Harary index [19],[57],[9]	$H_M(Id(Z_\rho)) = \sum_{\sigma\neq\varsigma} \frac{[\varpi(\sigma)\varpi(\varsigma)]}{\Upsilon(\sigma,\varsigma)}$

Theorem 3.7. *$W(Id(Z_1)) = 0$, $W(Id(Z_2)) = 1$, and for all $\rho \geq 3$, $W(Id(Z_\rho)) = \rho^2 - \frac{1}{2}(9\rho + 11)$.*

Proof. $W(Id(Z_1)) = 0$ because there is no pair of distinct vertices in $Id(Z_1)$, and $W(Id(Z_2)) = 1$ as there is only one pair of vertices in $Id(Z_2)$, and both vertices are adjacent to each other. For all $\rho \geq 3$, we have

$$W(Id(Z_\rho)) = \sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))} \Upsilon(\sigma,\varsigma)$$

$$= \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} \Upsilon(\sigma,\varsigma) + \sum_{(\sigma,\varsigma)\in E(Id(Z_\rho))} \Upsilon(\sigma,\varsigma).$$

Since all the vertices are adjacent to 1, so $\Upsilon(\sigma,\varsigma)=2$ for all $(\sigma,\varsigma)\notin E(Id(Z_\rho))$. For all the adjacent vertices, $\Upsilon(\sigma,\varsigma)=1$. Hence,

$$W(Id(Z_\rho)) = \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} 2 \; + \sum_{(\sigma,\varsigma)\in E(Id(Z_\rho))} 1.$$

According to Theorem 3.3,

$$\begin{aligned} W(Id(Z_\rho)) &= 2(\frac{(\rho-3)^2}{2}) + \frac{(3\rho-7)}{2} \\ W(Id(Z_\rho)) &= \rho^2 - \frac{1}{2}(9\rho+11). \end{aligned}$$

□

Theorem 3.8. *$WW(Id(Z_1))=0, WW(Id(Z_2))=1$, and for all $\rho\geq 3$, $WW(Id(Z_\rho))=\frac{3\rho^2}{2}-\frac{15\rho}{2}+10$.*

Proof. $WW(Id(Z_1))=0$ because there is no pair of distinct vertices in $Id(Z_1)$. $WW(Id(Z_2))=1$ because there is only one pair of vertices in $Id(Z_2)$, and both vertices are adjacent to each other. For $\rho>3$, we have

$$\begin{aligned} WW(Id(Z_\rho)) &= \sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))} (\Upsilon(\sigma,\varsigma)+\Upsilon(\sigma,\varsigma)^2) \\ &= \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} (\Upsilon(\sigma,\varsigma)+\Upsilon(\sigma,\varsigma)^2) + \sum_{(\sigma,\varsigma)\in E(Id(Z_\rho))} (\Upsilon(\sigma,\varsigma)+\Upsilon(\sigma,\varsigma)^2). \end{aligned}$$

Since all the vertices are adjacent to 1, so $\Upsilon(\sigma,\varsigma)=2$ for all $(\sigma,\varsigma)\notin E(Id(Z_\rho))$. For all the adjacent vertices, $\Upsilon(\sigma,\varsigma)=1$. Hence,

$$\begin{aligned} WW(Id(Z_\rho)) &= \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} (2+4) \; + \sum_{(\sigma,\varsigma)\in E(Id(Z_\rho))} (1+1) \\ &= \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} 6 \; + \sum_{(\sigma,\varsigma)\in E(Id(Z_\rho))} 2. \end{aligned}$$

According to Theorem 3.3,

$$\begin{aligned} WW(Id(Z_\rho)) &= \frac{1}{2}(6(\frac{(\rho-3)^2}{2}) + 2\frac{(3\rho-7)}{2}) \\ &= \frac{3\rho^2}{2} - \frac{15\rho}{2} + 10. \end{aligned}$$

□

Theorem 3.9. *$H(Id(Z_1))=0$, $H(Id(Z_2))=1$, and for all $\rho\geq 3$, $H(Id(Z_\rho))=\frac{1}{4}(\rho^2-5)$.*

Proof. $H(Id(Z_1)) = 0$ because there is no pair of distinct vertices in $Id(Z_1)$. $H(Id(Z_2)) = 1$ because there is only one pair of vertices in $Id(Z_2)$, and both vertices are adjacent to each other. For $\rho > 3$, we have

$$\begin{aligned} H(Id(Z_\rho)) &= \sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))} \frac{1}{\Upsilon(\sigma,\varsigma)} \\ &= \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} \frac{1}{\Upsilon(\sigma,\varsigma)} + \sum_{(\sigma,\varsigma)\in E(Id(Z_\rho))} \frac{1}{\Upsilon(\sigma,\varsigma)}. \end{aligned}$$

Since all the vertices are adjacent to 1, so $\Upsilon(\sigma,\varsigma) = 2$ for all $(\sigma,\varsigma) \notin E(Id(Z_\rho))$. For all the adjacent vertices, $\Upsilon(\sigma,\varsigma) = 1$. Hence,

$$H(Id(Z_\rho)) = \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} \frac{1}{2} + \sum_{(\sigma,\varsigma)\in E(Id(Z_\rho))} 1.$$

According to Theorem 3.3,

$$\begin{aligned} H(Id(Z_\rho)) &= (\frac{1}{2}(\frac{(\rho-3)^2}{2})) + (\frac{(3\rho-7)}{2}) \\ &= \frac{1}{4}(\rho^2 - 5). \end{aligned}$$

□

Theorem 3.10. *The $H_{old}(Id(Z_1)) = 0$, $H_{old}(Id(Z_2)) = 1$, and for all $\rho \geq 3$, $H_{old}(Id(Z_\rho)) = \frac{1}{4}(3\rho + \frac{1}{2}(\rho^2 - 19))$.*

Proof. $H_{old}(Id(Z_1)) = 0$ because there is no pair of distinct vertices in $(Id(Z_1))$. $H_{old}(Id(Z_2)) = 1$ because there is only one pair of vertices in $(Id(Z_2))$, and both vertices are adjacent to each other. For $\rho \geq 3$, we have

$$\begin{aligned} H_{old}(Id(Z_\rho)) &= \sum_{\{\sigma,\varsigma\}\subset V(Id(Z_\rho))} \frac{1}{\Upsilon(\sigma,\varsigma)^2} \\ &= \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} \frac{1}{\Upsilon(\sigma,\varsigma)^2} + \sum_{(\sigma,\varsigma)\in E(Id(Z_\rho))} \frac{1}{\Upsilon(\sigma,\varsigma)^2}. \end{aligned}$$

Since all the vertices are adjacent to 1, so $\Upsilon(\sigma,\varsigma) = 2$ for all $(\sigma,\varsigma) \notin E(Id(Z_\rho))$. For all the adjacent vertices, $\Upsilon(\sigma,\varsigma) = 1$. Hence,

$$H_{old}(Id(Z_\rho)) = \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} \frac{1}{4} + \sum_{(\sigma,\varsigma)\in E(Id(Z_\rho))} 1.$$

According to Theorem 3.3,

$$\begin{aligned} H_{old}(Id(Z_\rho)) &= (\frac{1}{4}(\frac{(\rho-3)^2}{2})) + (\frac{(3\rho-7)}{2}) \\ &= \frac{1}{4}(3\rho + \frac{1}{2}(\rho^2 - 19)). \end{aligned}$$

□

Theorem 3.11. *The* $DD(Id(Z_1)) = 0$, $DD(Id(Z_2)) = 2$, *and for all* $\rho \geq 3$, $DD(Id(Z_\rho)) = 5\rho^2 - 25\rho + 33$.

Proof. $DD(Id(Z_1)) = 0$ because there is no pair of distinct vertices in $(Id(Z_1))$. $DD(Id(Z_2)) = 2$ since there is only one pair of vertices in $(Id(Z_2))$ and both vertices are adjacent to each other. For $\rho \geq 3$, we have,

$$\begin{aligned} DD(Id(Z_\rho)) &= \sum_{\sigma \neq \varsigma} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma) \\ &= \sum_{(\sigma,\varsigma) \in E((Id(Z_\rho))} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma) + \sum_{(\sigma,\varsigma) \notin E((Id(Z_\rho))} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma). \end{aligned} \tag{3. 7}$$

As η, Ω, and $\{(1, \rho - 1)\}$ form a partition of $E(Id(Z_\rho))$, we have

$$\begin{aligned} \sum_{(\sigma,\varsigma) \in E((Id(Z_\rho))} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma) =& (\varpi(1) + \varpi(\rho - 1)) + \sum_{(\sigma,\varsigma) \in \Omega} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma) \\ &+ \sum_{(\sigma,\varsigma) \in \eta} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma). \end{aligned}$$

By Theorem 3.4,

$$\begin{aligned} \sum_{(\sigma,\varsigma) \in E((Id(Z_\rho))} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma) &= (\rho - 1 + 1) + (\rho - 3)(\rho - 1 + 2) + ((\frac{\rho - 3}{2})4) \\ &= \rho + (\rho - 3)(\rho + 1) + (2(\rho - 3)). \end{aligned} \tag{3. 8}$$

$$\sum_{(\sigma,\varsigma) \notin E((Id(Z_\rho))} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma) = \sum_{(\sigma,\varsigma) \in \beta} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma).$$

As $\beta - \gamma$ and γ form a partition of β, we have

$$\sum_{(\sigma,\varsigma) \notin E((Id(Z_\rho))} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma) = \sum_{(\sigma,\varsigma) \in \gamma} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma) + \sum_{(\sigma,\varsigma) \in \beta - \gamma} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma).$$

By Theorem 3.3,

$$\begin{aligned} \sum_{(\sigma,\varsigma) \notin E((Id(Z_\rho))} (\varpi(\sigma) + \varpi(\varsigma))\Upsilon(\sigma, \varsigma) &= (\rho - 3)(3)(2) + \frac{(\rho - 5)(\rho - 3)}{2}(4)(2) \\ &= 6(\rho - 3) + 4(\rho - 5)(\rho - 3). \end{aligned} \tag{3. 9}$$

Substituting the values of equations 3. 8 and 3. 9 into equation 3. 7 , we get

$$DD(Id(Z_\rho)) = 5\rho^2 - 25\rho + 33.$$

□

Theorem 3.12. *The* $Gut(Id(Z_1)) = 0$, $Gut(Id(Z_2)) = 1$, *and for all* $\rho \geq 3$, $Gut(Id(Z_\rho)) = 6\rho^2 - 33\rho + 47$.

Proof. $Gut(Id(Z_1)) = 0$ because there is no pair of distinct vertices in $Id(Z_1)$. $Gut(Id(Z_2)) = 1$ since there is only one pair of vertices in $(Id(Z_2))$ and both vertices are adjacent to each other. For $\rho \geq 3$, we have

$$\begin{aligned} Gut(Id(Z_\rho)) &= \sum_{\sigma\neq\varsigma}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma) \\ &= \sum_{(\sigma,\varsigma)\in E((Id(Z_\rho))}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma) + \sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma). \end{aligned} \tag{3. 10}$$

As η, Ω, and $\{(1, \rho-1)\}$ form a partition of $E(Id(Z_\rho))$, we have

$$\begin{aligned} \sum_{(\sigma,\varsigma)\in E((Id(Z_\rho))}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma) &= (\varpi(1)\varpi(\rho-1)) + \sum_{(\sigma,\varsigma)\in \Omega}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma) \\ &\quad + \sum_{(\sigma,\varsigma)\in \eta}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma). \end{aligned}$$

By Theorem 3.4,

$$\begin{aligned} \sum_{(\sigma,\varsigma)\in E((Id(Z_\rho))}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma) &= (\rho-1) + (\rho-3)(\rho-1)2 + (4(\frac{\rho-3}{2})) \\ &= (\rho-1) + (\rho-3)(\rho-1)2 + (2(\rho-3)). \end{aligned} \tag{3. 11}$$

$$\sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma) = \sum_{(\sigma,\varsigma)\in \beta}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma).$$

As $\beta - \gamma$ and γ form a partition of β, we have

$$\sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma) = \sum_{(\sigma,\varsigma)\in \gamma}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma) + \sum_{(\sigma,\varsigma)\in \beta-\gamma}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma).$$

By Theorem 3.3,

$$\begin{aligned} \sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))}(\varpi(\sigma)\varpi(\varsigma))\Upsilon(\sigma,\varsigma) &= (\rho-3)4 + \frac{(\rho-5)(\rho-3)}{2}(4)(2) \\ &= 4(\rho-3) + 4(\rho-5)(\rho-3). \end{aligned} \tag{3. 12}$$

Substituting the values of equations 3. 11 and 3. 12 into equation 3. 10 , we get

$$Gut(Id(Z_\rho)) = 6\rho^2 - 33\rho + 47.$$

□

Theorem 3.13. *The* $H_M(Id(Z_1)) = 0$, $H_M(Id(Z_2)) = 1$, *and for all* $\rho \geq 3$, $H_M(Id(Z_\rho)) = 3\rho^2 - 12\rho + 11$.

Proof. $H_M(Id(Z_1)) = 0$ because there is no pair of distinct vertices in $(Id(Z_1))$. $H_M(Id(Z_2)) = 1$ because there is only one pair of vertices in $(Id(Z_2))$ and both vertices are adjacent to each other. For $\rho \geq 3$, we have

$$\begin{aligned} H_M(Id(Z_\rho)) &= \sum_{\sigma \neq \varsigma} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} \\ &= \sum_{(\sigma,\varsigma)\in E((Id(Z_\rho))} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} + \sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)}. \end{aligned} \tag{3. 13}$$

As η, Ω, and $\{(1, \rho - 1)\}$ form a partition of $E(Id(Z_\rho))$, we have

$$\sum_{(\sigma,\varsigma)\in E((Id(Z_\rho))} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} = (\varpi(1)\varpi(\rho-1)) + \sum_{(\sigma,\varsigma)\in \Omega} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} + \sum_{(\sigma,\varsigma)\in \eta} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)}.$$

By Theorem 3.4,

$$\begin{aligned} \sum_{(\sigma,\varsigma)\in E((Id(Z_\rho))} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} &= (\rho-1) + (\rho-3)(\rho-1)2 + (4(\frac{\rho-3}{2})) \\ &= (\rho-1) + (\rho-3)(\rho-1)2 + (2(\rho-3)). \end{aligned} \tag{3. 14}$$

$$\sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} = \sum_{(\sigma,\varsigma)\in \beta} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)}.$$

As $\beta - \gamma$ and γ form a partition of β, we have

$$\sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} = \sum_{(\sigma,\varsigma)\in \gamma} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} + \sum_{(\sigma,\varsigma)\in \beta-\gamma} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)}.$$

By Theorem 3.3,

$$\sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))} \frac{(\varpi(\sigma)\varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} = (\rho-3) + (\rho-5)(\rho-3). \tag{3. 15}$$

Substituting the values of equations 3. 14 and 3. 15 into equation 3. 13 , we get

$$H_M(Id(Z_\rho)) = 3\rho^2 - 12\rho + 11.$$

□

Theorem 3.14. *The* $H_A(Id(Z_1)) = 0$, $H_A(Id(Z_2)) = 1$*, and for all* $\rho \geq 3$, $H_A(Id(Z_\rho)) = \frac{(4\rho-9)(\rho-1)}{2}$.

Proof. $H_A(Id(Z_1)) = 0$ because there is no pair of distinct vertices in $(Id(Z_1))$. $H_A(Id(Z_2)) = 1$ because there is only one pair of vertices in $(Id(Z_2))$, and both vertices are adjacent to each other. For $\rho \geq 3$, we have

$$H_A(Id(Z_\rho)) = \sum_{\sigma \neq \varsigma} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)}$$
$$= \sum_{(\sigma,\varsigma)\in E((Id(Z_\rho))} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} + \sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)}. \tag{3. 16}$$

As η, Ω, and $\{(1, \rho - 1)\}$ form a partition of $E(Id(Z_\rho))$, we have

$$\sum_{(\sigma,\varsigma)\in E((Id(Z_\rho))} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} = (\varpi(1)+\varpi(\rho-1)) + \sum_{(\sigma,\varsigma)\in\Omega} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} + \sum_{(\sigma,\varsigma)\in\eta} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)}.$$

By Theorem 3.4,

$$\sum_{(\sigma,\varsigma)\in E((Id(Z_\rho))} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} = (\rho - 1 + 1) + (\rho - 3)(\rho - 1 + 2) + ((\frac{\rho-3}{2})4)$$
$$= \frac{(4\rho - 9)(\rho - 1)}{2}. \tag{3. 17}$$

$$\sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} = \sum_{(\sigma,\varsigma)\in\beta} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)}.$$

As $\beta - \gamma$ and γ form a partition of β, we have

$$\sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} = \sum_{(\sigma,\varsigma)\in\gamma} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} + \sum_{(\sigma,\varsigma)\in\beta-\gamma} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)}.$$

By Theorem 3.3,

$$\sum_{(\sigma,\varsigma)\notin E((Id(Z_\rho))} \frac{(\varpi(\sigma) + \varpi(\varsigma))}{\Upsilon(\sigma,\varsigma)} = \frac{(\rho - 3)}{2} + (\rho - 5)(\rho - 3). \tag{3. 18}$$

Putting the values of equation 3. 17 and equation 3. 18 in equation 3. 16 , we get

$$H_A(Id(Z_\rho)) = \frac{(4\rho - 9)(\rho - 1)}{2}.$$

□

3.15. **The topological co-indices.** Topological co-indices are commonly defined based on nonadjacent pairings of vertices. The first Zagreb co-index, as presented in references [49] and [54], is defined as:

$$M_1(Id(Z_\rho)) = \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} [\varpi(\sigma) + \varpi(\varsigma)]. \tag{3. 19}$$

In reference [49], a lower bound on the first Zagreb co-index of trees is proposed. Moving on, the second Zagreb co-index, likewise defined in references [49] and [54], is given by:

TABLE 2. Mixed topological indices based on the distance between vertices and degrees of vertices

Name of the Index	Polynomial for index
Wiener index [21]	$W(Id(Z_\rho)) = \rho^2 - \frac{1}{2}(9\rho + 11$
Old Harary index [63]	$H_{old}(Id(Z_\rho)) = \frac{1}{4}(3\rho + \frac{1}{2}(\rho^2 - 19)$
Hyper Wiener index [41]	$WW(Id(Z_\rho)) = \frac{3\rho^2}{2} - \frac{15\rho}{2} + 10$
The Harary index [56],[17]	$H(Id(Z_\rho)) = \frac{1}{4}(\rho^2 - 5)$
The degree distance index or Schultz index [54]	$DD(Id(Z_\rho)) = 5\rho^2 - 25\rho + 33$
The reciprocal degree distance index or additively weighted Harary index[8]	$H_\alpha(Id(Z_\rho)) = \frac{(4\rho-9)(\rho-1)}{2}$
The multiplicatively weighted Harary index [19],[57],[9]	$H_M(Id(Z_\rho)) = 3\rho^2 - 12\rho + 11$

$$M_2(Id(Z_\rho)) = \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} [\varpi(\sigma)\varpi(\varsigma)]. \tag{3. 20}$$

The general Randic co-index, alternatively known as the product-connectivity index, is defined as:

$$\overline{R_\alpha}(Id(Z_\rho)) = \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} [\varpi(\sigma)\varpi(\varsigma)]^\alpha. \tag{3. 21}$$

where α is a real number. In reference [32], generalized formulations for the generic Randic co-index for line graphs and the subdivision graph are offered.

Furthermore, the universal sum-connectivity co-index, defined in reference [48], is stated as:

$$\overline{SCI_\alpha}(Id(Z_\rho)) = \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))} [\varpi(\sigma) + \varpi(\varsigma)]^\alpha. \tag{3. 22}$$

where α is a real number. In reference [48], generalized formulas for numerous graph structures, including wheel graphs, star graphs, broom graphs, lollipop graphs, double star graphs, multi-star graphs, and friendship graphs (also known as fan graphs), are introduced.

Theorem 3.16. *The* $M_1(Id(Z_1)) = 0$, $M_1(Id(Z_2)) = 0$, *and for all* $\rho \geq 3$, $M_1(Id(Z_\rho)) = 2\rho^2 - 13\rho + 21$.

TABLE 3. The topological co-indices

Name of the Index	Formula for index
The first Zagreb co-index [49],[54]	$\overline{M_1}(Id(Z_\rho)) = \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))}[\varpi(\sigma)+\varpi(\varsigma)]$
The 2nd co-index [49],[54]	$\overline{M_2}(Id(Z_\rho)) = \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))}[\varpi(\sigma)\varpi(\varsigma)]$
The general Randic co-index [32]	$\overline{R_\alpha}(Id(Z_\rho)) = \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))}[\varpi(\sigma)\varpi(\varsigma)]^\alpha$
The general sum-connectivity co-index [48]	$\overline{SCI_\alpha}(Id(Z_\rho)) = \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))}[\varpi(\sigma)+\varpi(\varsigma)]^\alpha$

Proof. $M_1(Id(Z_1)) = 0$ and $M_1(Id(Z_2)) = 0$ because there are no non-adjacent vertices in $M_1(Id(Z_1))$ and $M_1(Id(Z_2))$. For $\rho \geq 3$, we have

$$\begin{aligned}
M_1(Id(Z_\rho)) &= \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))}[\varpi(\sigma)+\varpi(\varsigma)] \\
&= \sum_{(\sigma,\varsigma)\in\beta}[\varpi(\sigma)+\varpi(\varsigma)] \\
&= \sum_{(\sigma,\varsigma)\in\gamma}[\varpi(\sigma)+\varpi(\varsigma)] + \sum_{(\sigma,\varsigma)\in\beta-\gamma}[\varpi(\sigma)+\varpi(\varsigma)].
\end{aligned}$$

By theorem 3.5 **(c)** and **(d)**,

$$\begin{aligned}
M_1(Id(Z_\rho)) &= \sum_{(\sigma,\varsigma)\in\gamma}[1+2] + \sum_{(\sigma,\varsigma)\in\beta-\gamma}[2+2] \\
&= 3(\rho-3) + 4(\frac{(\rho-5)(\rho-3)}{2}) \\
&= 2\rho^2 - 13\rho + 21.
\end{aligned}$$

□

Theorem 3.17. *The* $M_1(Id(Z_1)) = 0$, $M_1(Id(Z_2)) = 0$, *and for all* $\rho \geq 3$, $M_1(Id(Z_\rho)) = 2\rho^2 - 13\rho + 21$.

Proof. $M_1(Id(Z_1)) = 0$ and $M_1(Id(Z_2)) = 0$ because there are no non-adjacent vertices in $M_1(Id(Z_1))$ and $M_1(Id(Z_2))$. For $\rho \geq 3$, we have

$$\begin{aligned}
M_1(Id(Z_\rho)) &= \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))}[\varpi(\sigma)+\varpi(\varsigma)] \\
&= \sum_{(\sigma,\varsigma)\in\beta}[\varpi(\sigma)+\varpi(\varsigma)] \\
&= \sum_{(\sigma,\varsigma)\in\gamma}[\varpi(\sigma)+\varpi(\varsigma)] + \sum_{(\sigma,\varsigma)\in\beta-\gamma}[\varpi(\sigma)+\varpi(\varsigma)].
\end{aligned}$$

By Theorem 3.5 **(c)** and **(d)**,

$$\begin{aligned} M_1(Id(Z_\rho)) &= \sum_{(\sigma,\varsigma)\in\gamma}[1+2] + \sum_{(\sigma,\varsigma)\in\beta-\gamma}[2+2] \\ &= 3(\rho-3)+4(\frac{(\rho-5)(\rho-3)}{2}) \\ &= 2\rho^2-13\rho+21. \end{aligned}$$

□

Theorem 3.18. *The* $M_2(Id(Z_1))=0$, $M_2(Id(Z_2))=0$ *and for all* $\rho\geq 3$ $M_2(Id(Z_\rho))=2(\rho-3)(\rho-4)$.

Proof. $M_2(Id(Z_1)), M_2(Id(Z_2))=0$ because there are no nonadjacent vertices in $M_2(Id(Z_1))$ and $M_2(Id(Z_1))$. For $\rho\geq 3$ we have

$$\begin{aligned} M_2(Id(Z_\rho)) &= \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))}[\varpi(\sigma)\varpi(\varsigma)] \\ &= \sum_{(\sigma,\varsigma)\in\beta}[\varpi(\sigma)\varpi(\varsigma)] \\ &= \sum_{(\sigma,\varsigma)\in\gamma}[\varpi(\sigma)\varpi(\varsigma)] + \sum_{(\sigma,\varsigma)\in\beta-\gamma}[\varpi(\sigma)\varpi(\varsigma)]. \end{aligned}$$

By Theorem 3.5 **(c)** and **(d)** ,

$$\begin{aligned} M_2(Id(Z_\rho)) &= \sum_{(\sigma,\varsigma)\in\gamma}[1\times 2] + \sum_{(\sigma,\varsigma)\in\beta-\gamma}[2\times 2] \\ &= 2(\rho-3)+4(\frac{(\rho-5)(\rho-3)}{2}) \\ &= 2(\rho-3)(\rho-4). \end{aligned}$$

□

Theorem 3.19. *The* $\overline{R_\alpha}(Id(Z_1))=0$, $\overline{R_\alpha}(Id(Z_1))=0$, *and for all* $\rho\geq 3$, $\overline{R_\alpha}(Id(Z_\rho))=\frac{1}{2}(\rho-3)(4^\alpha\rho+2.2^\alpha-5.4^\alpha)$.

Proof. $\overline{R_\alpha}(Id(Z_1))=0$ and $\overline{R_\alpha}(Id(Z_2))=0$ because there are no non-adjacent vertices in $(Id(Z_1))$. For $\rho\geq 3$, we have

$$\begin{aligned} \overline{R_\alpha}(Id(Z_\rho)) &= \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))}[\varpi(\sigma)\varpi(\varsigma)]^\alpha \\ &= \sum_{(\sigma,\varsigma)\in\beta}[\varpi(\sigma)\varpi(\varsigma)]^\alpha \\ &= \sum_{(\sigma,\varsigma)\in\gamma}[\varpi(\sigma)\varpi(\varsigma)]^\alpha + \sum_{(\sigma,\varsigma)\in\beta-\gamma}[\varpi(\sigma)\varpi(\varsigma)]^\alpha. \end{aligned}$$

By Theorem 3.5 **(c)** and **(d)**,

$$\begin{aligned}\overline{R_\alpha}(Id(Z_\rho)) &= \sum_{(\sigma,\varsigma)\in\gamma}[1\times 2]^\alpha + \sum_{(\sigma,\varsigma)\in\beta-\gamma}[2\times 2]^\alpha\\ &= 2^\alpha(\rho-3)+4^\alpha(\frac{(\rho-5)(\rho-3)}{2})\\ &= \frac{1}{2}(\rho-3)(4^\alpha\rho+2.2^\alpha-5.4^\alpha).\end{aligned}$$

□

Theorem 3.20. *The* $\overline{SCI_\alpha}(Id(Z_1)) = 0$, $\overline{SCI_\alpha}(Id(Z_1)) = 0$, *and for all* $\rho \geq 3$, $\overline{SCI_\alpha}(Id(Z_\rho)) = \frac{1}{2}(\rho-3)(4^\alpha\rho+2.3^\alpha-5.4^\alpha)$.

Proof. $\overline{SCI_\alpha}(Id(Z_1)) = 0$ and $\overline{SCI_\alpha}(Id(Z_1)) = 0$ parts of this theorem are obvious as there are no nonadjacent vertices in $(Id(Z_1))$ and $(Id(Z_1))$. For $\rho > 3$, we have

$$\begin{aligned}\overline{SCI_\alpha}(Id(Z_\rho)) &= \sum_{(\sigma,\varsigma)\notin E(Id(Z_\rho))}[\varpi(\sigma)+\varpi(\varsigma)]^\alpha\\ &= \sum_{(\sigma,\varsigma)\in\beta}[\varpi(\sigma)+\varpi(\varsigma)]^\alpha\\ &= \sum_{(\sigma,\varsigma)\in\gamma}[\varpi(\sigma)+\varpi(\varsigma)]^\alpha + \sum_{(\sigma,\varsigma)\in\beta-\gamma}[\varpi(\sigma)+\varpi(\varsigma)]^\alpha.\end{aligned}$$

By Theorem 3.5 **(c)** and **(d)**,

$$\begin{aligned}\overline{SCI_\alpha}(Id(Z_\rho)) &= \sum_{(\sigma,\varsigma)\in\gamma}[1+2]^\alpha + \sum_{(\sigma,\varsigma)\in\beta-\gamma}[2+2]^\alpha\\ &= 3^\alpha(\rho-3)+4^\alpha(\frac{(\rho-5)(\rho-3)}{2})\\ &= \frac{1}{2}(\rho-3)(4^\alpha\rho+2.3^\alpha-5.4^\alpha).\end{aligned}$$

□

TABLE 4. The topological co-indices

Name of the Index	Formula for index
The first Zagreb co-index [49],[54]	$M_1(Id(Z_\rho)) = 2\rho^2-13\rho+21$
The 2nd co-index [49],[54]	$M_2(Id(Z_\rho)) = 2(\rho-3)(\rho-4)$
The general Randic co-index [32]	$\overline{R_\alpha}(Id(Z_\rho)) = \frac{1}{2}(\rho-3)(4^\alpha\rho+2.2^\alpha-5.4^\alpha)$
The general sum-connectivity co-index [48]	$\overline{SCI_\alpha}(Id(Z_\rho)) = \frac{1}{2}(\rho-3)(4^\alpha\rho+2.3^\alpha-5.4^\alpha)$

3.21. **Mixed topological indices based on total distances eccentricities and degrees of vertices.** The total distance of a vertex $\sigma \in V(Id(Z_\rho))$ is represented by $D(\sigma)$ and is defined as $D(\sigma) = \sum_{\varsigma \in V(Id(Z_\rho))} d(\sigma, \varsigma)$. Gupta Singh introduced the eccentric distance sum index (EDSI) for the first time in 2012 in reference [27], defining it as:

$$Ed(G) = \sum_{x \in V(G)} (e(x) + D(x)). \tag{3. 23}$$

EDSI belongs to the first generation of topological indices addressing the total distance of vertices. In reference [27], EDSI was applied to evaluate the anti-HIV capabilities of dehydrogenation and was found to be substantially more efficient than the Wiener index. In the same year, Sardana and Mardan invented the neighboring eccentric distance sum index, which is defined as:

$$SV(G) = \sum_{x \in V(G)} \frac{e(x)D(x)}{\varpi(x)}. \tag{3. 24}$$

In reference [2], both of these indices were applied to algebraic structures, notably to the complement of the subgroup graph of dihedral groups. We extend this work to the context of rings and apply both these indices to $Id(Z_\rho)$.

TABLE 5. Mixed topological indices based on total distances eccentricities and degrees of vertices

Name of the Index	Polynomials for index
The eccentric distance sum index [27]	$Ed(G) = \sum_{x \varepsilon V(G)} (e(x) + D(x))$
The adjacent eccentric distance sum index [27]	$SV(G) = \sum_{x \varepsilon V(G)} \frac{e(x)D(x)}{\varpi(x)}$

Theorem 3.22. *For $\rho \geq 5$*
(a)

$$Ed(Id(Z_\rho)) = \sum_{\sigma \varepsilon V(G)} (e(\sigma) + D(\sigma)) = 4\rho^2 - 19\rho + 24.$$

(b)

$$SV(Id(Z_\rho)) = \sum_{\sigma \varepsilon V(G)} \frac{e(\sigma)D(\sigma)}{\varpi(\sigma)} = 6\rho - 15.$$

Proof. **(a)**

$$\begin{aligned} Ed(Id(Z_\rho)) &= \sum_{\sigma \varepsilon V(G)} (e(\sigma) + D(\sigma)) \\ &= \sum_{\sigma \varepsilon A} (e(x) + D(x)) + \sum_{\sigma \varepsilon B} (e(\sigma) + D(\sigma)) + \sum_{\sigma \varepsilon C} (e(\sigma) + D(\sigma)) \\ &= (1)(2)(2\rho - 5) + 1(1)(\rho - 2) + (\rho - 3)(2)(2\rho - 6) \\ &= 4\rho^2 - 19\rho + 24. \end{aligned}$$

(b)

$$
\begin{aligned}
SV(Id(Z_\rho)) &= \sum_{x\varepsilon V(G)} \frac{e(\sigma)D(\sigma)}{\varpi(\sigma)} \\
&= \sum_{\sigma\varepsilon A} \frac{e(\sigma)D(\sigma)}{\varpi(\sigma)} + \sum_{\sigma\varepsilon B} \frac{e(\sigma)D(\sigma)}{\varpi(v)} + \sum_{\sigma\varepsilon C} \frac{e(\sigma)D(\sigma)}{\varpi(\sigma)} \\
&= \frac{2(2\rho-5)}{1} + \frac{(1)(\rho-2)}{\rho-2} + \frac{2(2\rho-6)}{2} \\
&= 4\rho - 10 + 1 + 2\rho - 6 \\
&= 6\rho - 15.
\end{aligned}
$$

□

TABLE 6. Mixed topological indices based on total distances eccentricities and degrees of vertices

Name of the Index	Polynomials for index
The eccentric distance sum index [27]	$Ed(G) = 4\rho^2 - 19\rho + 24$
The adjacent eccentric distance sum index [27]	$SV(G) = 6\rho - 15$

4. CONCLUSIONS AND RECOMMENDATIONS

The employment of graphs to depict interactions inside algebraic structures is an engaging and new method. Graphs provide a visual depiction of the complicated connections between many features of these structures. This hypothesis is particularly noteworthy because of the essential role that topological indices play in sciences like chemistry and biology. These indices serve as crucial instruments in identifying chemicals, enabling the categorization and characterization of molecular compounds based on their structural properties.

Moreover, the applicability of topological indices extends beyond the sphere of natural sciences. In the realm of mathematics, notably in algebra, these indices offer a valuable technique of reducing complicated arithmetic operations on structures like Z_ρ. Algebraic structures are a key aspect of mathematical study, and their features and behaviors have been widely examined in the current literature. However, even with the quantity of knowledge accessible, there remain unknown territories, particularly in the comprehension of basic groups and rings.

Herein lies the potential of graph theory. It acts as a powerful tool to expose hidden linkages and connections within algebraic structures. By modeling these structures as graphs, we may visualize how individual parts of a group or ring are interrelated. This depiction goes beyond the standard mathematical notation, providing insights into the structural and relational properties of these algebraic structures.

Furthermore, whereas topological indices have been well-established in domains like chemistry and biology, there exists an intriguing possibility to create algebraic topological indices customized exclusively for algebraic structures. These specific indices would promote a greater understanding of the relationships inside these structures and give a means of simplifying algebraic computations and manipulations.

The union of graph theory and algebraic structures provides us an intriguing area for investigation. It allows us to utilize the power of visualization to obtain better insights into the links between elements and characteristics within algebraic entities. By creating algebraic topological indices, we can better our understanding and streamline arithmetic operations within these structures, ultimately enhancing our comprehension of fundamental mathematical topics.

5. Conflict of interest

In accordance with ethical standards, it is essential to declare that there is no conflict of interest associated with this research article. The authors affirm that they have no financial, personal, or professional relationships that could influence or be perceived to influence the work reported in this manuscript. This commitment to transparency ensures the integrity of the research process and the unbiased dissemination of findings, fostering credibility and trust within the scholarly community.

6. Funding

This research was conducted without external funding. The authors independently pursued this study without financial support or sponsorship from any public or private entity. The absence of funding sources underscores the self-driven nature of this research endeavor. The authors have dedicated their personal resources and time to contribute valuable insights to the scientific community, emphasizing the intrinsic motivation and commitment to advancing knowledge in the absence of external financial assistance.

References

[1] A. Abdussakir. *Some Topological Indices of Subgroup Graph of Symmetric Group*. Mathematics and Statistics **7**, no. 4 (2019): 98-105.

[2] A. Abdussakir, A. E. Susanti, N. Hidayati, and N. M. Ulya. *Eccentric distance sum and adjacent eccentric distance sum index of complement of subgroup graphs of dihedral group*. In Journal of Physics: Conference Series, vol. **1375**, no. 1, (2019) p. 012065 .

[3] A. Abdussakir, L. A. Puspitasari, W. H.Irawan, and E. Alisah. *Eccentric connectivity index of identity graph of cyclic group and finite commutative ring with unity*. In Journal of Physics: Conference Series, vol. **1375**, no. 1,(2019), p. 012067.

[4] M. R. Adhikari, A. Adhikari. Rings: *Introductory Concepts*. Basic Modern Algebra with Applications (2014): 159-201.

[5] M. H. Aftab, M. Rafaqat, and M. Hussain. *Topological invariants for the general structure of grape seed proanthocyanidins*. Punjab University Journal of Mathematics **54**, no. 1 (2022).

[6] N. I. Alimon,, N. H. Sarmin, and A. Erfanian. *The topological indices of non-commuting graph for quasidihedral group*. APPLIED ANALYSIS and MATHEMATICAL MODELING, ICAAMM (2018): 77.

[7] N. I. Alimon,, N. H. Sarmin, and A. Erfanian. *Topological indices of non-commuting graph of dihedral groups*. Malaysian Journal of Fundamental and Applied Sciences (2018): 473-476.

[8] Y. Alizadeh, A. Iranmanesh, and T. Doslic. *Additively weighted Harary index of some composite graphs*.Elsevier **13** (2013): 26-34.

[9] M. An and L. Xiong. *Multiplicatively Weighted Harary index of some Composite graphs*. Filomat **29**, no. 4 (2015): 795-805.

[10] D. F. Anderson, J. Fasteen, and J. D. LaGrange. *The subgroup graph of a group. Arabian Journal of Mathematics* **1**, no. 1 (2012): 17-27.

[11] V.Andova, D. Dimitrov, J. Fink, and R. Skrekovski. *Bounds on Gutman index*. Match- Communications in Mathematical and Computer Chemistry **67**, no. 2 (2012): 515.

[12] M. Azari and A. Iranmanesh. *Computing the eccentric-distance sum for graph operations*. Discrete Applied Mathematics **161**, no. 18 (2013): 2827-2840.

[13] B. Basavanagoud., and S. Policepatil. *Integrity of wheel related graphs*. Punjab University Journal of Mathematics **53**, no. 5 (2021).

[14] H. Bielak and K., Broniszewska. *Eccentric distance sum index for some classes of connected graphs*. Annales Universitatis Mariae Curie-Sklodowska, sectio AMathematica **71**, no. 2 (2017).

[15] P. M. Cohn, *Introduction to ring theory*. Springer Science & Business Media, 2012.

[16] M. R. Darafsheh. *Computation of topological indices of some graphs*. Acta applicandae mathematicae **110** (2010): 1225-1235.

[17] K.C. Das, K. Xu, and I. Gutman. *On Zagreb and harary indices*. MATCH Commun. Math. Comput. Chem **70**, no. 1 (2013): 301-314.

[18] N. De. *On eccentric connectivity index and polynomial of thorn graph*. Scientific Research Publishing (2012): 931-934.

[19] H. Deng, B. Krishnakumari, Y. B. Venkatakrishnan, and S. Balachandran. *Multiplicatively weighted Harary index of graphs*. Journal of Combinatorial Optimization **30** (2015): 1125-1137.

[20] T. Doslic, and M. Saheli. *Eccentric connectivity index of composite graphs*. Util. Math **95** (2014): 3-22.

[21] M. Eliasi, G. Raeisi, and B.Taeri. *Wiener index of some graph operations*. Discrete Applied Mathematics **160**, no. 9 (2012): 1333-1344.

[22] A. Erfanian and B. Tolue. *Conjugate graphs of finite groups*. Discrete Mathematics, Algorithms and Applications **4**, no. 02 (2012): 1250035

[23] B.Eskender, and E.Vumar. *Eccentric connectivity index and eccentric distance sum of some graph operations*. Transactions on Combinatorics **2**, no. 1 (2013): 103-111.

[24] B. Furtula, A. Graovac, and D. Vukicevic. *Augmented zagreb index*. Journal of mathematical chemistry **48** (2010): 370-380.

[25] B. Lucic, N. Trinajstic, and B. Zhou. *Comparison between the sum-connectivity index and product connectivity index for benzenoid hydrocarbons*. Chemical Physics Letters **475**, no. 1-3 (2009): 146-148.

[26] X. Geng, S. Li, and M. Zhang. *Extremal values on the eccentric distance sum of trees*. Discrete Applied Mathematics **161**, no. 16-17 (2013): 2427-2439.

[27] S. Gupta, M. Singh, and A. K. Madan. *Eccentric distance sum: A novel graph invariant for predicting biological and physical properties*. Journal of Mathematical Analysis and Applications **275**, no. 1 (2002): 386-401.

[28] E. Hanson-Smith, *A brief history of CALL theory*. CATESOL Journal **15**, no. 1 (2003): 21-30.

[29] A. Ili, G.Yu, and L. Feng. *On the eccentric distance sum of graphs*. Journal of Mathematical Analysis and Applications **381**, no. 2 (2011): 590-600.

[30] M. Jahandideh, M. R. Darafsheh, and N. Shirali. *Computation of topological indices of non-commuting graphs*. Ital. J. Pure Appl. Math **34** (2015): 299-310.

[31] M. Jahandideh., N. H. Sarmin, S. M. S. Omer, I. Ahvaz, and L. Benghazi. *The topological indices of non-commuting graph of a finite group*. Int. J. Pure Appl. Math **105**, no. 1 (2015): 27-38.

[32] M. N. Jauhari, and F. Ali. *Survey on topological indices and graphs associated with a commutative ring*. In Journal of Physics: Conference Series,IOP Publishing, **1562**,no. 1,(2020) , p.012008. , .

[33] M. Javaid, and M. Ibraheem. *HZ-Coindex for the Sum-Graphs under Cartesian Product*. Punjab University Journal of Mathematics **54**, no. 3 (2022).

[34] V. Kandasamy, W. B., F.Smarandache, and K. Ilanthenral. *Special set linear Algebra and special set fuzzy linear Algebra*. arXiv e-prints (2009): arXiv-0912.

[35] M.R. R. Kanna, P. Kumar, and D. Soner Nandappa. *Computation of topological indices of windmill grahph*. Int. J. Pure Appl. Math. **119**, no. 1 (2018): 89-98

[36] D. Keppens, *On the history of ring geometry (with a thematical overview of literature)*. arXiv preprint arXiv:2003.02881 (2020).

[37] J.Li, and W. C. Shiu. The harmonic index of a graph. The Rocky Mountain Journal of Mathematics 44, no. 5 (2014): 1607-1620.

[38] S. C. Li, Y. Y. Wu, and L. L. Sun. *On the minimum eccentric distance sum of bipartite graphs with some given parameters*. Journal of Mathematical Analysis and Applications **430**, no. 2 (2015): 1149-1162.

[39] X. Li and Y. Shi. *A survey on the Randic index*. MATCH Commun. Math. Comput. Chem **59**, no. 1 (2008): 127-156.

[40] J. Liu. *On the harmonic index of triangle-free graphs*. Appl. Math **4**, no. 08 (2013): 1204-1206.

[41] M. Mirzargar, *Some Distance-Based Topological Indices of a Non-Commuting Graph*. Hacettepe Journal of Mathematics and Statistics **41**, no. 4 (2012): 515-526.

[42] M. Mogharrab. *Eccentric connectivity index of bridge graphs*. Optoelectron. Adv. Mater. Commun. **4** (2010): 1866-7.

[43] Y. A. Nacaroglu. A. R., and A.D. Maden. *On the eccentric connectivity index of unicyclic graphs*. Iranian Journal of Mathematical Chemistry **9**, no. 1 (2018): 47-56.

[44] T. Oner, and E. Erol. *On Cm-Supermagicness of Book-Snake Graphs*. Punjab University Journal of Mathematics **53**, no. 4 (2021).

[45] P. Padmapriya, and V. Mathad. *The eccentric-distance sum of some graphs*. Electronic Journal of Graph Theory and Applications (EJGTA) **5**, no. 1 (2017): 51-62.

[46] H. Qu, and S. Cao. *On the adjacent eccentric distance sum index of graphs*. Plos one, **10**, no. 6 (2015): e0129497.

[47] H. S. Ramane, and V. V. Manjalapur. *Reciprocal Wiener index and reciprocal complementary Wiener index of line graphs*. Indian J. Discrete Math **1**, no. 1 (2015): 23-32.

[48] H. S. Ramane, V.V. Manjalapur, and I. Gutman. *General sum-connectivity index, general product-connectivity index, general Zagreb index and coindices of line graph of subdivision graphs*. AKCE International Journal of Graphs and Combinatorics **14**, no. 1 (2017): 92-100.

[49] R. Rasi, S. M. Sheikholeslami, and A. Behmaram. *An upper bound on the first Zagreb index in trees*. Iranian Journal of Mathematical Chemistry **8**, no. 1 (2017): 71-82.

[50] Z. Raza and S. Faizi. *Non-commuting graph of finitely presented group*. Sci. Int.(Lahore) **25**, no. 4 (2013): 883-885.

[51] S. Sardana, and A. K. Madan. *Predicting anti-HIV activity of TIBO derivatives: a computational approach using a novel topological descriptor*. Molecular modeling annual **8** (2002): 258-265.

[52] N. H. Sarmin, N. I. Alimon, and A. Erfanian. *Topological indices of the non-commuting graph for generalised quaternion group*. Bulletin of the Malaysian Mathematical Sciences Society **43**, no. 5 (2020): 3361-3367.

[53] J. Vahidi and A. A. Talebi. *The commuting graphs on groups D2n and Qn*. J. Math. Comput. Sci **1**, no. 2 (2010): 123-127.

[54] H.H. Wiener and Schultz. *Molecular topological indices of graphs with specified cut edges*. Match **61**, no. 3 (2009): 643.

[55] R. Xing, Bo Zhou and F. Dong. *On atom bond connectivity index of connected graphs*. Discrete Applied Mathematics **159**, no. 15 (2011): 1617-1630.

[56] K. Xu,, and K.C. Das. *On Harary index of graphs*. Discrete applied mathematics **159**, no. 15 (2011): 1631-1640.

[57] K. Xu, J. Wang, K.C. Das, and S. Klavzar. *Weighted Harary indices of apex trees and k-apex trees*. Discrete Applied Mathematics **189** (2015): 30-40.

[58] Z., Xue, *Group Theory and Ring Theory*. In Journal of Physics: Conference Series IOP Publishing, **2386**, no. 1, (2022) p. 012024.

[59] G. Yu, L. Feng, and A. Ilic. *On the eccentric distance sum of trees and unicyclic graphs*. Journal of Mathematical Analysis and Applications **375**, no. 1 (2011): 99-107.

[60] X. Zhang, H.Yang, Y.Gao, and M. R. Farahani. *Some degree-based topological indices of base-3 Sierpiski graphs*. Science **5**, no. 3 (2017): 36-41.

[61] L. Zhong. *The harmonic index for graphs*. Applied mathematics letters **25**, no. 3 (2012): 561-566.

[62] L. Zhong, and K. Xu. *The harmonic index for bicyclic graphs*. Utilitas Math **90** (2013): 23-32.
[63] B. Zhou, X. Cai, and N. Trinajsti. *On harary index*. Journal of Mathematical Chemistry **44** (2008): 611-618.