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Intersections of Pell, Pell-Lucas Numbers and Sums of Two Jacobsthal Numbers

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Abstract. We use tools from Baker's linear forms theory to solve the two Diophantine equations $P_k = J_n + J_m$ and $Q_k = J_n + J_m$ where $\{P_k\}_{k\geq 0}$, $\{Q_k\}_{k\geq 0}$ and $\{J_k\}_{k\geq 0}$ are the sequences of Pell, Pell-Lucas and Jacobsthal numbers, respectively. The strategy depends mainly on the properties of the three sequences and Matveev's inequality which is an indispensable tool in the theory of linear forms. Also, we employ an inequality due to A. Dujella and A. Pethö to reduce the too large bounds obtained by Matveev's theorem.

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1. Introduction

The Pell sequence is defined recursively by $P_0 = 0, P_1 = 1$ and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 1$. A few terms of this sequence are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, ...$$

Pell-Lucas numbers are defined by $Q_0 = 2$, $Q_1 = 2$, and $Q_{n+1} = 2Q_n + Q_{n-1}$ for $n \geq 1$. Its first terms are

$$2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, ...$$

For some recent works related to Diophantine equations which includes Pell and Pell-Lucas numbers, see [1], [2], [5], [9] and [15].
Jacobsthal sequence is defined by $J_0 = 0$, $J_1 = 1$, and $J_{n+1} = J_n + 2J_{n-1}$ for $n \geq 1$. Its initial terms are

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, ...$$

There is a vast variety of applications of these three sequences in mathematics and applied sciences. We refer to [12] and [13] for a large number of these applications.

This work is motivated by the following remark: with a cursory glance, it turns out that

$$P_1 = J_1 + J_0, P_2 = J_1 + J_1$$

and

$$Q_0 = J_1 + J_1, Q_1 = J_1 + J_1.$$

The problem of expressing terms of an integer sequence as the sum, difference and product of terms of other integer sequences is an intensive research area. For instance, see [1], [8] and [16]. This article is devoted to investigate the solutions of the Diophantine equations:

$$P_k = J_n + J_m \tag{1. 1}$$

and

$$Q_k = J_n + J_m. \tag{1. 2}$$

The complete sets of solutions of the two equations are given in the following theorems.

Theorem 1.1. *Let $n \geq m$. Then the only non-negative triples (k, n, m) which satisfy Eq.(1. 1) are*

$$(1, 1, 0), (2, 2, 1), (2, 2, 2), (3, 4, 0), (4, 5, 1), (4, 5, 2).$$

Theorem 1.2. *Let $n \geq m$. Then the only non-negative triples (k, n, m) which satisfy Eq.(1. 2) are*

$$(0, 1, 1), (1, 1, 1), (1, 2, 1), (1, 2, 2), (2, 3, 3), (2, 4, 1), (2, 4, 2), (3, 5, 3), (0, 2, 1), (1, 2, 1).$$

The approach is to employ one of the best known variants of the Baker's theory due to Matveev to bound all the implied variables in terms of a single one. The upper bound obtained by Matveev's inequality is too large to be investigated. As a result, we consider a reduction inequality of Dujella and Pethö to cut down this upper bound. At last, we run Sage computations to determine all the solutions.

2. Number theoretic backgrounds

2.1. **Pell and Pell-Lucas sequences.** The characteristic equation of Pell and Pell-Lucas sequences is

$$\Psi(x) := x^2 - 2x - 1 = 0,$$

and their Binet formula is, assuming $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$,

$$P_k = \frac{\gamma^k - \delta^k}{2\sqrt{2}} \quad \text{for all} \quad k \geq 0. \tag{2. 3}$$

and

$$Q_k = \gamma^k + \delta^k \quad \text{for all} \quad k \geq 0. \tag{2. 4}$$

It is straightforward to show that

$$\gamma^{k-2} \leq P_k \leq \gamma^{k-1} \quad \text{holds for all} \quad k \geq 1. \tag{2. 5}$$

and

$$\gamma^{k-1} < Q_k < \gamma^{k+1} \quad \text{holds for all} \quad k \geq 0. \tag{2. 6}$$

2.2. **Jacobsthal sequence.** The characteristic polynomial of the Jacobsthal sequence is

$$G(x) = x^2 - x - 2.$$

Its Binet formula is

$$J_n = \frac{2^n - (-1)^n}{3}. \tag{2. 7}$$

A direct induction argument shows that

$$2^{n-2} \leq J_n \leq 2^{n-1} \quad \text{for all} \quad n \geq 1. \tag{2. 8}$$

Basic properties of the Jacobsthal numbers can be found in [10] and [13].

2.3. **Linear forms in logarithms.** Consider an algebraic number κ. Suppose the minimal polynomial (over $\mathbb{Z}$) of κ has degree m and let $\kappa^{(i)}$'s be the conjugates of κ. Then the minimal polynomial of κ can be written as

$$c_0 x^m + c_1 x^{m-1} + ... + c_m = c_0 \prod_{i=1}^{m} (x - \kappa^{(i)}),$$

where c_0 is a positive integer . The logarithmic Weil height (over an algebraic real field) of κ is given by

$$h(\alpha) := \frac{1}{m} \left(\log c_0 + \sum_{i=1}^{m} \log \left(\max \left\{ \left| \kappa^{(i)} \right|, 1 \right\} \right) \right)$$

The function h of logarithmic height satisfies the following properties (see [3] for proofs):

$$\begin{aligned} h(\kappa_1 \pm \kappa_2) &\leq h(\kappa_1) + h(\kappa_2) + \log 2; \\ h\left(\kappa_1 \kappa_2^{\pm 1}\right) &\leq h(\kappa_1) + h(\kappa_1); \\ h(\kappa^s) &= |s|\, h(\kappa) \quad (s \in \mathbb{Z}). \end{aligned}$$

The next inequality of linear forms is fundamental. Bugeaud, Mignotte and Siksek deduced it, see [4], from Matveev's theorem [14].

Theorem 2.4. $(\mathbf{Matveev})$ *Let $\mathbb{A}$ be a real algebraic number field of degree $d_{\mathbb{A}}$ and $\kappa_1, ..., \kappa_r$ be positive real algebraic numbers in $\mathbb{A}$. Suppose that $t_1, ..., t_r$ are non zero integers such that the quantity*

$$\Lambda_1 := \kappa_1^{t_1} \kappa_2^{t_2} ... \kappa_r^{t_r} - 1 \neq 0.$$

Then,

$$\log |\Lambda| > -1.4 \cdot 30^{r+3} \cdot r^{4.5} \cdot d_{\mathbb{A}}^2 \cdot (1 + \log d_{\mathbb{A}}) \cdot (1 + \log B) A_1 ... A_r, \tag{2. 9}$$

where

$$B \geq \max\{|t_1|, ..., |t_r|\},$$

and

$$A_i \geq \max\{d_{\mathbb{A}} h(\kappa_i), |\log \kappa_i|, 0.16\}, \quad \textit{for all} \quad i = 1, ..., r.$$

2.5. **Dujella and Pethö reduction lemma.** Let λ be a real number. Set $||\lambda||:= min\{|\lambda - n| : n \in \mathbb{Z}\}$. A. Dujella and A. Pethö proved, in[7], the following result . It plays a basic rule in reducing the upper bound obtained in many Diophantine equations.

Lemma 2.6. *Let $M > 0$ be an integer. Let $\tau, \mu, A > 0, B > 1$ be given real numbers. Suppose $\frac{p}{q}$ is a convergent of τ with $q > 6M$ and $\epsilon := ||\mu q|| - M||\tau q|| > 0$. If (n, m, ω) is to*

$$0 < |n\tau - m + \mu| < \frac{A}{B^{\omega}}$$

with $n, m, \omega > 0, n \leq M$, then

$$\omega < \frac{\log\left(\frac{Aq}{\epsilon}\right)}{\log B}.$$

Example 2.7. *In solving the problem of determining all the balancing numbers which are expressible as the sum of two Jacobsthal numbers ($B_k = J_n + J_m$.) we arrive at the inequality*

$$0 < \left| \frac{\log(\frac{3}{2\sqrt{8}})}{\log 2} - n + k\left(\frac{\log\rho}{\log 2}\right)\right| < \frac{8}{2^{n-m}\log 2} < \frac{12}{2^{n-m}}, \tag{2. 10}$$

where $\rho = 3 + \sqrt{8}$. We apply the reduction lemma with $M = 6 \times 10^{29}$ ($M > 2n > k$) , $\tau = \frac{\log\rho}{\log 2}$, $\mu = \frac{\log(\frac{3}{2\sqrt{8}})}{\log 2}$, $A = 12$, $B = 2$. Writing τ as a continued fraction $[a_0, a_1, ...]$, we see that $q_{61} = 6332847229674209482244367144203 > 6M$. Computing

$$\epsilon = ||\mu q_{61}|| - M\,||\tau q_{61}|| > 0.4.$$

It follows that $n - m < 108$.

2.8. **Legendre theorem.** The following theorem is due to Legendre and will be used in some cases of our investigation. Further details can be found in [6].

Theorem 2.9. *Let x be a real number, let $p, q \in \mathbb{Z}$ and let $x = [a_0, a_1, ...]$. If*

$$\left|\frac{p}{q} - x\right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$ is a convergent continued fraction of x. Furthermore, let M and n be non-negative integers with $q_n > M$. Put $b = \max\{a_i : i = 0, 1, 2, .., n\}$, then

$$\frac{1}{(b+2)\,q^2} < \left|\frac{p}{q} - x\right|.$$

Example 2.10. *Suppose that we have the inequality*

$$\rho^k 2^{-(m+3)} - 1 < \frac{1}{2^m},$$

with $\rho = 3 + \sqrt{8}$, $m < 3 \times 10^{29}$ and $k < 2m$. This gives, for $m \geq 3$, that

$$\left|k\frac{\log\rho}{\log 2} - (m+3)\right| < \frac{4}{2^m} < \frac{1}{4}.$$

Employing that $16m < 2^m$ *for* $m \geq 7$, *we deduce that* $\frac{4}{2^m} < \frac{1}{2k^2}$. *Then* $\left|\frac{\log \rho}{\log 2} - \frac{m+3}{k}\right| < \frac{1}{2k^2}$. *So, by Legendre's theorem,* $\frac{m+3}{k}$ *is a convergent of* $\frac{\log \rho}{\log 2}$. *If* $k < M = 6 \times 10^{29}$, *then some computations show that*

$$q_{58} < M < q_{59} \quad \textit{and} \quad b := \max\{a_i : i = 0, 1, 2, ..., 59\} = 200.$$

Consequently,

$$\frac{1}{202k} < \frac{4}{2^m}.$$

3. Solution of $P_k = J_n + J_m$

3.1. **Bounding the variables.** Applying the inequalities (2. 8) and (2. 5) to establish the relationship between k and n, we get

$$\gamma^{k-2} \leq P_k \leq 2^n \text{ and } 2^{n-2} \leq P_k \leq \gamma^{k-1}. \tag{3. 11}$$

These imply that

$$(n-2)\frac{\log 2}{\log \gamma} + 1 \leq k \leq n\frac{\log 2}{\log \gamma} + 2. \tag{3. 12}$$

We can consider $k < 2n$. Using the Binet formulas of the Pell and Jacobsthal sequences in Eq.(1. 1), we obtain

$$\frac{\gamma^k - \delta^k}{2\sqrt{2}} = \frac{2^n - (-1)^n}{3} + \frac{2^m - (-1)^m}{3}. \tag{3. 13}$$

Then

$$\left|\frac{\gamma^k}{2\sqrt{2}} - \frac{2^n}{3}\right| = \left|\frac{2^m}{3} - \frac{((-1)^n + (-1)^m)}{3} + \frac{\delta^k}{2\sqrt{2}}\right|. \tag{3. 14}$$

It follows that

$$\left|\frac{\gamma^k}{2\sqrt{2}} - \frac{2^n}{3}\right| < \frac{4 \cdot 2^m}{3}. \tag{3. 15}$$

Thus,

$$\left|\frac{3\gamma^k 2^{-n}}{2\sqrt{2}} - 1\right| < \frac{4}{2^{n-m}}. \tag{3. 16}$$

Let

$$\Lambda_1 = \frac{3\gamma^k 2^{-n}}{2\sqrt{2}} - 1, \quad r = 3, \quad \kappa_1 = \frac{3}{2\sqrt{2}}, \quad \kappa_2 = \gamma, \quad \kappa_3 = 2, \quad t_1 = 1, \quad t_2 = k, \quad t_3 = -n.$$

If $\Lambda_1 = 0$, then $3\gamma^k = 2^n.2\sqrt{2}$. Consider the automorphism σ such that $\sigma(\gamma) = \delta$. Then $\left|3\delta^k\right| = 2^n.2\sqrt{2}$. But $\left|3\delta^k\right| < 3$, then $2^n.2\sqrt{2} < 3$ which is a contradiction. So, $\Lambda_1 \neq 0$.

Take $\mathbb{A} = \mathbb{Q}(\gamma)$. Then $d_{\mathbb{A}} = 2$. The logarithmic heights are

$$h(\kappa_1) \leq h(3) + h(2\sqrt{2}) \leq \log 3 + \frac{3}{2}\log 2;$$

$$h(\kappa_2) = \frac{1}{2}\log \gamma;$$

$$h(\kappa_3) = \log 2.$$

Taking

$$A_1 = 2\log 3 + 3\log 2, \quad A_2 = \log \gamma, \quad \text{and} \quad A_3 = 2\log 2, B = 2n$$

we obtain

$\log|\Lambda_1| > -1.4\times 30^6\times 3^{4.5}\times 4\times(1+\log 2)(1+\log 2n)(2\log 3+3\log 2)(2\log 2\log\gamma)$.

Then

$$\log|\Lambda_1| > -6\times 10^{12}(1+\log 2n). \tag{3. 17}$$

Also, from Eq.(3. 16) we have,

$$\log|\Lambda_1| < \log 4 + (m-n)\log 2. \tag{3. 18}$$

Comparing inequalities in (3. 17) and (3. 18)gives

$$(n-m)\log 2 - \log 4 < 6\times 10^{12}(1+\log 2n). \tag{3. 19}$$

Hence,

$$m\log 2 > n\log 2 - 6\times 10^{12}(1+\log 2n) - \log 4. \tag{3. 20}$$

Eq.(3. 13) is the same as

$$\frac{\gamma^k}{2\sqrt{2}} - \frac{2^n(1+2^{m-n})}{3} = \frac{\delta^k}{2\sqrt{2}} - \frac{(-1)^n-(-1)^m}{3}, \tag{3. 21}$$

Therefore

$$\left|\frac{3\gamma^k 2^{-n}}{2\sqrt{2}(1+2^{m-n})} - 1\right| = \left|\frac{3\cdot 2^{-n}}{1+2^{m-n}}\left(\frac{\delta^k}{2\sqrt{2}} - \frac{(-1)^n-(-1)^m}{3}\right)\right|. \tag{3. 22}$$

Hence,

$$\left|\frac{3\gamma^k 2^{-n}}{2\sqrt{2}(1+2^{m-n})} - 1\right| < \frac{5}{2^m}. \tag{3. 23}$$

Let $\Lambda_2 = \frac{3\gamma^k 2^{-n}}{2\sqrt{2}(1+2^{m-n})} - 1$. Then

$$\log|\Lambda_2| < \log 5 - m\log 2. \tag{3. 24}$$

Let

$$\kappa_1 = \frac{3}{2\sqrt{2}(1+2^{m-n})},\quad \kappa_2=\gamma,\quad \kappa_3 = 2,\quad r=3,\quad t_1=1,\quad t_2=k,\quad t_3=-n, B=2n.$$

First we show that $\Lambda_2 \neq 0$. If $\Lambda_2 = 0$, then $3\gamma^k = 2\sqrt{2}(2^n+2^m)$. Consider the automorphism σ such that $\sigma(\gamma)=\delta$. Then $\left|3\delta^k\right| = 2\sqrt{2}(2^n+2^m)$. But $\left|3\delta^k\right| < 3$, which is a contradiction. Then we take $\mathbb{A}=\mathbb{Q}(\gamma)$, for which $d_{\mathbb{A}}=2$. The logarithmic heights are computed as follows:

$$h(\kappa_1) \le h(3)+h(2)+h(2\sqrt{2})+h(1+2^{m-n}) \le \log 3 + \frac{5}{2}\log 2 + (n-m)\log 2;$$
$$h(\kappa_2) = \frac{1}{2}\log\gamma;$$
$$h(\kappa_3) = \log 2.$$

We take

$$A_1 = 2\log 3 + 2(n-m)\log 2 + 5\log 2,\quad A_2 = \log\gamma,\quad \text{and}\quad A_3 = 2\log 2.$$

By Matveev's inequality, we get

$$\log|\Lambda_2| > c(1+\log 2n)(2\log 3 + 2(n-m)\log 2 + 5\log 2)(2\log 2\log\gamma),$$

where $c = -1.4 \times 30^6 \times 3^{4.5} \times 4 \times (1 + \log 2)$. Using Eqs.(3. 19),(3. 20),(3. 24) and some simple manipulations, we find

$$n \log 2 < 6 \times 10^{13}(1 + \log 2n) + 24 \times 10^{27}(1 + \log 2n)^2 + 3. \tag{3. 25}$$

A Mathematica computation reveals that

$$n < 2 \times 10^{29}. \tag{3. 26}$$

3.2. **Reducing the upper bound.** Now, we use the inequality of A. Dujella and A. Pethö to cut down the previous on n. Let

$$\Gamma_1 = \log(\frac{3}{2\sqrt{2}}) + k \log \gamma - n \log 2.$$

Eq.(3. 16) entails that

$$|\Lambda_1| = \left|e^{\Gamma_1} - 1\right| < \frac{4}{2^{n-m}} < \frac{1}{4}, n - m > 4. \tag{3. 27}$$

This implies that

$$|\Gamma_1| < \frac{1}{2}. \tag{3. 28}$$

Then $|\Gamma_1| < 2\left|e^{\Gamma_1} - 1\right|$. Therefore we get

$$|\Gamma_1| < \frac{8}{2^{n-m}}. \tag{3. 29}$$

We observe that $\Gamma_1 \neq 0$. Then

$$0 < \left|\frac{\log(\frac{3}{2\sqrt{2}})}{\log 2} - n + k\left(\frac{\log \gamma}{\log 2}\right)\right| < \frac{8}{2^{n-m}\log 2} < \frac{12}{2^{n-m}}. \tag{3. 30}$$

Let $M = 4 \times 10^{29}$ $(M > 2n > k)$, $\tau = \frac{\log \alpha}{\log 2}$, $\mu = \frac{\log(\frac{3}{2\sqrt{2}})}{\log 2}$, $A = 12$, $B = 2$. Writing τ as a continued fraction , we get $q_{65} = 2427228558134035529638808203392547 > 6M$. We compute

$$\epsilon = ||\mu q_{65}|| - M\,||\tau q_{65}|| > 0.1.$$

Thus $n - m < 118$. Now we let

$$\Gamma_2 = \log\left(\frac{3}{2\sqrt{2}(1 + 2^{m-n})}\right) + k \log \gamma - n \log 2.$$

Then we have from Eq.(3. 23) that, for $m \geq 5$,

$$|\Lambda_2| = \left|e^{\Gamma_2} - 1\right| < \frac{5}{2^m} < \frac{1}{4}. \tag{3. 31}$$

Consequently,

$$|\Gamma_2| < \frac{1}{2}. \tag{3. 32}$$

Then $|\Gamma_2| < 2\left|e^{\Gamma_2} - 1\right|$. Therefore

$$|\Gamma_2| < \frac{10}{2^m}. \tag{3. 33}$$

We observe that $\Gamma_2 \neq 0$ since $\Lambda_2 \neq 0$. Then

$$0 < \left| \frac{\log\left(\frac{3}{2\sqrt{2}(1+2^{m-n})}\right)}{\log 2} - n + k\left(\frac{\log\gamma}{\log 2}\right) \right| < \frac{15}{2^m}. \qquad (3.\ 34)$$

We apply lemma 2.6 with $M = 4\times 10^{29}$ $(M > 2n > k)$, $\tau = \frac{\log\gamma}{\log 2}$, $\mu = \frac{\log(\frac{3}{2\sqrt{2}(1+2^{m-n})})}{\log 2}$, $A = 15$ and $B = 2$. It can be shown that $q_{65} = 242722855813403552963880820339254 7 > 6M$. Using Sage to investigate all the values of ϵ such that $n - m < 118$, we see that

$$\epsilon = ||\mu q_{65}|| - M\,||\tau q_{65}|| > 0.01.$$

Thus ,by Lemma 2.6, it follows that $m < 122$. So, $n < 240$ and $k < 480$. Solving Eq.(1. 1) for $m < 122$, $n < 240$ and $k < 480$, we get the results in Theorem (1.1).

4. Solution of $Q_k = J_n + J_m$

By symmetry of Eq.(1. 2), we assume that $n \geq m$.

4.1. **Bounding the variables.** By (2. 6) and (2. 8), we have

$$\alpha^{k-2} \leq R_k \leq 2^n \text{ and } 2^{n-2} \leq R_k \leq \alpha^{k+1}. \qquad (4.\ 35)$$

We conclude that

$$(n-2)\frac{\log 2}{\log\gamma} - 1 \leq k \leq (n-1)\frac{\log 2}{\log\alpha} + 1. \qquad (4.\ 36)$$

We can take $k < 2n$. Replacing the Pell-Lucas and Jacobsthal sequences in Eq.(1. 2) by their Binet formulas, we get

$$\gamma^k + \delta^k = \frac{2^n - (-1)^n}{3} + \frac{2^m - (-1)^m}{3}. \qquad (4.\ 37)$$

Then

$$\left|\gamma^k - \frac{2^n}{3}\right| = \left|\frac{2^m}{3} - \frac{((-1)^n + (-1)^m)}{3} - \delta^k\right|. \qquad (4.\ 38)$$

Easily, it follows that

$$\left|\gamma^k - \frac{2^n}{3}\right| \leq \frac{2^m}{3} + \frac{2}{3} + \left|\delta^k\right|.$$

Then

$$\left|\gamma^k - \frac{2^n}{3}\right| < \frac{4\cdot 2^m}{3} \qquad (4.\ 39)$$

So

$$\left|\frac{3\gamma^k}{2^n} - 1\right| < \frac{4}{2^{n-m}}. \qquad (4.\ 40)$$

Consider the following:

$$\Lambda_3 = 3\gamma^k 2^{-n} - 1,\ \ r = 3,\ \ \kappa_1 = 3,\ \ \kappa_2 = \gamma,\ \ \kappa_3 = 2,\ \ t_1 = 1,\ \ t_2 = k,\ \ t_3 = -n.$$

Again $\Lambda_3 \neq 0$. Let $\mathbb{A} = \mathbb{Q}(\gamma)$. Then

$$h(\kappa_1) = \log 3;$$

$$h(\kappa_2) = \frac{1}{2}\log\gamma;$$

$$h(\kappa_3) = \log 2.$$

We take,

$$A_1 = 2\log 3, \quad A_2 = \log\gamma, \quad \text{and} \quad A_3 = 2\log 2.$$

Let $B = 2n$. Then Theorem(2.4) shows that

$$\log|\Lambda_3| > -1.4 \times 30^6 \times 3^{4.5} \times 4 \times (1+\log 2)(1+\log 2n)(2\log 3)(2\log 2\log\gamma).$$

Consequently,

$$\log|\Lambda_3| > -3 \times 10^{12}(1+\log 2n). \tag{4. 41}$$

Then also from (4. 40) we have

$$\log|\Lambda_3| < \log 4 + (m-n)\log 2. \tag{4. 42}$$

Thus by comparing inequalities in (4. 41) and (4. 42) we get

$$(n-m)\log 2 - \log 6 < 3 \times 10^{12}(1+\log 2n). \tag{4. 43}$$

Hence,

$$m\log 2 > n\log 2 - 3\times 10^{12}(1+\log 2n) - \log 4. \tag{4. 44}$$

Eq.(4. 37) is equivalent to

$$\gamma^k - \frac{2^n(1+2^{m-n})}{3} = -\delta^k - \frac{(-1)^n - (-1)^m}{3}. \tag{4. 45}$$

So

$$\left|\frac{3\gamma^k 2^{-n}}{1+2^{m-n}} - 1\right| = \left|\frac{3\cdot 2^{-n}}{1+2^{m-n}}\left(-\delta^k - \frac{(-1)^n-(-1)^m}{3}\right)\right|. \tag{4. 46}$$

Then

$$\left|\frac{3\gamma^k 2^{-n}}{1+2^{m-n}} - 1\right| < \frac{5}{2^m}. \tag{4. 47}$$

Let $\Lambda_4 = \frac{3}{1+2^{m-n}}\gamma^k 2^{-n} - 1$. Hence,

$$\log|\Lambda_4| < \log 5 - m\log 2. \tag{4. 48}$$

Set

$$\eta_1 = \frac{3}{1+2^{m-n}}, \quad \eta_2 = \gamma, \quad \eta_3 = 2, \quad r = 3, \quad t_1 = 1, \quad t_2 = k, \quad t_3 = -n.$$

As before we can show that $\Lambda_4 \neq 0$. Let $\mathbb{A} = \mathbb{Q}(\gamma)$. Then,

$$h(\kappa_1) \leq \log 3 + (n-m)\log 2 + \log 2;$$

$$h(\kappa_2) = \frac{1}{2}\log\gamma;$$

$$h(\kappa_3) = \log 2.$$

Taking

$$A_1 = 2\log 3 + 2(n-m)\log 2 + 2\log 2, \quad A_2 = \log\gamma, \quad \text{and} \quad A_3 = 2\log 2 \; and \; B = 2n,$$

we obtain

$$\log|\Lambda_4| > -1.4\times 30^6\times 3^{4.5}\times 4\times(1+\log 2)(1+\log 2n)(2\log 3+2\log 2+2(n-m)\log 2)(2\log 2\log\gamma),$$

Eqs.(4. 43), (4. 44) and (4. 48) with routine computations give

$$n\log 2 < 11\times 10^{12}(1+\log 2n) + 12\times 10^{24}(1+\log 2n)^2 + 3. \tag{4. 49}$$

Hence,

$$n < 3 \times 10^{28}. \tag{4. 50}$$

4.2. **Reducing the upper bound.** Assume that $n - m \geq 5$. Let

$$\Gamma_3 = \log(3) + k \log \gamma - n \log 2.$$

By Eq.(4. 40), we have

$$|\Lambda_3| = \left|e^{\Gamma_3} - 1\right| < \frac{4}{2^{n-m}} < \frac{1}{4}, \tag{4. 51}$$

So

$$|\Gamma_3| < \frac{1}{2}. \tag{4. 52}$$

Then, $|\Gamma_3| < 2\left|e^{\Gamma_3} - 1\right|$. So

$$|\Gamma_3| < \frac{8}{2^{n-m}}. \tag{4. 53}$$

We observe that $\Gamma_3 \neq 0$. It follows that

$$0 < \left|\frac{\log 3}{\log 2} - n + k\left(\frac{\log \gamma}{\log 2}\right)\right| < \frac{8}{2^{n-m}\log 2} < \frac{12}{2^{n-m}}. \tag{4. 54}$$

Let $M = 6 \times 10^{28}$ $(M > 2n > k)$, $\tau = \frac{\log \gamma}{\log 2}$, $\mu = \frac{\log 3}{\log 2}$, $A = 12$, $B = 2$. Considering the continued fraction of τ, we find that $q_{65} > 6M$. We compute

$$\epsilon = ||\mu q_{65}|| - M\,||\tau q_{65}|| > 0.3.$$

Thus, by Lemma 2.6, we get $n - m < 117$. Set

$$\Gamma_4 = \log\left(\frac{3}{1 + 2^{m-n}}\right) + k \log \gamma - n \log 2.$$

and let $m > 5$. Then we have from Eq.(4. 47) that

$$|\Lambda_4| = \left|e^{\Gamma_4} - 1\right| < \frac{5}{2^m} < \frac{1}{4}. \tag{4. 55}$$

We conclude that

$$|\Gamma_4| < \frac{1}{2}. \tag{4. 56}$$

Thus $|\Gamma_4| < 2\left|e^{\Gamma_4} - 1\right|$. Therefore we get

$$|\Gamma_4| < \frac{10}{2^m}. \tag{4. 57}$$

We observe that $\Gamma_4 \neq 0$. So

$$0 < \left|\frac{\log\left(\frac{3}{1+2^{m-n}}\right)}{\log 2} - n + k\left(\frac{\log \gamma}{\log 2}\right)\right| < \frac{15}{2^m}. \tag{4. 58}$$

Let $M = 6 \times 10^{28}$ $(M > 2n > k)$, $\tau = \frac{\log \gamma}{\log 2}$, $\mu = \frac{\log\left(\frac{3}{1+2^{m-n}}\right)}{\log 2}$, $A = 15$, $B = 2$. We have
$q_{65} > 6M$. We consider the values of ϵ in the following two cases
Case I: if $n - m < 117$ and $n - m \neq 1$, we find that

$$\epsilon = ||\mu q_{65}|| - M\,||\tau q_{65}|| > 0.01.$$

Thus by Lemma 2.6, we get $m < 122$. Then $n < 239$ and $k < 478$.

Case II: for $n - m = 1$ we get ϵ always negative. So, we solve Eq.(1. 2) if $n - m = 1$. In this case, Eq.(1. 2) can be written as

$$Q_k = 2^m. \tag{4. 59}$$

Then $k < 2m$ and from Eq.(4. 50) we get $m < 3 \times 10^{28}$. As before, We can prove that

$$\alpha^k 2^{-m} - 1 < \frac{1}{2^m},$$

This gives, for $m \geq 3$, that

$$\left| k\frac{\log\gamma}{\log 2} - m \right| < \frac{4}{2^m} < \frac{1}{4}.$$

Using the relation $16m < 2^m$ for $m \geq 7$, we deduce that $\frac{4}{2^m} < \frac{1}{2k^2}$. Then $\left|\frac{\log\gamma}{\log 2} - \frac{m}{k}\right| < \frac{1}{2k^2}$. So, by Legendre's theorem, $\frac{m}{k}$ is a convergent of $\frac{\log\gamma}{\log 2}$. Using $k < M$ and some Sage computations we find that

$$q_{53} < M < q_{54} \quad \text{and} \quad b := \max\{a_i : i = 0, 1, 2, ..., 54\} = 100.$$

Consequently,

$$\frac{1}{(2+100)\,k} < \frac{4}{2^m}.$$

Thus

$$2^m < 3 \cdot 10^{31}.$$

Then $m \leq 104$. Solutions of Eq.(4. 59) for $m \leq 104$. and Eq.(1. 2) for $m < 122$, $n < 239$ and $k < 478$ give the results in Theorem (1.2).

5. Conclusion

The present paper is a contribution to the broad area of the theory of Diophantine equations through the solutions of the two equations $P_k = J_n + J_m$ and $Q_k = J_n + J_m$. We revealed all the Pell and Pell-Lucas numbers that are sums of two Jacobsthal numbers. The approach we used is mainly dependent on Matveev's theorem which is a basic technique in the theory of linear forms in logarithms of algebraic numbers. Also, we employed a reduction lemma due to Dujella and Pethö to greatly decrease the obtained upper bound. We found that there are six Pell numbers and ten Pell-Lucas numbers which are expressible as the sum of two Jacobsthal numbers. It opens the door to the investigation of expressing other integer sequences (Balancing, Co-balancing, k-Fibonacci, k-Lucs, ...) as sums of Jacobsthal numbers.

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References

[1] C. Adegbindin, F. Luca and A. Togbé, *Pell and Pell-Lucas numbers as sums of two repdigits*, Bull. Malays. Math. Sci. Soc. **43**, No.1 (2020)1253-1271.
[2] J. Bravo and F. Luca, *Perfect Pell and Pell-Lucas numbers*, Studia Sci. Math. Hungar. **56**, No.4 (2019)381-387.
[3] Y. Bugeaud, *Linear forms in logarithms and applications*, Zurich, European Mathematical Society, 2018.
[4] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas powers*, Ann. Math. **163**, No.3 (2006)969-1018.
[5] A. Cağman, *An approach to Pillai's problem with the Pell sequence and the powers of 3*, Miskolc Math. Notes **22**, No.2 (2021)599-610.
[6] H. Cohen, *Number theory. volume 1.*, New York, Springer, 2007.
[7] A. Dujella and A. Pethö, *A generalization of a theorem of Baker and Davenport*, Q. J. Math. **49**, No.195 (1998)291-306.
[8] F. Erduvan, R. Keskin and F. Luca *Fibonacci and Lucas numbers as difference of two repdigits*, Rendiconti del Circolo Matematico di Palermo Series 2. **71**, No.2 (2022)575-589.
[9] B. Faye and B. Edjeou, *On the problem of Pillai with Pell numbers, Pell-Lucas numbers and powers of 3*, Int. J. Number Theory. **2** , No.1(2022)1-22.
[10] A. F. Horadam, *Jacobsthal representation numbers*, Fibonacci Quart. **34**, No.1 (1996)40-54.
[11] B. Kafle, S. E. Rihane and A. Togbé, *Pell and Pell-Lucas numbers of the form* $x^a \pm x^b + 1$, Bol. Soc. Mat. Mex. **26**, No.3 (2020)879-893.
[12] T. Koshy, *Pell and Pell-Lucas numbers with applications*, Berlin, Springer, 2014.
[13] T. Koshy, *Fibonacci and Lucas numbers with applications*, New Jersey, Wiley, 2019.
[14] E. M. Matveev, *An explicit lower bound for a homogeneous rational form in the logarithms of algebraic numbers. II*, Izv. Math. **64**, No.6 (2000)1217-1269.
[15] Y. Y. Qu and J. W. Zeng, *Pell and Pell-Lucas numbers of the form* $-2^a - 3^b + 5^c$, Czechoslovak Math. J. **70**, No.1 (2020)281-289 .
[16] S. E. Rihane, *k-Fibonacci and k-Lucas numbers as product of two repdigits*, Results Math. **76**, No.4 (2021) Article 208.