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Shape Preservation of Ternary 4-point Non-Stationary Interpolating Subdivision Scheme

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Abstract.: The paper suggests conditions for preserving the shape properties of the original data using the ternary 4-point non-stationary interpolating subdivision scheme. Sufficient conditions with a suitable selection of the starting tension parameter are determined, which guarantee to retain the properties of positivity, monotonicity and convexity from the initial data to the curves generated for a limited number of iterations. The obtained results are significantly extended in the limiting case for maintaining such shape properties in the limit functions. The geometric interpretation of results is depicted through different numerical examples.

AMS (MOS) Subject Classification Codes: 65D17; 65D10; 68U07
Key Words: Shape preserving, Subdivision schemes, Non-stationary, Positivity, Monotonicity, Convexity.

1. INTRODUCTION

Subdivision schemes offer an efficient algorithm with the intention of creating a curve from a given polygon or a surface from a given polyhedral mesh by an iterative application of a refinement rule. The interpolating subdivision schemes define a new set of vertices in between the old ones, as a consequence, the limit shape passes through the original data. Thus, interpolating schemes are capable to construct the final shape in the most predictable manner from a given initial data.

The earlier analysis in the field of subdivision was confined to binary stationary schemes [7, 8, 19]. In general, the availability of tension control and conic reproduction are important attributes of the non-stationary schemes [3, 4, 11]. Smaller support and higher order smoothness are the special features that are attainable for just ternary schemes [3, 10] in comparison with binary schemes [4, 8].

The notion of shape properties generally includes positivity, monotonicity and convexity, and refers to the geometric behavior of the limit shapes. During the past few years, a great interest has been revealed in literature [5, 12, 9, 15, 2] to deal with the problem of shape preservation. In the method proposed by Dyn et al. [9], a parameter was chosen depending on the univariate convex initial data to preserve convexity of the scheme [8].

Cai [6] suggested an efficient and easy approach to develop conditions on the initial data for the convexity preservation of the C^2 ternary scheme [10]. Along the same lines, several attempts conducted in [18, 14, 17] to investigate the monotonicity or the convexity preservation of many stationary subdivision schemes. Marinov et al. [13] analyzed the properties for the shape preservation of a 4-point non-stationary scheme by modifying the tension parameter locally according to the geometric behavior of the control polygon. In Akram et al. [1] analyzed the hyperbolic form of a C^1 binary non-stationary interpolating scheme [4] to preserve the shape properties satisfying suitable conditions on the original data. In order to overcome the limitation of the non-stationary scheme [4] due to the lower order smoothness, the shape preservation of C^2 ternary non-stationary interpolating scheme et al. [3] is examined in this paper. Moreover, the motivation of the ternary schemes is the higher regularity in comparison with binary schemes.

In particular, the refinement rule changes from one level to another in a non-stationary scheme. Therefore, the aim of our work is to identify the conditions for retaining the shape characteristics of the curves that are generated after a finite number of iterations. The appropriate conditions are obtained for the shape preservation of the ternary non-stationary interpolating scheme by choosing adequately and a posteriori value of initial parameter β_0.

2. Ternary Non-Stationary Interpolating Subdivision Scheme

We consider a class of ternary 4-point non-stationary interpolating subdivision scheme, defined by

$$\begin{cases} f_{3i}^{k+1} = & f_i^k, \\ f_{3i+1}^{k+1} = & \alpha_0^k f_{i-1}^k + \alpha_1^k f_i^k + \alpha_2^k f_{i+1}^k + \alpha_3^k f_{i+2}^k, \\ f_{3i+2}^{k+1} = & \alpha_3^k f_{i-1}^k + \alpha_2^k f_i^k + \alpha_1^k f_{i+1}^k + \alpha_0^k f_{i+2}^k, \end{cases} \tag{2. 1}$$

where $\{(x_i^k, f_i^k)\}_{i\in\mathbb{Z}}$, $\forall k \in \mathbb{Z}_+$ is the set of control points, obtained from the given set of initial data $\{(x_i^0, f_i^0) \in \mathbb{R}^d\}_{i\in\mathbb{Z}}$. Moreover,

$$\begin{aligned} \alpha_0^k &= \frac{1}{60}\left(-90\gamma_{k+1} - 1\right), \\ \alpha_1^k &= \frac{1}{60}\left(90\gamma_{k+1} + 43\right), \\ \alpha_2^k &= \frac{1}{60}\left(90\gamma_{k+1} + 17\right), \\ \alpha_3^k &= \frac{1}{60}\left(-90\gamma_{k+1} + 1\right), \end{aligned}$$

with

$$\gamma_{k+1} = -\frac{1}{3(1-(\beta_{k+1})^2)(1+\beta_{k+1})} \tag{2. 2}$$

and the parameter β_{k+1} satisfies the recurrence relation

$$\beta_{k+1} = \sqrt{2+\beta_k},\ \beta_k \in [-2,\infty)\setminus\{-1\} \quad \forall k \in \mathbb{Z}_+. \tag{2. 3}$$

2.1. **Convergence and Regularity.** Beccari et al. [3] investigated the convergence properties of the non-stationary scheme (2. 1) by showing its asymptotic equivalence to the stationary scheme of Hassan et al. [10], for $\mu = \frac{1}{10}$. This sufficient condition for asymptotic equivalence has been thoroughly studied (see section 3, [3]).

Based on Riouls method [16], Zheng et al. [20] obtained the Hölder regularity $R(\mu)$ against μ of the stationary scheme [10], as

$$R(\mu) = \begin{cases} 2-\log_3\left(\frac{3-15\mu}{2}\right), & \frac{1}{15} < \mu < \frac{1}{11} \\ 2-\log_3(9\mu), & \frac{1}{11} \le \mu < \frac{1}{9}, \end{cases}$$

The Hölder regularity at $\mu = \frac{1}{10}$ can be obtained as $R(\frac{1}{10}) = 2.0959$. Hence, the holder regularity of the non-stationary scheme (2. 1) is 2.0959.

The non-stationary scheme (2. 1) generates C^2-limit functions. Moreover, for the initial tension parameter β_0, it holds:

- if $\beta_0 \in [-2,2)\setminus\{-1\}$, then $\beta_k \in [0,2)\ \forall k > 0$ and the sequence $\{\beta_k\}_{k\in\mathbb{N}}$ is strictly increasing
- if $\beta_0 = 2$, then $\beta_k = 2\ \forall k > 0$ and the sequence $\{\beta_k\}_{k\in\mathbb{N}}$ converges to 2
- if $\beta_0 \in (2,\infty)$, then $\beta_k \in (2,\infty)\ \forall k > 0$ and the sequence $\{\beta_k\}_{k\in\mathbb{N}}$ is strictly decreasing

Since γ_{k+1} is defined in terms of tension parameter β_{k+1} through Eq.(2. 2), thus it holds:

- $\gamma_{k+1} > \frac{1}{27} \Leftrightarrow \beta_{k+1} \in (1,2)$
- $\gamma_{k+1} = \frac{1}{27} \Leftrightarrow \beta_{k+1} = 2$
- $\gamma_{k+1} < \frac{1}{27} \Leftrightarrow \beta_{k+1} \in [0,1) \bigcup (2,\infty)$

The following discussion is about the shape preservation of the ternary non-stationary scheme (2. 1), which is analyzed by restricting our choice of the initial parameter $\beta_0 \in [2,\ 6.62149)$, such that the sequence $\{\beta_k\}_{k\in\mathbb{N}}$ is strictly decreasing and the parameter γ_{k+1} ranges in the interval $\left(\frac{1}{90},\frac{1}{27}\right]$ for all k subdivision levels.

3. Positivity preservation

The preservation of positivity for a subdivision scheme can be attained when the new set of control points are positive at each subdivision level for a given positive initial data. Using the coefficients in the refinement rule (2. 1), it may be observed that not all of them are positive for $\gamma_{k+1} \in (\frac{1}{90},\frac{1}{27}]$, as a consequence, the new control points after some finite number of iterations may not preserve the positivity of initial data. However, satisfying sufficient condition on the initial data, the positivity preservation of the subdivision scheme (2. 1) can be guaranteed in the curves achieved after some specific number of iterations.

Firstly, some inequalities are established in Lemma 3.1 for breaking down more complicated arguments of Lemma 3.2 into simple steps. The proof of this lemma is omitted as can be directly verified.

Lemma 3.1. *For any* $n \in \mathbb{Z}_+$, *let* $\lambda_n = min\left\{\frac{-90\gamma_{n+1}+43}{90\gamma_{n+1}+1}, \frac{90\gamma_{n+1}+17}{90\gamma_{n+1}-1}\right\}$ *and* $\gamma_{n+1} \in \left(\frac{1}{90},\frac{1}{27}\right]$, *then the following inequalities hold*

(i) $(-90\gamma_{n+1}-1)\lambda_n+(90\gamma_{n+1}+43)>0$
(ii) $(90\gamma_{n+1}+17)\lambda_n+(-90\gamma_{n+1}+1)>0$
(iii) $(90\gamma_{n+1}-43)\lambda_n+(90\gamma_{n+1}+1)<0$
(iv) $(90\gamma_{n+1}+1)\lambda_n+(-90\gamma_{n+1}+1)>0$
(v) $(90\gamma_{n+1}-1)\lambda_n-(90\gamma_{n+1}+1)<0$ *for* $\gamma_{n+1}\in\left(\frac{1}{90},\frac{1}{180}(21-\sqrt{353})\right]$
(vi) $(90\gamma_{n+1}-1)\lambda_n-(90\gamma_{n+1}+1)>0$ *for* $\gamma_{n+1}\in\left(\frac{1}{180}(21-\sqrt{353}),\frac{1}{27}\right]$
(vii) $(90\gamma_{n+1}-1)\lambda_n-(90\gamma_{n+1}+17)\leq 0$ *for* $\gamma_{n+1}\in\left(\frac{1}{180}(21-\sqrt{353}),\frac{1}{27}\right]$
(viii) $(-90\gamma_{n+1}+1)\lambda_n^2+16\lambda_n+(90\gamma_{n+1}+43)>0$
(ix) $(90\gamma_{n+1}+1)\lambda_n^2-(180\gamma_{n+1}+42)\lambda_n+(90\gamma_{n+1}+17)<0$
(x) $(90\gamma_{n+1}+1)\lambda_n^2-(90\gamma_{n+1}+43)\lambda_n+60>0$ *for* $\gamma_{n+1}\in\left(\frac{1}{90},\frac{1}{30}\right]$
(xi) $(90\gamma_{n+1}+1)\lambda_n^2-(90\gamma_{n+1}+43)\lambda_n+60<0$ *for* $\gamma_{n+1}\in\left(\frac{1}{30},\frac{1}{27}\right]$
(xii) $(90\gamma_{n+1}-1)\lambda_n^2+(90\gamma_{n+1}-17)\lambda_n-60<0$ *for* $\gamma_{n+1}\in\left(\frac{1}{30},\frac{1}{27}\right]$

The level dependent positivity preserving condition can be derived in the following manner:

Lemma 3.2. *Define,* $p_i^k=\frac{f_{i+1}^k}{f_i^k}$ and $P^k=max_i\{p_i^k,\frac{1}{p_i^k}\},\ i\in\mathbb{Z},\ k\in\mathbb{Z}_+$. *For any* $n\in\mathbb{Z}_+$, *if the initial data* $\{(x_i^0,f_i^0):i\in\mathbb{Z}\}$ *is positive, i.e.,* $f_i^0>0,\ i\in\mathbb{Z}$, *such that* $\beta_0\in[2,\ 6.62149)$, $\gamma_{n+1}\in\left(\frac{1}{90},\frac{1}{27}\right]$ *and*

$$P^0<\lambda_n=min\left\{\frac{-90\gamma_{n+1}+43}{90\gamma_{n+1}+1},\frac{90\gamma_{n+1}+17}{90\gamma_{n+1}-1}\right\}, \tag{3. 4}$$

then the control points generated up to n *iterations by the non-stationary subdivision scheme (2. 1) preserve the positivity of the initial data.*

When $\gamma_{k+1}\in\left(\frac{1}{90},\frac{1}{27}\right]$, it can be obtained that $\gamma_k\leq\gamma_n$ for all $k<n$. Precisely, $\{\gamma_k\}_{k\in\mathbb{N}}$ is an increasing sequence for the successive proceedings of iterations. In particular,

$$\lambda_k=\frac{-90\gamma_{k+1}+43}{90\gamma_{k+1}+1}\text{ for }\gamma_{k+1}\in\left(\frac{1}{90},\frac{1}{30}\right]$$

and

$$\lambda_k=\frac{90\gamma_{k+1}+17}{90\gamma_{k+1}-1}\text{ for }\gamma_{k+1}\in\left(\frac{1}{30},\frac{1}{27}\right].$$

Ultimately, subsequent levels of refinement leads to the fact that $\{\lambda_k\}_{k\in\mathbb{N}}$ is a decreasing sequence. That is to say, λ_n is the smallest finite value achieved after n iterations of the scheme, which allows us to consider λ_n in the positivity preserving condition to be imposed on the initial control points.

Proof. To prove Lemma 3.2, mathematical induction on n is used.

(i) By hypothesis, the statement is true for $n=0$, *i.e.*, $f_i^0>0,\ P^0<\lambda_n,\ i\in\mathbb{Z}$.
(ii) Assume by inductive hypothesis $f_i^n>0$ and $P^n<\lambda_n,\ i\in\mathbb{Z}$ for any $n\in\mathbb{Z}_+$, then $\frac{1}{\lambda_n}<p_i^n<\lambda_n$. Therefore, it is to be verified that $f_i^{n+1}>0$ and $P^{n+1}<\lambda_n$.

By the definition of scheme (2. 1),

$$f_{3i}^{n+1}>0. \tag{3. 5}$$

Moreover, in view of Lemma 3.1(i) and (viii), f^{n+1}_{3i+1} satisfies

$$\begin{aligned}
f^{n+1}_{3i+1} &= \frac{f^n_{i+1}}{60}\Bigg((-90\gamma_{n+1}-1)\frac{1}{p^n_{i-1}p^n_i}+(90\gamma_{n+1}+43)\frac{1}{p^n_i} \\
&\quad +(90\gamma_{n+1}+17)+(-90\gamma_{n+1}+1)p^n_{i+1}\Bigg) \\
&> \frac{f^n_{i+1}}{60}\Bigg(\Big((-90\gamma_{n+1}-1)\lambda_n+(90\gamma_{n+1}+43)\Big)\frac{1}{p^n_i} \\
&\quad +(90\gamma_{n+1}+17)+(-90\gamma_{n+1}+1)\lambda_n\Bigg) \\
&> \frac{f^n_{i+1}}{60}\Bigg(\Big((-90\gamma_{n+1}-1)\lambda_n+(90\gamma_{n+1}+43)\Big)\frac{1}{\lambda_n} \\
&\quad +(90\gamma_{n+1}+17)+(-90\gamma_{n+1}+1)\lambda_n\Bigg) \\
&= \frac{f^n_{i+1}}{60\lambda_n}\Big((-90\gamma_{n+1}+1)\lambda_n^2+16\lambda_n+(90\gamma_{n+1}+43)\Big) \\
&> 0. \qquad (3.\ 6)
\end{aligned}$$

Similarly, it can be verified that

$$f^{n+1}_{3i+2} > 0. \qquad (3.\ 7)$$

Combining Eqs.(3. 5), (3. 6) and (3. 7), it follows that $f^{n+1}_i > 0$.
In order to prove $P^{n+1} < \lambda_n$, it is sufficient to satisfy that $\frac{1}{\lambda_n} < p^{n+1}_i < \lambda_n$. Precisely, it is to be verified that $\frac{1}{\lambda_n} < p^{n+1}_{3i+j} < \lambda_n$ for $j = 0, 1, 2$.
Taking into account Lemma 3.1 (ii) and (iii), it gives

$$\begin{aligned}
f^{n+1}_{3i+1}-\lambda_n f^{n+1}_{3i} &= \frac{f^n_i}{60}\Bigg((90\gamma_{n+1}+43)-60\lambda_n-(90\gamma_{n+1}+1)\frac{1}{p^n_{i-1}} \\
&\quad +(90\gamma_{n+1}+17)p^n_i+(-90\gamma_{n+1}+1)p^n_{i+1}p^n_i\Bigg) \\
&< \frac{f^n_i}{60}\Bigg((90\gamma_{n+1}+43)-60\lambda_n-(90\gamma_{n+1}+1)\frac{1}{\lambda_n} \\
&\quad +\Big((90\gamma_{n+1}+17)+(-90\gamma_{n+1}+1)\frac{1}{\lambda_n}\Big)p^n_i\Bigg) \\
&= \frac{f^n_i}{60}\Bigg((90\gamma_{n+1}+43)-60\lambda_n-(90\gamma_{n+1}+1)\frac{1}{\lambda_n} \\
&\quad +\Big((90\gamma_{n+1}+17)\lambda_n+(-90\gamma_{n+1}+1)\Big)\frac{p^n_i}{\lambda_n}\Bigg)
\end{aligned}$$

$$\begin{aligned}
&< \frac{f_i^n}{60}\bigg((90\gamma_{n+1}+43)-60\lambda_n-(90\gamma_{n+1}+1)\frac{1}{\lambda_n}\\
&\quad +(90\gamma_{n+1}+17)\lambda_n+(-90\gamma_{n+1}+1)\bigg)\\
&= \frac{f_i^n}{60\lambda_n}\Big((90\gamma_{n+1}-43)\lambda_n^2+44\lambda_n-(90\gamma_{n+1}+1)\Big)\\
&= \frac{f_i^n}{60\lambda_n}(\lambda_n-1)\Big((90\gamma_{n+1}-43)\lambda_n+(90\gamma_{n+1}+1)\Big)\\
&< 0,
\end{aligned}$$

which shows that $p_{3i}^{n+1} < \lambda_n$.
For $\gamma_{n+1} \in \big(\frac{1}{90}, \frac{1}{180}(21-\sqrt{353})\big]$, using Lemma 3.1 (iv), (v) and (ix), it gives

$$\begin{aligned}
f_{3i+2}^{n+1}-\lambda_n f_{3i+1}^{n+1} &= \frac{f_{i+1}^n}{60}\bigg(\Big((90\gamma_{n+1}+1)\lambda_n+(-90\gamma_{n+1}+1)\Big)\frac{1}{p_{i-1}^n p_i^n}\\
&\quad +\Big(-(90\gamma_{n+1}+43)\lambda_n+(90\gamma_{n+1}+17)\Big)\frac{1}{p_i^n}\\
&\quad +\Big(-(90\gamma_{n+1}+17)\lambda_n+(90\gamma_{n+1}+43)\Big)\\
&\quad +\Big((90\gamma_{n+1}-1)\lambda_n-(90\gamma_{n+1}+1)\Big)p_{i+1}^n\bigg) \qquad (3.\ 8)\\
&< \frac{f_{i+1}^n}{60}\bigg(\Big((90\gamma_{n+1}+1)\lambda_n^2-(180\gamma_{n+1}+42)\lambda_n\\
&\quad +(90\gamma_{n+1}+17)\Big)\frac{1}{p_i^n}-(90\gamma_{n+1}+17)\lambda_n+(90\gamma_{n+1}+43)\\
&\quad +\Big((90\gamma_{n+1}-1)\lambda_n-(90\gamma_{n+1}+1)\Big)\frac{1}{\lambda_n}\bigg)\\
&< \frac{f_{i+1}^n}{60}\bigg(\Big((90\gamma_{n+1}+1)\lambda_n^2-(180\gamma_{n+1}+42)\lambda_n\\
&\quad +(90\gamma_{n+1}+17)\Big)\frac{1}{\lambda_n}-(90\gamma_{n+1}+17)\lambda_n+(90\gamma_{n+1}+43)\\
&\quad +\Big((90\gamma_{n+1}-1)\lambda_n-(90\gamma_{n+1}+1)\Big)\frac{1}{\lambda_n}\bigg)\\
&= -\frac{4f_{i+1}^n}{15\lambda_n}(\lambda_n^2-1)\\
&< 0.
\end{aligned}$$

For $\gamma_{n+1} \in \big(\frac{1}{180}(21-\sqrt{353}), \frac{1}{27}\big]$, using Lemma 3.1 (iv), (vi), (vii) and (ix), Eq.(3. 8) reduces to

$$\begin{aligned}
f_{3i+2}^{n+1}-\lambda_n f_{3i+1}^{n+1} &< \frac{f_{i+1}^n}{60}\bigg(\Big((90\gamma_{n+1}+1)\lambda_n^2-(180\gamma_{n+1}+42)\lambda_n\\
&\quad +(90\gamma_{n+1}+17)\Big)\frac{1}{p_i^n}-(90\gamma_{n+1}+17)\lambda_n+(90\gamma_{n+1}+43)
\end{aligned}$$

$$\begin{aligned}
&+(90\gamma_{n+1}-1)\lambda_n^2-(90\gamma_{n+1}+1)\lambda_n\Big)\\
<\ & \frac{f_{i+1}^n}{60}\bigg(\Big((90\gamma_{n+1}+1)\lambda_n^2-(180\gamma_{n+1}+42)\lambda_n+(90\gamma_{n+1}+17)\Big)\frac{1}{\lambda_n}\\
&+(90\gamma_{n+1}-1)\lambda_n^2-(180\gamma_{n+1}+18)\lambda_n+(90\gamma_{n+1}+43)\bigg)\\
=\ & \frac{f_{i+1}^n}{60\lambda_n}\Big((90\gamma_{n+1}-1)\lambda_n^3-(90\gamma_{n+1}+17)\lambda_n^2\\
&-(90\gamma_{n+1}-1)\lambda_n+(90\gamma_{n+1}+17)\Big)\\
=\ & \frac{f_{i+1}^n}{60\lambda_n}(\lambda_n^2-1)\Big((90\gamma_{n+1}-1)\lambda_n-(90\gamma_{n+1}+17)\Big)\\
\leq\ & 0.
\end{aligned}$$

Thus, $p_{3i+1}^{n+1}<\lambda_n$ for $\gamma_{n+1}\in\left(\frac{1}{90},\frac{1}{27}\right]$.
For $\gamma_{n+1}\in\left(\frac{1}{90},\frac{1}{30}\right]$, considering Lemma 3.1 (x), it can be written as

$$\begin{aligned}
f_{3i+3}^{n+1}-\lambda_n f_{3i+2}^{n+1} =\ & \frac{f_i^n}{60}\bigg((90\gamma_{n+1}-1)\frac{\lambda_n}{p_{i-1}^n}-(90\gamma_{n+1}+17)\lambda_n\\
&+\Big(60-(90\gamma_{n+1}+43)\lambda_n\Big)p_i^n+(90\gamma_{n+1}+1)\lambda_n p_{i+1}^n p_i^n\bigg)\\
<\ & \frac{f_i^n}{60}\bigg((90\gamma_{n+1}-1)\lambda_n^2-(90\gamma_{n+1}+17)\lambda_n\\
&+\Big((90\gamma_{n+1}+1)\lambda_n^2-(90\gamma_{n+1}+43)\lambda_n+60\Big)p_i^n\bigg) \qquad (3.\ 9)\\
<\ & \frac{f_i^n}{60}\bigg((90\gamma_{n+1}-1)\lambda_n^2-(90\gamma_{n+1}+17)\lambda_n\\
&+\Big((90\gamma_{n+1}+1)\lambda_n^2-(90\gamma_{n+1}+43)\lambda_n+60\Big)\lambda_n\bigg)\\
=\ & \frac{f_i^n\lambda_n}{60}\Big((90\gamma_{n+1}+1)\lambda_n^2-44\lambda_n-(90\gamma_{n+1}-43)\Big)\\
=\ & \frac{f_i^n\lambda_n}{60}(\lambda_n-1)\Big((90\gamma_{n+1}+1)\lambda_n+(90\gamma_{n+1}-43)\Big)\\
=\ & 0.
\end{aligned}$$

The latter relation holds by the definition of λ_n.
For $\gamma_{n+1}\in\left(\frac{1}{30},\frac{1}{27}\right]$, using Lemma 3.1 (xi) and (xii), Eq.(3. 9) satisfies

$$\begin{aligned}
f_{3i+3}^{n+1}-\lambda_n f_{3i+2}^{n+1} <\ & \frac{f_i^n}{60}\bigg((90\gamma_{n+1}-1)\lambda_n^2-(90\gamma_{n+1}+17)\lambda_n\\
&+\Big((90\gamma_{n+1}+1)\lambda_n^2-(90\gamma_{n+1}+43)\lambda_n+60\Big)\frac{1}{\lambda_n}\bigg)
\end{aligned}$$

$$\begin{aligned} &= \frac{f_i^n}{60\lambda_n}\Big((90\gamma_{n+1}-1)\lambda_n^3 - 16\lambda_n^2 - (90\gamma_{n+1}+43)\lambda_n + 60\Big) \\ &= \frac{f_i^n}{60\lambda_n}(\lambda_n - 1)\Big((90\gamma_{n+1}-1)\lambda_n^2 + (90\gamma_{n+1}-17)\lambda_n - 60\Big) \\ &< 0. \end{aligned}$$

Hence, $p_{3i+2}^{n+1} < \lambda_n$ for $\gamma_{n+1} \in \left(\frac{1}{90}, \frac{1}{27}\right]$.
Similarly, it can be verified that $p_{3i+j}^{n+1} > \frac{1}{\lambda_n}$ for $j = 0, 1, 2$.
Since, $P^{n+1} = max_i\{p_i^{n+1}, \frac{1}{p_i^{n+1}}\}$, therefore $P^{n+1} < \lambda_n$. □

In Lemma 3.2, the derived condition of positivity holds for some finite number of iterations which may not satisfy for the limiting case. Hence, Theorem 3.3 is given for preserving the positivity of the limit functions, when $n \to \infty$.
Since, the recurrence relation in Eq.(2. 3) satisfies

$$\lim_{n\to\infty} \beta_{n+1} = 2.$$

Therefore, from Eq.(2. 2), it gives

$$\lim_{n\to\infty} \gamma_{n+1} = \frac{1}{27}.$$

Moreover, $\lim_{n\to\infty} \lambda_n = \frac{61}{7}$ in Eq.(3. 4) and on that account the proof of Theorem 3.3 can be verified using the same arguments of Lemma 3.2.

Theorem 3.3. *Let the initial data* $\{(x_i^0, f_i^0) : i \in \mathbb{Z}\}$ *be positive, such that* $\beta_0 \in [2,\ 6.62149)$ *and*

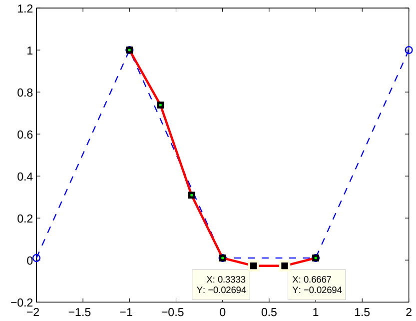
$$P^0 < \frac{61}{7},$$

then the limit function of the non-stationary subdivision scheme (2. 1) preserves positivity.

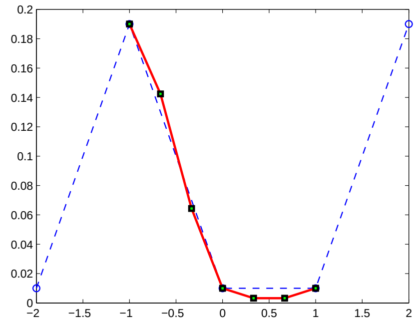
Therefore, under appropriate condition on the initial data, one can always maintain positivity with this scheme for any n-*th* subdivision level by choosing adequately the value of β_0 such that $\gamma_{n+1} \in \left(\frac{1}{90}, \frac{1}{27}\right]$. The value of λ_n at n-th level, defined in Eq.(3. 4), is calculated by first evaluating the parameters β_{n+1} and γ_{n+1} for the n-th level of refinement. In order to have a geometric interpretation of positivity preserving condition suggested in this section, some numerical results of the required parameters for the first four iterations are inferred. For instance, selecting the value of initial parameter $\beta_0 = 6$, the results of $\beta_{n+1},\ \gamma_{n+1}$ and λ_n for the first four subdivision levels are given in Table 1. In Figure 1

Table 1: the results of $\beta_{n+1},\ \gamma_{n+1}$ and λ_n for the first four subdivision levels, when $\beta_0 = 6$.

	First level	Second level	Third level	Forth level
β_{n+1}	2.82843	2.19737	2.04875	2.01215
γ_{n+1}	0.012438	0.02723	0.03420	0.03630
λ_n	19.76015	11.75066	9.66399	8.94069



(a)



(b)

Figure 1: The control polygons obtained after first iteration for two different positive initial data.

(a) and (b), the refined polygons after one application of refinement rule are obtained for two different positive initial control polygons. In fact, the initial data taken in Figure 1 (a) does not satisfy the required condition $P^0 < 19.76015$ for $\beta_0 = 6$. As a consequence, the highlighted control points attained after first iteration are not positive in the refined polygon shown by a red solid line. However, satisfying sufficient condition of positivity ($P^0 < 19.76015$), the positive sequence of control points after first iteration are generated in Figure 2.

The limit function depicted in Figure 3, using the non-stationary scheme (2. 1), clearly preserves positivity where the initial data is configured under the additional constraints of Lemma 3.3.

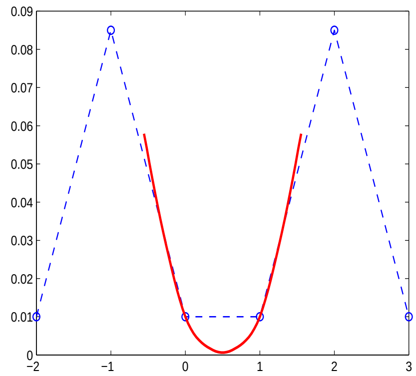


Figure 2: The limit function of the non-stationary scheme (2. 1) shown along with the initial control polygon by preserving positivity.

4. Monotonicity preservation

Divided differences of first order are defined by $D_i^k = 3^k(f_{i+1}^k - f_i^k),\ i \in \mathbb{Z},\ k \in \mathbb{Z}_+$. Divided difference of first order for the non-stationary scheme (2. 1) can be expressed as

$$D_{3i}^{k+1} = \frac{1}{20}\Big((90\gamma_{k+1}+1)D_{i-1}^k + 18D_i^k + (-90\gamma_{k+1}+1)D_{i+1}^k\Big), \quad (4.\ 10)$$

$$D_{3i+1}^{k+1} = \frac{1}{20}\Big(-2D_{i-1}^k + 24D_i^k - 2D_{i+1}^k\Big), \quad (4.\ 11)$$

$$D_{3i+2}^{k+1} = \frac{1}{20}\Big((-90\gamma_{k+1}+1)D_{i-1}^k + 18D_i^k + (90\gamma_{k+1}+1)D_{i+1}^k\Big). \quad (4.\ 12)$$

In order to analyze the monotonicity preserving property of the scheme (2. 1), sufficient condition on the initial data is derived in the Theorem 4.2. Firstly, some inequalities are provided in Lemma 4.1 with the attempt to shorten the proof of Lemma 4.2. The proof of this lemma can be easily verified by the direct calculations.

Lemma 4.1. *For any* $n \in \mathbb{Z}_+$, *let* $\gamma_{n+1} \in \left(\frac{1}{90}, \frac{1}{27}\right]$ *and* $\alpha = \frac{9}{2}$, *then the following inequalities hold*

(i) $(90\gamma_{n+1}+1)\alpha + 2 > 0$
(ii) $2\alpha - (90\gamma_{n+1}-1) > 0$
(iii) $2\alpha + (90\gamma_{n+1}+1) > 0$
(iv) $(90\gamma_{n+1}-1)\alpha - 2 < 0$ *for* $\gamma_{n+1} \in \left(\frac{1}{90}, \frac{13}{810}\right]$
(v) $(90\gamma_{n+1}-1)\alpha - 2 > 0$ *for* $\gamma_{n+1} \in \left(\frac{13}{810}, \frac{1}{27}\right]$
(vi) $(-90\gamma_{n+1}+1)\alpha^2 + 18\alpha + (90\gamma_{n+1}+1) > 0$
(vii) $(90\gamma_{n+1}-1)\alpha^2 + (90\gamma_{n+1}-21)\alpha + 2 < 0$ *for* $\gamma_{n+1} \in \left(\frac{13}{810}, \frac{1}{27}\right]$
(viii) $(90\gamma_{n+1}+1)\alpha^2 - 18\alpha + (90\gamma_{n+1}-1) < 0$ *for* $\gamma_{n+1} \in \left(\frac{1}{90}, \frac{247}{7650}\right]$
(ix) $(90\gamma_{n+1}+1)\alpha^2 - 18\alpha + (90\gamma_{n+1}-1) > 0$ *for* $\gamma_{n+1} \in \left(\frac{247}{7650}, \frac{1}{27}\right]$

(x) $(90\gamma_{n+1}-1)\alpha^2-18\alpha+(90\gamma_{n+1}-1)<0$ *for* $\gamma_{n+1}\in\left(\frac{247}{7650},\frac{1}{27}\right]$

Lemma 4.2. *Taking,* $q_i^k=\frac{D_{i+1}^k}{D_i^k}$ *and* $Q^k=max_i\{q_i^k,\frac{1}{q_i^k}\},\ i\in\mathbb{Z},\ k\in\mathbb{Z}_+$*. For any* $n\in\mathbb{Z}_+$*, if the initial data* $\{(x_i^0,f_i^0):\ i\in\mathbb{Z}\}$ *is strictly monotonic increasing, i.e.,* $D_i^0>0,\ i\in\mathbb{Z}$*, such that* $\beta_0\in[2,\ 6.62149)$*,* $\gamma_{n+1}\in\left(\frac{1}{90},\frac{1}{27}\right]$ *and*

$$Q^0<\alpha=\frac{9}{2}, \tag{4. 13}$$

then the control points generated up to n *iterations by the non-stationary subdivision scheme (2. 1) preserve monotonicity of the initial data.*

Proof. To prove Lemma 4.2, induction on n is used.

(i) By hypothesis, the statement is true for $n=0$, *i.e.*, $D_i^0>0,\ Q^0<\alpha,\ i\in\mathbb{Z}$.
(ii) For any $n\in\mathbb{Z}_+$, assume that $D_i^n>0$ and $Q^n<\alpha,\ i\in\mathbb{Z}$, then $\frac{1}{\alpha}<q_i^n<\alpha$. Therefore, our goal is to show that $D_i^{n+1}>0$ and $Q^{n+1}<\alpha$.

From Eq.(4. 10), considering Lemma 4.1 (vi), it follows that

$$\begin{aligned}
D_{3i}^{n+1} &= \frac{D_i^n}{20}\left((90\gamma_{n+1}+1)\frac{1}{q_{i-1}^n}+18+(-90\gamma_{n+1}+1)q_i^n\right)\\
&> \frac{D_i^n}{20}\left((90\gamma_{n+1}+1)\frac{1}{\alpha}+18+(-90\gamma_{n+1}+1)\alpha\right)\\
&= \frac{D_i^n}{20\alpha}\Big((-90\gamma_{n+1}+1)\alpha^2+18\alpha+(90\gamma_{n+1}+1)\Big)\\
&> 0.
\end{aligned} \tag{4. 14}$$

Using Eq.(4. 11), D_{3i+1}^{n+1} satisfies

$$\begin{aligned}
D_{3i+1}^{n+1} &= \frac{D_i^n}{20}\left(-\frac{2}{q_{i-1}^n}+24-2q_i^n\right)\\
&> \frac{D_i^n}{20}\Big(24-4\alpha\Big)\\
&> 0.
\end{aligned} \tag{4. 15}$$

Similarly, by the same discussion as in Eq.(4. 14), it can be verified that D_{3i+2}^{n+1} satisfies

$$D_{3i+2}^{n+1} > 0. \tag{4. 16}$$

Hence, combining Eqs.(4. 14), (4. 15) and (4. 16) leads to $D_i^{n+1}>0$.
In order to prove $Q^{n+1}<\alpha$, it is to be verified that $\frac{1}{\alpha}<q_i^{n+1}<\alpha$. Therefore, the result consists in satisfying $\frac{1}{\alpha}<q_{3i+j}^{n+1}<\alpha$ for $j=0,1,2$.

For $\gamma_{n+1} \in \big(\frac{1}{90}, \frac{13}{810}\big]$, taking into account Lemma 4.1 (i) and (iv), it follows that

$$\begin{aligned}
D^{n+1}_{3i+1} - \alpha D^{n+1}_{3i} &= \frac{D^n_i}{20}\bigg(24 - 18\alpha - \Big((90\gamma_{n+1}+1)\alpha + 2\Big)\frac{1}{q^n_{i-1}} \\
&\quad + \Big((90\gamma_{n+1}-1)\alpha - 2\Big)q^n_i\bigg) \qquad (4.\ 17)\\
&< \frac{D^n_i}{20}\bigg(24 - 18\alpha - \Big((90\gamma_{n+1}+1)\alpha + 2\Big)\frac{1}{\alpha} \\
&\quad + \Big((90\gamma_{n+1}-1)\alpha - 2\Big)\frac{1}{\alpha}\bigg) \\
&= \frac{D^n_i}{20\alpha}\Big(-18\alpha^2 + 22\alpha - 4\Big) \\
&= -\frac{9D^n_i}{10\alpha}(\alpha - 1)\Big(\alpha - \frac{2}{9}\Big) \\
&< 0.
\end{aligned}$$

For $\gamma_{n+1} \in \big(\frac{13}{810}, \frac{1}{27}\big]$, considering Lemma 4.1 (i),(v) and (vii), Eq.(4. 17) reduces to

$$\begin{aligned}
D^{n+1}_{3i+1} - \alpha D^{n+1}_{3i} &< \frac{D^n_i}{20}\bigg(24 - 18\alpha - \Big((90\gamma_{n+1}+1)\alpha + 2\Big)\frac{1}{\alpha} \\
&\quad + \Big((90\gamma_{n+1}-1)\alpha - 2\Big)\alpha\bigg) \\
&= \frac{D^n_i}{20\alpha}\Big((90\gamma_{n+1}-1)\alpha^3 - 20\alpha^2 + (-90\gamma_{n+1}+23)\alpha - 2\Big) \\
&= \frac{D^n_i}{20\alpha}(\alpha - 1)\Big((90\gamma_{n+1}-1)\alpha^2 + (90\gamma_{n+1}-21)\alpha + 2\Big) \\
&< 0.
\end{aligned}$$

Thus, $q^{n+1}_{3i} < \alpha$ for $\big(\frac{1}{90}, \frac{1}{27}\big]$.
In view of Lemma 4.1 (ii) and (iii),

$$\begin{aligned}
D^{n+1}_{3i+2} - \alpha D^{n+1}_{3i+1} &= \frac{D^n_i}{20}\bigg(18 - 24\alpha + \Big(2\alpha - (90\gamma_{n+1}-1)\Big)\frac{1}{q^n_{i-1}} \\
&\quad + \Big(2\alpha + (90\gamma_{n+1}+1)\Big)q^n_i\bigg) \\
&< \frac{D^n_i}{20}\bigg(18 - 24\alpha + \Big(2\alpha - (90\gamma_{n+1}-1)\Big)\alpha \\
&\quad + \Big(2\alpha + (90\gamma_{n+1}+1)\Big)\alpha\bigg) \\
&= \frac{D^n_i}{20}\Big(4\alpha^2 - 22\alpha + 18\Big) \\
&= \frac{D^n_i}{5}(\alpha - 1)\Big(\alpha - \frac{9}{2}\Big) \\
&= 0.
\end{aligned}$$

Thus, $q^{n+1}_{3i+1} < \alpha$.
For $\gamma_{n+1} \in \left(\frac{1}{90}, \frac{247}{7650}\right]$, considering Lemma 4.1 (viii), it can be obtained that

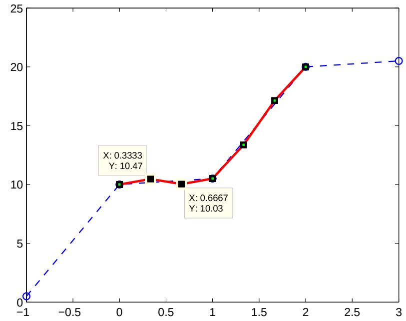
$$\begin{aligned} D^{n+1}_{3i+3} - \alpha D^{n+1}_{3i+2} &= \frac{D^n_i}{20}\Bigg((90\gamma_{n+1}+1) - 18\alpha + (90\gamma_{n+1}-1)\frac{\alpha}{q^n_{i-1}} \\ &\quad + \Big(18 - (90\gamma_{n+1}+1)\alpha\Big)q^n_i - (90\gamma_{n+1}-1)q^n_{i+1}q^n_i\Bigg) \\ &< \frac{D^n_i}{20}\Bigg((90\gamma_{n+1}+1) - 18\alpha + (90\gamma_{n+1}-1)\alpha^2 \\ &\quad + \Big(18 - (90\gamma_{n+1}+1)\alpha - (90\gamma_{n+1}-1)\frac{1}{\alpha}\Big)q^n_i\Bigg) \\ &= \frac{D^n_i}{20}\Bigg((90\gamma_{n+1}+1) - 18\alpha + (90\gamma_{n+1}-1)\alpha^2 \\ &\quad - \Big((90\gamma_{n+1}+1)\alpha^2 - 18\alpha + (90\gamma_{n+1}-1)\Big)\frac{q^n_i}{\alpha}\Bigg) \qquad (4.\ 18) \\ &< \frac{D^n_i}{20}\Bigg((90\gamma_{n+1}+1) - 18\alpha + (90\gamma_{n+1}-1)\alpha^2 \\ &\quad -(90\gamma_{n+1}+1)\alpha^2 + 18\alpha - (90\gamma_{n+1}-1)\Bigg) \\ &= -\frac{D^n_i}{10}(\alpha^2 - 1) \\ &< 0. \end{aligned}$$

For $\gamma_{n+1} \in \left(\frac{247}{7650}, \frac{1}{27}\right]$, in view of Lemma 4.1 (ix) and (x), Eq.(4. 18) takes the form

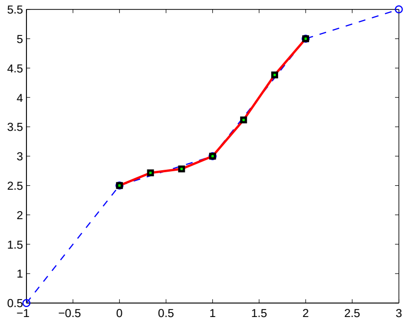
$$\begin{aligned} D^{n+1}_{3i+3} - \alpha D^{n+1}_{3i+2} &< \frac{D^n_i}{20}\Bigg((90\gamma_{n+1}+1) - 18\alpha + (90\gamma_{n+1}-1)\alpha^2 \\ &\quad - \Big((90\gamma_{n+1}+1)\alpha^2 - 18\alpha + (90\gamma_{n+1}-1)\Big)\frac{1}{\alpha^2}\Bigg) \\ &= \frac{D^n_i}{20\alpha^2}\Big((90\gamma_{n+1}-1)\alpha^4 - 18\alpha^3 + 18\alpha - (90\gamma_{n+1}-1)\Big) \\ &= \frac{D^n_i}{20\alpha^2}(\alpha^2-1)\Big((90\gamma_{n+1}-1)\alpha^2 - 18\alpha + (90\gamma_{n+1}-1)\Big) \\ &< 0. \end{aligned}$$

Hence, $q^{n+1}_{3i+2} < \alpha$ for $\gamma_{n+1} \in \left(\frac{1}{90}, \frac{1}{27}\right]$. In the same manner, it can be verified that $q^{n+1}_{3i+j} > \frac{1}{\alpha}$ for $j = 0, 1, 2$.
Since, $Q^{n+1} = max_i\{q^{n+1}_i, \frac{1}{q^{n+1}_i}\}$, thus $Q^{n+1} < \alpha$. □

Since the condition established in Eq.(4. 13) for preserving monotonicity is independent of n. Therefore, the condition developed for a finite n subdivision levels also holds for the limit function. For the case of monotonicity preservation the condition attained in Lemma 4.2 is quite satisfactory, which allows the user to construct different shapes preserving the monotonicity of initial sequence. In this context, the refined polygon after first iteration are



(a)



(b)

Figure 3: The control polygons obtained after first iteration for two different monotonic increasing initial data.

obtained in Figure 3(a) and (b) for $\beta_0 = 6$. In Figure 3(a), the monotonic increasing initial data is taken in such a way that $Q^0 \not< \frac{9}{2}$. In this case, D_i^1 defined by two indicated vertices is not positive in the refined polygon. Alternatively, imposing sufficient condition $Q^0 < \frac{9}{2}$ on the initial data, the monotonic increasing refined polygon is constructed in Figure 3(b). The initial data is considered in Figure 4 satisfying the additional condition of Theorem 4.2. Moreover, the monotonic increasing limit function produced by the non- stationary scheme (2. 1) is depicted.

5. Convexity preservation

Define the divided differences of second order by $d_i^k = 3^{2k}(f_{i-1}^k - 2f_i^k + f_{i+1}^k),\ i \in \mathbb{Z},\ k \in \mathbb{Z}_+$.
Divided difference of second order for the non-stationary scheme (2. 1) can be expressed

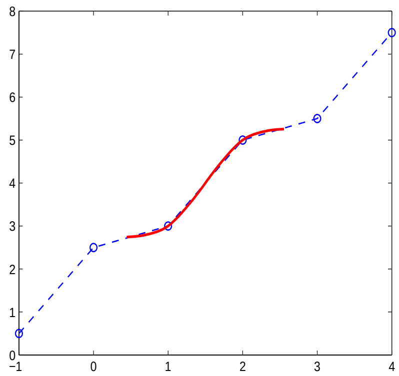


Figure 4: The limit function of the non-stationary scheme (2. 1) shown along with the initial control polygon by preserving monotonicity.

as,

$$d_{3i}^{k+1} = \frac{3}{20}\Big((-90\gamma_{k+1}+1)(d_{i-1}^k + d_{i+1}^k) + (-180\gamma_{k+1}+18)d_i^k\Big), \quad (5.\ 19)$$

$$d_{3i+1}^{k+1} = \frac{3}{20}\Big((90\gamma_{k+1}+3)d_i^k + (90\gamma_{k+1}-3)d_{i+1}^k\Big), \quad (5.\ 20)$$

$$d_{3i+2}^{k+1} = \frac{3}{20}\Big((90\gamma_{n+1}-3)d_i^k + (90\gamma_{n+1}+3)d_{i+1}^k\Big). \quad (5.\ 21)$$

Firstly, Lemma 5.1 is proposed for a straightforward proof of Lemma 5.2. The proof of this lemma can be easily verified by the direct substitutions.

Lemma 5.1. *For any* $n \in \mathbb{Z}_+$, *let* $\nu_n = \frac{21-270\gamma_{n+1}+\sqrt{15(31-852\gamma_{n+1}+540\gamma_{n+1}^2)}}{4(-1+90\gamma_{n+1})}$ *and* $\gamma_{n+1} \in \left[\frac{3}{90}, \frac{1}{27}\right]$, *then the following inequalities hold*

(i) $(-180\gamma_{n+1}+2)\nu_n + (-180\gamma_{n+1}+18) > 0$
(ii) $(90\gamma_{n+1}-1)\nu_n + (90\gamma_{n+1}-3) > 0$
(iii) $(-90\gamma_{n+1}+3)\nu_n + (-90\gamma_{n+1}+1) < 0$
(iv) $-(90\gamma_{n+1}+3)\nu_n^2 - (270\gamma_{n+1}-21)\nu_n - (180\gamma_{n+1}-2) < 0$

In the same fashion, the sufficient condition of convexity is developed for the finite n subdivision levels.

Lemma 5.2. *Taking,* $r_i^k = \frac{d_{i+1}^k}{d_i^k}$, *and* $R^k = max_i\{r_i^k, \frac{1}{r_i^k}\},\ k \geq 0,\ k \in \mathbb{Z}$. *For any* $n \in \mathbb{Z}_+$, *if the initial data* $\{(xi^0, f_i^0) :\ i \in \mathbb{Z}\}$ *is strictly convex, i.e.,* $d_i^0 > 0,\ i \in \mathbb{Z}$, *such that* $\beta_0 \in [2,\ 2.26297]$, $\gamma_{n+1} \in \left[\frac{3}{90}, \frac{1}{27}\right]$ *and*

$$R^0 < \nu_n = \frac{21-270\gamma_{n+1}+\sqrt{15(31-852\gamma_{n+1}+540\gamma_{n+1}^2)}}{4(-1+90\gamma_{n+1})}, \quad (5.\ 22)$$

then the control points generated up to n iterations by the non-stationary subdivision scheme (2. 1) preserve convexity of the initial data.

Since $\gamma_{k+1} \in \left[\frac{3}{90}, \frac{1}{27}\right]$, thus $\{\gamma_k\}_{k\in\mathbb{N}}$ is an increasing sequence for the successive progress of iterations. Correspondingly, $\nu_n \leq \nu_k$, $\forall k < n$. This indicates, ν_n is the smallest finite value attained after n iterations of the scheme, thus the initial data must satisfy the condition including ν_n.

Proof. We use the induction method on n to prove Lemma 5.2.

(i) By hypothesis, the result holds for $n = 0$, *i.e.*, $d_i^0 > 0,\ R^0 < \nu_n,\ i \in \mathbb{Z}$.

(ii) For any $n \in \mathbb{Z}_+$, assume that $d_i^n > 0$ and $R^n < \nu_n,\ i \in \mathbb{Z}$, then $\frac{1}{\nu_n} < r_i^n < \nu_n$. Therefore, it is to be verified that $d_i^{n+1} > 0$ and $R^{n+1} < \nu_n$.

From Eq.(5. 19), in view of Lemma 5.1 (i), d_{3i}^{n+1} satisfies

$$\begin{aligned} d_{3i}^{n+1} &= \frac{3d_i^n}{20}\left((-90\gamma_{n+1}+1)\frac{1}{r_{i-1}^n} + (-180\gamma_{n+1}+18) + (-90\gamma_{n+1}+1)r_i^n\right) \\ &> \frac{3d_i^n}{20}\left((-180\gamma_{n+1}+2)\nu_n + (-180\gamma_{n+1}+18)\right) \\ &> 0. \end{aligned} \tag{5. 23}$$

From Eqs.(5. 20) and (5. 21),

$$d_{3i+1}^{n+1} > 0 \text{ and } d_{3i+2}^{n+1} > 0 \tag{5. 24}$$

Combining Eqs.(5. 23) and (5. 24), leads to $d_i^{n+1} > 0$.
Taking into account Lemma 5.1 (ii),

$$\begin{aligned} d_{3i+1}^{n+1} - \nu_n d_{3i}^{n+1} &= \frac{3d_i^n}{20}\Bigg((180\gamma_{n+1}-18)\nu_n + (90\gamma_{n+1}+3) + (90\gamma_{n+1}-1)\frac{\nu_n}{r_{i-1}^n} \\ &\quad + \Big((90\gamma_{n+1}-1)\nu_n + (90\gamma_{n+1}-3)\Big)r_i^n\Bigg) \\ &< \frac{3d_i^n}{20}\Bigg((180\gamma_{n+1}-18)\nu_n + (90\gamma_{n+1}+3) + (90\gamma_{n+1}-1)\nu_n^2 \\ &\quad + \Big((90\gamma_{n+1}-1)\nu_n + (90\gamma_{n+1}-3)\Big)\nu_n\Bigg) \\ &= \frac{3d_i^n}{20}\Big((180\gamma_{n+1}-2)\nu_n^2 + (270\gamma_{n+1}-21)\nu_n + (90\gamma_{n+1}+3)\Big) \\ &= \frac{3d_i^n}{10}(90\gamma_{n+1}-1) \\ &\quad \times\left(\nu_n - \frac{21-270\gamma_{n+1}-\sqrt{15(31-852\gamma_{n+1}+540\gamma_{n+1}^2)}}{4(-1+90\gamma_{n+1})}\right) \\ &\quad \times\left(\nu_n - \frac{21-270\gamma_{n+1}+\sqrt{15(31-852\gamma_{n+1}+540\gamma_{n+1}^2)}}{4(-1+90\gamma_{n+1})}\right) \\ &= 0. \end{aligned}$$

The latter relation holds by the definition of ν_n, which shows that $r_{3i}^{n+1} < \nu_n$.
Since,

$$\begin{aligned} d_{3i+2}^{n+1} - \nu_n d_{3i+1}^{n+1} &= \frac{3d_i^n}{20}\Big((90\gamma_{n+1}-3)(1-\nu_n r_i^n) + (90\gamma_{n+1}+3)(r_i^n - \nu_n)\Big) \\ &< 0. \end{aligned}$$

Thus, $r_{3i+1}^{n+1} < \nu_n$.
Using Lemma 5.1 (iii) and (iv), it gives

$$\begin{aligned} d_{3i+3}^{n+1} - \nu_n d_{3i+2}^{n+1} &= \frac{3d_{i+1}^n}{20}\Bigg(\Big((-90\gamma_{n+1}+3)\nu_n + (-90\gamma_{n+1}+1)\Big)\frac{1}{r_i^n} \\ &\quad +(-180\gamma_{n+1}+18) - (90\gamma_{n+1}+3)\nu_n + (-90\gamma_{n+1}+1)r_{i+1}^n\Bigg) \\ &< \frac{3d_{i+1}^n}{20}\Bigg(\Big((-90\gamma_{n+1}+3)\nu_n + (-90\gamma_{n+1}+1)\Big)\frac{1}{\nu_n} \\ &\quad +(-180\gamma_{n+1}+18) - (90\gamma_{n+1}+3)\nu_n + (-90\gamma_{n+1}+1)\frac{1}{\nu_n}\Bigg) \\ &= \frac{3d_{i+1}^n}{20\nu_n}\Big(-(90\gamma_{n+1}+3)\nu_n^2 - (270\gamma_{n+1}-21)\nu_n - (180\gamma_{n+1}-2)\Big) \\ &< 0. \end{aligned}$$

Thus, $r_{3i+2}^{n+1} < \nu_n$.
Using the same arguments, it can be verified that $r_{3i+j}^{n+1} > \frac{1}{\nu n}$ for $j = 0, 1, 2$.
Since, $R^{n+1} = max_i\{r_i^{n+1}, \frac{1}{r_i^{n+1}}\}$, therefore $R^{n+1} < \nu_n$. □

Lemma 5.2 discusses the convexity preserving property of the scheme (2. 1) for a specific n number of iterations, which does not hold for the limit function. Therefore, Theorem 5.3 is given to derive the condition for the convexity preservation of the limit functions by using the fact that $\lim_{n\to\infty} \nu_n = \frac{19}{14}$ for the limiting case.

Theorem 5.3. *Suppose that the initial data* $\{(x_i^0, f_i^0) :\ i \in \mathbb{Z}\}$ *is strictly convex, such that* $\beta_0 \in [2,\ 2.26297]$ *and*

$$R^0 < \frac{19}{14},$$

then the limit function of the non-stationary subdivision scheme (2. 1) preserves convexity.

Remark 5.4. *The non-stationary scheme (2. 1) converges to its stationary counterpart* [10] *for* $\mu = \frac{1}{10}$*. Thus, the condition in the limiting case is same as the result attained in* [6] *about the convexity of stationary scheme* [10].

Remark 5.5. *Following the technique of Lemma 5.2 for* $\gamma_{n+1} \in \left(\frac{1}{90}, \frac{3}{90}\right)$*, it can be easily checked that the non-stationary subdivision scheme (2. 1) preserves the convexity of initial data, only when* $\nu_n = 1$*. In this case, the conditions are much restrictive for preserving convexity in the limit functions. They only cover a very particular case, when the initial data satisfy* $d_{i+1}^k = d_i^k$ *for all* i*, which means that the data coming from parabolas only satisfy it.*

In many practical applications, a good approximation of the final limit shape can be attained just after a few numbers of iterations. The technique of utilizing the condition for preserving convexity up to a limited number of iterations is satisfactory in many curve designs. Therefore, for a suitable choice of β_0 such that $\gamma_{n+1} \in \left[\frac{3}{90}, \frac{1}{27}\right]$, the convexity of this scheme can be preserved up to n-*th* subdivision level by satisfying sufficient condition on the initial data.
In order to work out ν_n up to first four subdivision levels, the results are given in Table 2 by choosing the initial parameter $\beta_0 = 2.26$.

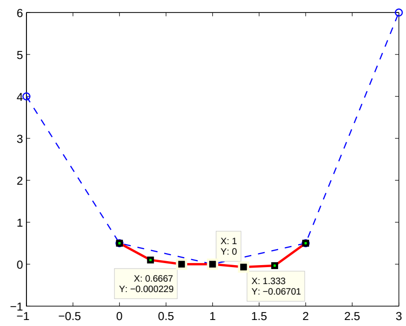
Table 2: the results of $\beta_{n+1},\ \gamma_{n+1}$ and ν_n for the first four subdivision levels, when $\beta_0 = 2.26$.

	First level	Second level	Third level	Forth level
β_{n+1}	2.06398	2.01593	2.00398	2.00099
γ_{n+1}	0.03337	0.036072	0.03679	0.03698
ν_n	2.35647	1.67743	1.45642	1.38477

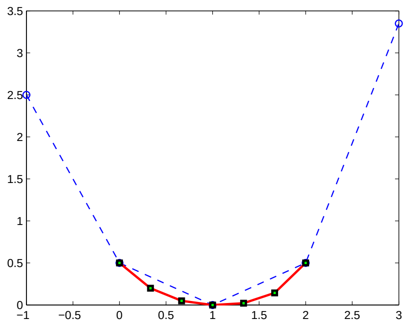
To elaborate the effectiveness of condition developed for convexity in Lemma 5.2, the polygons after first subdivision level are obtained for two different strictly convex initial data. It can be observed in Table 2 as depicted for $\beta_0 = 2.26$, the condition $R^0 < 2.35647$ must be satisfied with the initial data in order to generate convex sequence of control points after first level of refinement. Figure 5(a) displays the refined control polygon obtained by an initial data without the fulfillment of the above mentioned condition. Thus, d_i^1 defined by three indicated vertices is not positive in the refined polygon, which exposes the occurrence of an inflection point in this polygon. When the initial data is aligned under the required additional condition, new set of vertices $\{(0,\ 0.5),\ (0.3333,\ 0.1999),\ (0.6667,\ 0.04986),\ (1,\ 0),\ (1.333,\ 0.02147),\ (1.667,\ 0.1431),\ (2,\ 0.5)\}$ that are attained after first iteration generate strictly convex refined polygon, as revealed in Figure 5(b).
The initial data is considered under the condition of Theorem 5.3 in Figure 6. Moreover, the strictly convex limit function generated by the non-stationary scheme (2. 1) is depicted.
In Figure 7, construction of a car model is illustrated along with the initial control points using the non-stationary scheme (2. 1). For instance, it is always required for several parts on the top surface of the car models to be smoothly shaped. The least amount of folds in the covering will result in the unpleasant appearance of the car. To control unwanted oscillations, the top of car is obtained through a strictly convex data points as shown by the red solid line.

6. Conclusion

The preservation of shape properties is analyzed in the curves generated by the ternary 4-point non-stationary interpolating scheme. The conditions on the initial data are obtained for some suitable choice of initial parameter $\beta_0 \in [2,\ 6.62149)$, such that $\gamma_{k+1} \in \left(\frac{1}{90}, \frac{1}{27}\right]$, $\forall k \in \mathbb{Z}$. A level dependant proposal of shape preserving conditions is the most



(a)



(b)

Figure 5: The control polygons obtained after first iteration for two different strictly convex initial data.

significant feature, which makes it possible to preserve the shape properties of initial sequence in the curves generated after desired number of iterations. The results are also generalized in the limiting case using the fact that $\lim_{n\to\infty}\gamma_{n+1} = \frac{1}{27}$. Experimental results reveal that the allocation of initial data under the derived conditions is sufficient criteria for the preservation of shape properties in the generated subdivision curves.

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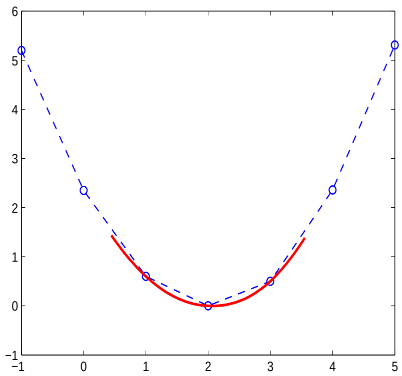

Figure 6: The limit function of the non-stationary scheme (2. 1) shown along with the initial control polygon by preserving convexity.

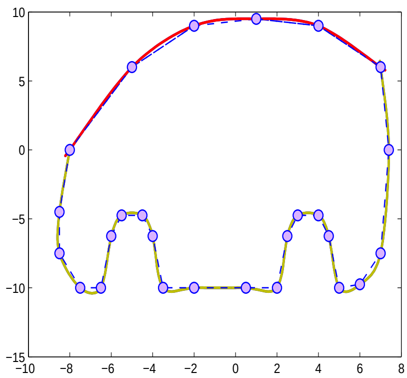

Figure 7: The construction of a car model using the non-stationary scheme (2. 1) shown along with the initial control polygon.

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