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Grothendieck's Quot Scheme and Moduli of Quotient Sheaves

Hafiz Syed Husain[a] and Mariam Sultana[b]
[a,b] Department of Mathematical Sciences,
Federal Urdu University of Arts, Science & Technology, Karachi-75300, Pakistan,
Email: [a]hsyed.hussain@fuuast.edu.pk, [b]marium.sultana@fuuast.edu.pk

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Abstract.: This paper is an exposition on how Grothendieck's Quot scheme can be seen as a solution to the moduli problem of quotient sheaves. These schemes as corresponding moduli spaces play a significant role, not only in solving problems within geometry in general, but they are also applicable to problems of classification in algebra. These Quot schemes are introduced in context of the setting up of a solution to a moduli problem. This is then followed by its application to particular problems of classifying continuously varying families of quotient sheaves with a fixed Hilbert polynomial. The purpose of this investigation is to bridge the often abstract settings of moduli theory as pioneered by Grothendieck, and its concrete application through solving particular moduli problems in concrete projective geometric setting. In particular, this study will present how algebraic geometry through moduli theory helps reveal significant information about the family of objects which it parameterizes.

AMS (MOS) Subject Classification Codes: 14D20; 14M15; 16D80
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1. Introduction

Roughly speaking, a moduli problem amounts to parameterizing an interesting class $\mathcal{A}$ of objects from a category $\mathcal{C}$ with the help of points of some geometric object, say X, up to an equivalence relation. This X under certain conditions is called the fine moduli space. It has its significance in problems pertaining to classification of objects in $\mathcal{A}$ up to the corresponding equivalence with the expectation that the geometry of X will produce valuable information for our understanding of $\mathcal{A}$. For instance, it is a well known fact that the class of all hypersurfaces of degree d in an n-dimensional projective space $\mathbb{P}^n$ corresponds to $\mathbb{P}^N$ such that $N = \binom{d+n}{n} - 1$ ([5] IX.2). For $n = 2$, it helps us better understand why two points determine a line, five points a conic, nine points a cubic, so on and so forth ([1] 10, pp. 67-81). In other words, in this case, the number of N points, which is a finite data, sufficiently determines the possibility of existence of any degree d hypersurface in $\mathbb{P}^n$, which is a classic example of application of moduli problems to the problems in combinatorics and enumerative geometry.[1] On the other hand, moduli problems can also help us state the classification problems of algebraic objects. For instance, it has been identified that if A is a $\mathbb{Z}$-graded finitely generated algebra isomorphic to the exterior algebra $\Lambda(V_K)$ of some finite dimensional vector space V over K, then every $\mathbb{Z}$-graded A-module M generated by $(n+1)$ elements which is also a finite K-dimensional vector space can be obtained from an algebraic vector bundle on $\mathbb{P}^n$ ([3] Appendix A).

An outline of the construction of Quot schemes is presented in section 2. There are two ways moduli problems can be stated, depending upon the representability or corepresentability of the corresponding moduli functors. There are many such moduli functors. For instance, there is one which determines families of closed subschemes of a given scheme under certain conditions called Hilbert functor, and there is one which determines families of subsheaves or quotient sheaves on a fixed scheme satisfying some conditions called Quot functor. The functors are found to be representable, and the representing objects are called Hilbert and Quot schemes, respectively ([8], [12]). We will restrict our attention to the case when the moduli functor is the Quot functor. In such a case, the representing object which solves the moduli problem is called fine moduli space. All schemes will be projective over some algebraically closed field K. Section 3 will present explicit calculations involving moduli problems and their solutions, i.e. moduli spaces, such that these moduli spaces are special cases of Quot scheme in concrete projective geometric setting.

2. Grothendieck's Quot Scheme as a Moduli Space

Let X be some projective scheme over an algebraically closed field K such that it comes equipped with the usual ample line bundle $\mathcal{O}_X(1)$.[2] Then for any coherent sheaf $\mathcal{H}$ on X, Euler characteristic and Hilbert polynomial of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$ and $P_{\mathcal{H}}$ respectively, are related by the formula

$$P_{\mathcal{H}}(t) = \chi(\mathcal{H} \otimes \mathcal{O}_X(t)) = \sum_{j=0}^{\dim(X)} (-1)^j \dim_K(H^j(X, \mathcal{H} \otimes \mathcal{O}_X(t))).$$

Define a contravariant functor

$$\mathcal{Q}uot^P_{X,\mathcal{E}} : \mathrm{Sch}_K \longrightarrow \mathrm{Set}$$

from the category of schemes over K to the category of sets with a fixed Polynomial $P \in \mathbb{Q}[t]$ and a coherent sheaf $\mathcal{E}$ on X, such that $\forall S \in \mathrm{Ob}(\mathrm{Sch}_K)$,

$$\mathcal{Q}uot^P_{X,\mathcal{E}}(S) = \{< \mathcal{Q}, q > \mid \text{ condition } \alpha \text{ is satisfied}\}$$

whereas we may state condition α and the designation of $< \mathcal{Q}, q >$ in terms of the following [12]:

[1] This was known to early Greek mathematicians, provided we fix $\mathcal{C}$ to be the category of algebraic curves over $\mathbb{R}$. These number of points is exactly the dimension of the moduli space $\mathbb{P}^N$ corresponding to the hypersurfaces of degree 1, 2, . . ., respectively.

[2] A more general construction is also permissible. For instance, we may allow X to be Noetherian and of 'finite type' over some Noetherian scheme S ([4], 2.1).

$\forall S \in \mathrm{Sch}_K$, if $X \times_K S$ is the fiber product of X and S over K,[3] with π_X and π_S being the canonical projections onto first and second factors respectively, then $< \mathcal{Q}, q >$ denotes the equivalence class of short exact sequences in $\mathrm{Coh}_K(X \times S)$ (i.e. category of coherent sheaves on $(X \times S)$) of the form

$$0 \longrightarrow \mathcal{K}_{\mathcal{Q},q} \longrightarrow \pi_X^*(\mathcal{E}) = \mathcal{E}_S \xrightarrow{q} \mathcal{Q} \longrightarrow 0 \tag{2. 1}$$

with $\mathcal{Q}$ flat over S having the Hilbert polynomial $P_{\mathcal{Q}} = P$, q being the surjective $\mathcal{O}_{X\times S}$-linear homomorphism with kernel $\mathcal{K}_{\mathcal{Q},q}$, and the equivalence class is determined by the equivalence relation: $< \mathcal{Q}, q >=< \mathcal{Q}', q' > \Longleftrightarrow$ the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_{\mathcal{Q},q} & \longrightarrow & \mathcal{E}_S & \xrightarrow{q} & \mathcal{Q} & \longrightarrow & 0 \\ & & \Big| \simeq & & \Big| \mathrm{id} & & \Big\downarrow \simeq & & \\ 0 & \longrightarrow & \mathcal{K}_{\mathcal{Q}',q'} & \longrightarrow & \mathcal{E}_S & \xrightarrow{q'} & \mathcal{Q}' & \longrightarrow & 0. \end{array} \tag{2. 2}$$

These classes may further satisfy some further conditions, depending upon the problem.[4] One may wonder whether the functor $\mathcal{Q}uot^P_{X,\mathcal{E}}$ is representable (cf. [14] III.2, pp. 22-23 for the definition of representability of a functor). Grothendieck (cited in [12]) proved that this functor is representable by a projective scheme over K which can be seen to be either the Grassmannian scheme itself or a projective scheme embedded inside Grassmannian. Let the projective scheme which represents this functor be denoted by $\overline{\mathrm{Quot}}^P_{X,\mathcal{E}}$. As a result, from Yoneda'a Lemma ([14], p. 61), we obtain the natural isomorphism,

$$\mathcal{Q}uot^P_{X,\mathcal{E}} \overset{\beta}{\simeq} \mathrm{Hom}_{Sch_K}(\,-\,, \overline{\mathrm{Quot}}^P_{X,\mathcal{E}}),$$

such that this isomorphism comes equipped with the commutative diagram:

$$\begin{array}{ccccc} \mathcal{Q}uot^P_{X,\mathcal{E}}(\overline{\mathrm{Quot}}^P_{X,\mathcal{E}}) & \xrightarrow{\beta_{\overline{\mathrm{Quot}}^P_{X,\mathcal{E}}}} & \mathrm{Hom}_{Sch_K}(\overline{\mathrm{Quot}}^P_{X,\mathcal{E}}, \overline{\mathrm{Quot}}^P_{X,\mathcal{E}}) & & S \\ \Big\downarrow {\scriptstyle \mathcal{Q}uot^P_{X,\mathcal{E}}(f)} & & \Big\downarrow {\scriptstyle f^*} & & \Big\downarrow {\scriptstyle f} \\ \mathcal{Q}uot^P_{X,\mathcal{E}}(S) & \xrightarrow{\beta_S} & \mathrm{Hom}_{Sch_K}(S, \overline{\mathrm{Quot}}^P_{X,\mathcal{E}}) & & \overline{\mathrm{Quot}}^P_{X,\mathcal{E}} \end{array} \tag{2. 3}$$

with both $\beta_{\overline{\mathrm{Quot}}^P_{X,\mathcal{E}}}, \beta_S$ being the invertible components of the natural isomorphism β. By the standard construction from category theory, there exists a universal object (or equivalently universal arrow), say $\overline{\mathcal{Q}}$ on $X \times \overline{\mathrm{Quot}}^P_{X,\mathcal{E}}$, which is the image $\beta^{-1}_{\overline{\mathrm{Quot}}^P_{X,\mathcal{E}}}(1_{\overline{\mathrm{Quot}}^P_{X,\mathcal{E}}}) \in \mathcal{Q}uot^P_{X,\mathcal{E}}(\overline{\mathrm{Quot}}^P_{X,\mathcal{E}})$. This corresponds to the continuously varying family of quotient sheaves over X parameterized by the points of $\overline{\mathrm{Quot}}^P_{X,\mathcal{E}}$ obtained as the restriction of $\overline{\mathcal{Q}}$ over the fibers $\pi^{-1}_{\overline{\mathrm{Quot}}^P_{X,\mathcal{E}}}(x) \subset X \times \overline{\mathrm{Quot}}^P_{X,\mathcal{E}}$ with universal property that if $\mathcal{Q}uot^P_{X,\mathcal{E}}(S)$

[3]From this point onwards, we will drop K as subscript and simply write $X \times S$.

[4]We may keep this arbitrary right now. It can be equality, it can be a condition on chern classes of coherent sheaves, etc. For instance, Okonek et al. [3] discussed the explicit construction of the moduli space of rank 2 stable vector bundles on $\mathbb{P}^2$ such that their first and second chern classes c_1 and c_2 respectively, satisfy the condition: 'either c_1 is odd, or, c_1 is even and $c_2 - c_1^2/4$ is odd.'

is any other family of quotients parameterized by points of S, which are restrictions of $\mathcal{Q}_{\mathcal{S}} \in \mathrm{Coh}_K(X \times S)$ (given by (2. 1) above) over the fiber $\pi_S^{-1}(s) \subset X \times S$, which are flat over S, then there exists a unique $f \in \mathrm{Hom}_{Sch_K}(S, \overline{\mathrm{Quot}}^P_{X,\mathcal{E}})$ such that the above diagram (2. 3) commutes. It is in this setting, we say that $\overline{\mathrm{Quot}}^P_{X,\mathcal{E}}$ is the fine moduli space to the fine moduli problem of classifying all quotient sheaves over X up to a given Hilbert polynomial. Solving this fine moduli problem amounts to determining this represented object $\overline{\mathrm{Quot}}^P_{X,\mathcal{E}}$ up to isomorphism.

3. Moduli Spaces of Quotient Sheaves

We now present explicit solutions to some fine moduli problems involving coherent quotient sheaves on $\mathbb{P}^1$ such that the corresponding moduli spaces are examples of Quot scheme as projective varieties. We will also derive some significant results which are correlatively motivated by these moduli problems. In this section, we are interested in parameterizing some interesting classes of quotient sheaves with a given Hilbert polynomial with points of Grassmannian. We first consider the class of sheaves given by the family $\mathcal{A}$ of short exact sequences (s.e.s.) of the form:

$$0 \longrightarrow \mathcal{F} \longrightarrow (\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})_S \xrightarrow{q_S} \mathcal{Q} \longrightarrow 0 \tag{3. 4}$$

on $\mathbb{P}^1 \times S$ obtained by pulling $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ via the functor $\pi^*_{\mathbb{P}^1}(\ ) \otimes \pi^*_S(\mathcal{O}_S)$, where $\mathcal{F}$ is any subsheaf of $(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})_S$ on $\mathbb{P}^1 \times S$. This helps us see $\mathcal{F}$ as family of subsheaves on $\mathbb{P}^1 \times S$, varying with points $s \in S$ such that it gives rise to a corresponding family of subsheaves $\mathcal{F}(s), \forall s \in S$, where $\mathcal{F}(s)$ is the pullback of $\mathcal{F}$ over the fiber $i^{-1}(s)$ via the canonical embedding $i : \mathbb{P}^1 \to \mathbb{P}^1 \times S$, giving us, $\forall s \in S$, the s.e.s. on $\mathbb{P}^1$,

$$0 \longrightarrow \mathcal{F}(s) \longrightarrow (\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}) \xrightarrow{q} \mathcal{Q}(s) \longrightarrow 0. \tag{3. 5}$$

In what follows, for notational brevity, we will keep writing $\mathcal{F}$ over $\mathbb{P}^1$ for $\mathcal{F}(s)$ and $\mathcal{Q}$ over $\mathbb{P}^1$ for $\mathcal{Q}(s)$ and presume that the context will make everything clear. We fix Hilbert polynomial for the quotient $\mathcal{Q}$ to be $P_{\mathcal{Q}}(t) = t + 3$. Then from Section 2 above, we can restate our problem in terms of a fine moduli problem or functor as follows: let $X = \mathbb{P}^1, P = t + 3, \mathcal{E}_S = (\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})_S,\ \forall S \in \mathrm{Ob}(\mathrm{Sch}_K)$, to obtain the moduli Quot functor as

$$\mathcal{Q}uot^{P_{\mathcal{Q}}}_{\mathbb{P}^1}(S) = \{< (\mathcal{Q}_S, q_S >\}. \tag{3. 6}$$

We have already seen that this Quot functor is representable by the Quot scheme $\overline{\mathrm{Quot}}^{P_{\mathcal{Q}}}_{\mathbb{P}^1}$ as a projective scheme. We solve this moduli problem by determining this projective scheme up to isomorphism.

Proposition 3.1. *$\overline{\mathrm{Quot}}^{P_{\mathcal{Q}}}_{\mathbb{P}^1}$ is isomorphic to $\mathbb{P}^3$.*

Proof. From ([11]4.4) we can define a map

$$\Phi : \overline{\mathrm{Quot}}^{P_{\mathcal{Q}}}_{\mathbb{P}^1} \longrightarrow G(P_{\mathcal{Q}}(n), H^0(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}(n)))$$

where $G(P_{\mathcal{Q}}(n), H^0(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}(n)))$ is the Grassmannian variety, consisting of $P_{\mathcal{Q}}(n)$-codimensional subspaces of the vector space $H^0(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}(n))$ for some particular choice of $n \geq 0$, such that $\Sigma = \mathrm{Im}(\Phi)$ satisfies the condition that $\forall x \in \Sigma$, we have

$$\dim_K((H^0(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}(m)))/(x \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m-n)))) = P_{\mathcal{Q}}(m), \forall m \geq n. \tag{3. 7}$$

The map Φ becomes an embedding if for some $n \geq 0$, we can show that $ker(q) = \mathcal{F}$ in s.e.s. (3. 5) satisfies the following two conditions [11](4.4.10):

(a) $H^1(\mathbb{P}^1, \mathcal{F}(n+k)) = 0, \ \forall k \geq 0$,
(b) $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) \otimes H^0(\mathbb{P}^1, \mathcal{F}(n)) = H^0(\mathbb{P}^1, \mathcal{F}(n+k)), \ \forall k \geq 0.$

We first note, from the additivity of Hilbert polynomial, that $P_{\mathcal{F}} + P_{\mathcal{Q}} = P_{\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}}$. Since $P_{\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}}(t) = 2t + 4$ this gives $P_{\mathcal{F}} = t + 1$. This shows that rank of $\mathcal{F}$ is 1 and is of the form $\mathcal{O}_{\mathbb{P}^1}$. This implies $H^1(\mathbb{P}^1, \mathcal{F}(k)) = 0, \forall k \geq 0$ ([13] III.5). This gives us full freedom for our choice of n. We fix $n = 0$, then both conditions (a) and (b) are satisfied by $\mathcal{F}$. Thus, $\overline{\mathrm{Quot}}_{\mathbb{P}^1}^{P_{\mathcal{Q}}}$ sits inside the Grassmannian $G(3, 4) = \mathbb{P}^3$.[5] However, we have to show that map Φ surjects, i.e. $\Sigma = \mathbb{P}^3$. In other words, we must show that $\forall x \in \mathbb{P}^3$, x satisfies condition (3. 7).
Consider $m \geq 0$, and the long exact sheaf cohomology sequence corresponding to s.e.s. (3. 5), we have

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{P}^1, \mathcal{F}(m)) &\longrightarrow H^0(\mathbb{P}^1, H^0(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}(m)) \\ \longrightarrow H^0(\mathbb{P}^1, \mathcal{Q}(m)) &\longrightarrow H^1(\mathbb{P}^1, \mathcal{F}(m)) \longrightarrow \dots . \end{aligned}$$

From the additivity of $\dim_K(\ )$ as a function on long exact sequence above, we obtain,

$$\dim_K(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}(m)) - \dim_K(x \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})) = m + 3 = P_{\mathcal{Q}}(m), \forall m \geq 0$$

therefore, $\Sigma = \mathbb{P}^3$. □

As already remarked above in Section 1, there are deeper connections between moduli theory of sheaves and classification problems in algebra. In order to work out a concrete example to motivate how such a connection makes sense, we start with some observations. Let A be an associative $\mathbb{Z}$-graded algebra over K, which is finitely generated by degree 1 component. Let M be the corresponding $\mathbb{Z}$-graded finitely generated left-module over A which is also a $\mathbb{Z}$-graded finite-dimensional K-vector space. We define the Grassmannian variety $G(n, M)$ to be the space which parameterizes all n-codimensional subspaces of M ([6], 1.5). But since M is also a finitely generated A-module, we may define a variant of $G(n, M)$ to be the $G_A(n, M) \subset G(n, M)$ such that $G_A(n, M)$ is the parameter space which parameterizes all $\mathbb{Z}$-graded left-A submodules of M. This is a projective subvariety of $G(n, M)$ [9]. In case $M = \oplus_{i \in \mathbb{Z}} M_i$ such that all M_i are finite dimensional K-vector spaces without (possibly) M itself being so, we may still have a variant of $G_A(n, M)$ construction by considering the truncations of M as follows. Define $M_{\geq p}$ to be the submodule of M consisting of all degree q-components such that $q \geq p$. Let $M_{[p,q]}$ be the submodule consisting of M_j such that $M_j \neq 0, p \leq j \leq q$, or equivalently, $M_{[p,q]} = M_p / M_q$. In particular, if $\mathcal{Q}uot_X^{P_{\mathcal{Q}}}(S)$ is the Quot functor, with X projective over K, parameterizing all quotient sheaves with fixed Hilbert polynomial $P_{\mathcal{Q}}$ which are flat over S, then this is representable by $\overline{\mathrm{Quot}}_X^{P_{\mathcal{Q}}}$ which is isomorphic to the projective limit of the inverse system $(G_q)_{q \geq p \gg 0}$, with

$$G_q := G_A(P_{\mathcal{Q}}, M_{[p,q]}) = \Pi_{r=p}^{r=q} G_A(P_{\mathcal{Q}}(r), M_r), p \leq r \leq q,$$

[5] $G(3, 4)$ corresponds to 3-codimensional subspaces inside the 4-dimensional vector space $H^0(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$.

where p is some sufficiently large integer satisfying some cohomology conditions [9] (cf. 1.2-1.3). Ciocan-Fontanine and Mikhail Kapranov [9] have shown that for every such A and M, there exists p and q with $0 \ll p \leq q$ such that,

$$\overline{\mathrm{Quot}}_X^{P_{\mathcal{Q}}} \simeq \varprojlim (G_q)_{q\geq p\gg 0} \simeq \Pi_{r=p}^{r=q} G_A(P_{\mathcal{Q}}(r), M_r),$$

provided X is projective with ample line bundle $\mathcal{O}_X(1), \mathcal{Q} \simeq \mathcal{M}/\mathcal{K}$, for some $\mathcal{M}$ and $M = \oplus_{i\in\mathbb{Z}} H^0(X, \mathcal{M}(i))$. Let's call the inverse system $(G_q)_{q\geq p\gg 0}$, A-Grassmannian. In this context, solving a moduli problem amounts to determining the algebraic data (A, M, p, q) which will then completely determine the projective limit up to isomorphism. As a particular application to this, we have the following statement.

Corollary 3.2. *Let $\mathbb{P}^3$ be the moduli space for the moduli problem determined by* (3. 6)*. Then there exists A-Grassmannian inverse system $(G_q)_{q\geq p\gg 0}$ such that $\mathbb{P}^3 = \varprojlim (G_q)_{q\geq p\gg 0}$.*

Proof. Let $A = \bigoplus_{j\in\mathbb{Z}} (H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j)))$ be the associative $\mathbb{Z}$-graded algebra generated by the degree 1 component A_1. Let $M = \bigoplus_{j\in\mathbb{Z}} (H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}(j)))$. Then M is a finitely generated $\mathbb{Z}$-graded left A-module. However, M is not finite K-dimensional vector space. Consider the truncation of M to $M_{[p,q]}$, then this truncation becomes finite K-dimensional. From [6](1.5), [7] and [9], we obtain $n_0 = 0$ such that $\forall p \geq n_0$, p satisfies the required cohomology conditions, with $q = 2p + 4$. So we can now consider $(G_q)_{q\geq p\gg 0}$. Then from [9] (1.4) we have $\varprojlim (G_q)_{q\geq p\gg 0} \simeq \mathbb{P}^3$. □

Let us pull back the s.e.s. (3. 5) to $\mathbb{P}^1 \times \overline{\mathrm{Quot}}_{\mathbb{P}^1}^{P_{\mathcal{Q}}(t)} \simeq \mathbb{P}^1 \times \mathbb{P}^3$ which gives us

$$\mathcal{Q}uot_{\mathbb{P}^1}^{P_{\mathcal{Q}}}(\mathbb{P}^3) = \{< (\mathcal{Q})_{\mathbb{P}^3}, q_{\mathbb{P}^3} >\}$$

Let $\overline{\mathcal{Q}}$ be the universal quotient on $\mathbb{P}^1 \times \mathbb{P}^3$. From this point onwards, unless stated otherwise, $\mathbb{P}^3$ will always be considered as the fine moduli space which solved the moduli problem stated in terms of moduli functor $\mathcal{Q}uot_{\mathbb{P}^1}^{P_{\mathcal{Q}}}$. Let $\pi_{\mathbb{P}^3}$ be the projection onto the second factor, with fibers $\pi_{\mathbb{P}^3}^{-1}(x), \forall x \in \mathbb{P}^3$, such that $\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}$ denotes the restriction of $\overline{\mathcal{Q}}$ onto the corresponding fiber. Also, from this point onwards, set $K = \mathbb{C}$. Then we want to determine when the quotient sheaves $\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}$ over the fiber $\pi_{\mathbb{P}^3}^{-1}(x) \simeq \mathbb{P}^1 \times \{x\}$ are algebraic vector bundles on $\mathbb{P}^1$ for $x \in \mathbb{P}^3$. By algebraic, we mean that the transition functions are algebraic. This is determined by the following cohomology condition.

Proposition 3.3. *Let $x \in \mathbb{P}^3$ such that $H^1(\pi_{\mathbb{P}^3}^{-1}(x), \overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}(-3)) \not\simeq \mathbb{C}$. Then $\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}$ is an algebraic vector bundle on $\mathbb{P}^1$.*

Proof. We prove it contrapositively. Let $x \in \mathbb{P}^3$ such that $\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}$ is not an algebraic vector bundle. Since all coherent sheaves on projective algebraic curves are locally modules over principal ideal domains, therefore, there exists an affine open neighborhood U of $\pi_{\mathbb{P}^3}^{-1}(x)$ such that,

$$\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}|_U = (K[X])^{\oplus a} \oplus T, a \geq 0.$$

This implies, there always exists a maximal torsion subsheaf, say $\mathcal{G}$, such that $\mathcal{G}|_U = T$ which is the torsion part, giving us locally,

$$(\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}|_U)/(\mathcal{G}|_U) \simeq (K[X]^{\oplus a}),$$

where X is the affine coordinate corresponding to U. Gluing up affine neighborhoods, we obtain a maximal torsion subsheaf $\mathcal{G}x$, for every $\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}$ not locally free such that the quotient $(\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)})/\mathcal{G}x = Q$ is torsion free, and thus locally free on $\mathbb{P}^1$. This gives us the following s.e.s.

$$0 \longrightarrow \mathcal{G}x \longrightarrow \overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)} \longrightarrow Q \longrightarrow 0. \tag{3. 8}$$

Then from the additivity of Hilbert polynomials, we get, $P_{\mathcal{G}x}(t) \geq 1$ and $P_Q(t) \leq t+2$, which follows from the fact that Hilbert polynomial is locally constant on flat families ([4] 2.1). The Hilbert polynomial of Q shows that it is a line bundle on $\mathbb{P}^1$, hence, it must be of the type $\mathcal{O}_{\mathbb{P}^1}(k), k \leq 1$ ([13] II.6). From s.e.s. (3. 8) we obtain the long exact cohomology sequence,

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{G}x(-3)) \longrightarrow H^0(\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}(-3)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(k-3)) \longrightarrow H^1(\mathcal{G}x(-3)) \\ \longrightarrow H^1(\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}(-3)) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^1}(k-3)) \longrightarrow 0 \longrightarrow \ldots \end{aligned}$$

Since $H^1(\mathcal{G}x(-3)) = 0$, which follows from the fact that,

$$\chi(\mathcal{G}x(t-3)) = \chi(\mathcal{G}x(t)) = P_{\mathcal{G}x}(t),$$

where χ denotes the Euler characteristic (cf. section 2 above),[6] we get

$$H^1(\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}(-3)) \simeq H^1(\mathcal{O}_{\mathbb{P}^1}(k-3)).$$

Then applying Serre's Duality ([13] III.7) on right hand side, we obtain

$$H^1(\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}(-3)) \simeq H^0(\mathcal{O}_{\mathbb{P}^1}(1-k))^* \tag{3. 9}$$

So, $H^1(\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}(-3)) \not\simeq 0$, only for $1-k \geq 0$. This leaves us with two choices for k: i.e. $k = 0, 1$. We must show that k is only equal to 1. From s.e.s. (3. 5) and (3. 8), we obtain the following commutative diagram with exact rows and columns([2] 16.3):

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{G}x & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} & \longrightarrow & \overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & Q & = & Q & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{3. 10}$$

[6]One can also get $H^1(\mathcal{G}x(-3)) = 0$ by observing that $\mathcal{G}x$ is a torsion sheaf on $\mathbb{P}^1$, therefore, its support is zero-dimensional. Authors are thankful to one of the anonymous reviewer of this paper for bringing this into their notice as an alternative to their reasoning.

This gives $P_{\mathcal{P}}(t) = P_{\mathcal{F}}(t) + P_{\mathcal{G}x}(t) \Rightarrow P_{\mathcal{P}}(t) \geq (t+1)+n$. Also, since $\mathcal{P}$ is a submodule of $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$, we have $n \geq 1$. This gives $\mathcal{P} \simeq \mathcal{O}_{\mathbb{P}^1}(n)$. Also, from $P_{\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}}(t) = P_{\mathcal{P}}(t) + P_{Q}(t) \Rightarrow 2t+4 = [(t+1)+n] + (t+k+1) \Rightarrow k+n = 2$. On the other hand, diagram (3. 10) gives the chain of injections,

$$\mathcal{F} \hookrightarrow \mathcal{P} \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2},$$

which helps give,

$$\deg(\mathcal{F}) \leq \deg(\mathcal{P}) \leq \deg(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$$

where 'deg( )' denotes the degree of coherent sheaf on any projective curve X, given by the formula,

$$\deg(\mathcal{H}) = \chi(\mathcal{H}) - r\chi(\mathcal{O}_X),$$

where r denotes the rank of $\mathcal{H}$ ([13] IV.1). Taking $X = \mathbb{P}^1$, we obtain,

$$\deg(\mathcal{P}) = \chi(\mathcal{P}) - r\chi(\mathcal{O}_{\mathbb{P}^1}) = 0, 1.$$

Now suppose $\deg(\mathcal{P}) = 0 \Rightarrow \chi(\mathcal{P}) = 1 \Rightarrow n = 0$, which is impossible since in that case $\mathcal{G}x = 0$ which in turn would give $\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}$ locally free, but this is a contradiction, for we have already chosen $x \in \mathbb{P}^3$ such that $\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}$ is not locally free. Therefore, $\deg(\mathcal{P}) = 1$. Hence, $\chi(\mathcal{P}) - r = 1 \Rightarrow n = 1 \Rightarrow k = 1$. Putting this $k = 1$ in the isomorphism (3. 9), we obtain the result. □

The next question that naturally comes to one's mind after proving Propositions 3.1 & 3.3, is this: What exactly is the geometry of the points $x \in \mathbb{P}^3$ for which the restriction of universal quotient onto the fiber $\pi_{\mathbb{P}^3}^{-1}(x) \subset \mathbb{P}^1 \times \mathbb{P}^3$ is not an algebraic vector bundle? Or, more significantly, how is being or not being an algebraic vector bundle reflected in the geometry of the moduli space $\mathbb{P}^3$?

Proposition 3.4. *Let $Y \subset \mathbb{P}^3$ such that $\forall x \in Y$, $\overline{\mathcal{Q}}|_{\pi_{\mathbb{P}^3}^{-1}(x)}$ is not an algebraic vector bundle. Then Y is a quadric surface in $\mathbb{P}^3$.*

Proof. From diagram (3. 10) above, which characterizes Y as subset of $\mathbb{P}^3$, we get the following commutative square,

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{h} & \mathcal{O}_{\mathbb{P}^1}(1) \\ \Big\| \text{id} & & \Big\downarrow (\alpha,\beta) \\ \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{(f,g)} & \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \end{array} \qquad (3.\ 11)$$

such that all sheaf morphisms are injections. It is obvious that f, g and h are all global sections of $\mathcal{O}_{\mathbb{P}^1}(1)$, i.e. $f, g, h \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. On the other hand, we have

$$\begin{aligned} (\alpha, \beta) \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}) &\simeq \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}) \\ &\simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}) \\ &\simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)). \end{aligned}$$

Thus, from the commutativity of diagram (3. 11), we get $(f, g) = (\alpha h, \beta h)$. giving us the morphism,

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(1)) \qquad (3.\ 12)$$

$$(\alpha, \beta) \otimes h \longrightarrow (\alpha h, \beta h).$$

Projectivizing (3. 12), we obtain the Segre embedding

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

defined by

$$[\alpha, \beta] \times [u_0, u_1] \xrightarrow{\sigma} [\alpha u_0, \alpha u_1, \beta u_0, \beta u_1]$$

giving us a homogeneous quadratic polynomial $P(z_0, z_1, z_2, z_3) = z_0 z_3 - z_1 z_1$ which determines a quadric surface in $\mathbb{P}^3$ ([10] 2.11). □

4. Conclusion

In this paper, we have presented explicit calculations for solving a particular moduli problem involving quotient sheaves on $\mathbb{P}^1$ such that the solution $\mathbb{P}^3$ obtained was a particular case of Grothendieck Quot scheme. This was our Proposition 3.1. We have also discussed how this problem solving relates with algebra. This was our Corollary 3.2. In particular, we presented how this parametrization of quotient sheaves by the points of the moduli space $\mathbb{P}^3$ reflected significant information about these quotient sheaves. This information had two aspects. One algebraic and the other geometric. With focus on universal quotient, the algebraic aspect helped us understand how we can further classify or distinguish which of these quotient sheaves form an algebraic vector bundle. This was our Proposition 3.3. The geometric aspect showed how this information is reflected in the geometry of the moduli space. This was our Proposition 3.4.

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