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Extension in the Geometry of Goncharov Motivic and Configuration Chain Complexes

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Abstract. This study aims to propose extensions of morphisms in the geometry of configuration and Goncharov motivic chain complexes. First, the geometry of these complexes will be extended for weight 3 by introducing some interesting morphisms and then this extension for weight 4 and 5 will be presented. The commutativity of these generalized diagrams will also be proven.

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1. INTRODUCTION

Grassmannian complex was first introduced by Suslin [11, 12, 17]. Goncharov [5–7] defined trilogarithmic group $\mathcal{B}_3(F)$ and generalized Bloch-Suslin complex known as Goncharov's motivic Complex, then connected the Grassmannian configuration with the Bloch-Suslin complex and Goncharov motivic complex for weight 2 and 3, while proving the associated diagrams to be bicomplex and commutative. Cathelineau [1, 2] used a derivation map to introduce a variant of Goncharov complex in two ways, one was infinitesimal, while the other was in a tangential setting. Khalid et al. [9, 10, 13] defined a new geometry for the configuration and variant of Cathelineau complexes up to weight $n = N$. The same author also defined some extensions in the geometry of variant of Cathelineau and Grassmannian chain complexes for weight 4 and 5 [14, 15]. In this paper, the proposed work of Khalid et al. in [13] is extended in the geometry of Goncharov motivic and configuration complexes.

This paper has the following structure: Section 2 describes the concepts of configuration chain complexes, cross ratio, projected cross ratio, triple cross ratio, classical polylogarithmic groups complexes and Goncharov's generalized polylog chain complex. Section

3 presents the geometry, and some extensions for the geometry, of Grassmannian configuration and Goncharov motivic chain complexes for weight 3 up to weight 5 to produce commutative diagrams. The last section concludes the entire research work.

2. Preliminary and Background

This section discusses the preliminary background relevant to this research in detail. It covers the Grassmannian complex, cross ratio, classical polylog groups and Goncharov motivic polylog chain complexes, all of which are crucial for this research study.

2.1. **Grassmannian Complex.** Let $G_{n+1}(n)$ be a free abelian group generated by configuration of $(n+1)$ points in n-dimensional vector space V^n over some arbitrary number field F. These groups form Grassmannian chain complex as given below

$$\begin{array}{ccccc} G_{n+4}(n+2) & \xrightarrow{d} & G_{n+3}(n+2) & \xrightarrow{d} & G_{n+2}(n+2) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_{n+3}(n+1) & \xrightarrow{d} & G_{n+2}(n+1) & \xrightarrow{d} & G_{n+1}(n+1) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_{n+2}(n) & \xrightarrow{d} & G_{n+1}(n) & \xrightarrow{d} & G_{n}(n) \end{array} \tag{A}$$

Lemma 2.2. *The diagram A is bi-complex and commutative* (see [17]).

2.3. **Cross Ratio.** Cross ratio of four points is defined as

$$r(v_0, v_1, v_2, v_3) = \frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)},$$

where $\triangle(v_0, v_3) = (v_3 - v_0)$ shows determinant of points v_0 and v_3 in 2-dimensional vector space. Siegel [16] introduced the following most important property of cross ratio:

$$1 - r(v_0, v_1, v_2, v_3) - r(v_0, v_2, v_1, v_3) = 0. \tag{2.1}$$

2.3.1. *Projected Cross Ratio.* Goncharov [5] defined the following projected cross ratio of four points with single projected point as

$$r(v_i|(v_0, v_2, v_1, v_3) = \frac{\triangle(v_i|v_0, v_3)\triangle(v_i|v_1, v_2)}{\triangle(v_i|v_0, v_2)\triangle(v_i|v_1, v_3)} = \frac{\triangle(v_i, v_0, v_3)\triangle(v_i, v_1, v_2)}{\triangle(v_i, v_0, v_2)\triangle(v_i, v_1, v_3)}, \tag{2.2}$$

where $\triangle(v_i|v_0, v_3)$ shows determinant of points v_i, v_0 and v_3.

2.3.2. *Triple Cross Ratio.* Goncharov [5] generalized cross ratio as a triple cross ratio of six points, given by

$$r(v_0, ..., v_5) = \frac{\triangle(v_0, v_1, v_5)\triangle(v_1, v_2, v_3)\triangle(v_2, v_0, v_4)}{\triangle(v_0, v_1, v_3)\triangle(v_1, v_2, v_4)\triangle(v_2, v_0, v_5)}. \tag{2.3}$$

Theorem 2.4. *The ratio of two projected cross ratio of four points can be written in the form of triple cross ratio of six points.*

Proof. Let us assume $(v_0, ..., v_5)$ to be six points with two projected points v_1 and v_2, then

$$\frac{r(v_2|v_0,v_1,v_3,v_5)}{r(v_1|v_0,v_2,v_4,v_5)} = \frac{\triangle(v_2,v_0,v_5)\triangle(v_2,v_1,v_3)}{\triangle(v_2,v_0,v_3)\triangle(v_2,v_1,v_5)} / \frac{\triangle(v_1,v_0,v_5)\triangle(v_1,v_2,v_4)}{\triangle(v_1,v_0,v_4)\triangle(v_1,v_2,v_5)}$$
$$= \frac{\triangle(v_2,v_0,v_5)\triangle(v_2,v_1,v_3)\triangle(v_1,v_0,v_4)}{\triangle(v_2,v_0,v_3)\triangle(v_1,v_0,v_5)\triangle(v_1,v_2,v_4)}. \quad (2.4)$$

□

2.5. **Classical Polylog Groups and Complexes.** The classical p-logarithms series is expressed as $Li_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n^p}$, in the unit disc. For $p = 1$, $Li_1(x) = -Li(1-x)$ with generalized form $\log x + \log y = \log xy$. F is a field and $F^{\bullet\bullet} = F - \{0, 1\}$. let $Z[\mathbf{P}^1_F/\{0, 1, \infty\}]$ is a free abelian group generated by [x], where [x] means logarithms of an element x. The group $\mathcal{B}(F)$ is a quotient of $Z[\mathbf{P}^1_F]$ by its subgroup generated by Abel's five term relation $[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-y^{-1}}{1-x^{-1}}\right] + \left[\frac{1-y}{1-x}\right]$, where x$\neq y$ and $x, y \neq 0, 1$ [3,4,8].

2.6. **Goncharov Motivic Polylog Chain Complexes.** Let $\mathcal{B}_2(F) = Z[\mathbf{P}^1_F/\{0, 1, \infty\}]/ < \mathsf{R}_2(F) >$, where $< \mathsf{R}_2(F) >$ is generated by group $\mathsf{R}_2(F) = \sum_{i=0}^{4}(-1)^i r(v_0, ..., \hat{v}_i, ..., v_4)$, which is a five term relation of cross ratio. Construct a chain

$$\mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^\times \ .$$

Where, δ is a morphism defined as $\delta : [x]_2 \to (1 - x) \wedge x$. This complex is called Bloch-Suslin complex for weight 2. For weight 3, Goncharov [5] defined a seven-term relation of triple cross ratio of six points given as

$$\mathsf{R}_3(F) = \sum_{i=0}^{6}(-1)^i Alt_6\Big[r(v_0, ..., \hat{v}_i, ..., v_6)\Big]. \quad (2.5)$$

Goncharov [5] introduced a subgroup group $\mathcal{B}_3(F) = Z[\mathbf{P}^1_F/\{0, 1, \infty\}]/ < \mathsf{R}_3(F) >$. Following is a chain complex for weight 3

$$\mathcal{B}_3(F) \xrightarrow{\delta} \mathcal{B}_2(F) \otimes F^\times \xrightarrow{\delta} \wedge^3 F^\times \ .$$

Finally, Goncharov [5] generalized the subgroup $\mathcal{B}_n(F) = Z[\mathbf{P}^1_F/\{0, 1, \infty\}]/ < \mathsf{R}_n(F) >$, then introduced a generalized chain complex expressed as

$$\mathcal{B}_n(F) \xrightarrow{\delta} \mathcal{B}_{n-1}(F) \otimes F^\times \xrightarrow{\delta} \mathcal{B}_{n-2}(F) \otimes \wedge^2(F) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{B}_2(F) \otimes \wedge^{n-2}(F) \xrightarrow{\delta} \frac{\wedge^n F^\times}{2 - torsion}. \quad (2.6)$$

3. Geometry and Extension upto Weight 6 of Grassmannian and Goncharov Complexes

3.1. Geometry for Weight 2. As defined in [13], the geometry of Grassmannian configuration and Goncharov motivic in weight-2 is represented as

$$\begin{array}{ccccc} G_6(3) & \xrightarrow{d} & G_5(3) & & \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\ G_5(2) & \xrightarrow{d} & G_4(2) & \xrightarrow{g_1^2} & \mathcal{B}_2(F) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \delta} \\ G_4(1) & \xrightarrow{d} & G_3(1) & \xrightarrow{g_0^2} & \wedge^2 F^\times \end{array} \tag{B}$$

where,

$$g_0^2(v_0, ..., v_2) = -\triangle(v_1) \wedge \triangle(v_2) + \triangle(v_0) \wedge \triangle(v_2) - \triangle(v_0) \wedge \triangle(v_1) \tag{3.7}$$

and

$$g_1^2(v_0, ..., v_3) = [r(v_0, ..., v_3)]_2. \tag{3.8}$$

Lemma 3.2. *The diagram B is bi-complex and commutative* [13].

3.3. Geometry for Weight 3. The geometry of Grassmannian and Goncharov motivic for weight-3 is presented in [13] as follows:

$$\begin{array}{ccccc} G_7(3) & \xrightarrow{d} & G_6(3) & & \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\ G_6(2) & \xrightarrow{d} & G_5(2) & \xrightarrow{g_1^3} & \mathcal{B}_2(F) \otimes F^\times \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \delta} \\ G_5(1) & \xrightarrow{d} & G_4(1) & \xrightarrow{g_0^3} & \wedge^3 F^\times \end{array} \tag{C}$$

where,

$$g_0^3(v_0, v_1, v_2, v_3) \to \sum_{i=j+1}^{3} (-1)^{i+1} \bigwedge_{j \neq i}^{3} \triangle(v_j) \ (mod\ 4) \tag{3.9}$$

and

$$g_1^3(v_0, v_1, ..., v_4) \to -\frac{1}{3} \sum_{i=0}^{4} (-1)^i [r(v_0, ..., \hat{v}_i, ..., v_4)]_2 \otimes \prod_{i \neq r}^{4} \triangle(v_i, v_r) \ (mod\ 5). \tag{3.10}$$

Lemma 3.4. *The diagram C is bi-complex and commutative* [13].

3.4.1. *Extension in Geometry for Weight 3.* For weight 3, morphism g_2^3 is introduced to connect Grassmannian and Goncharov complexes and extend the commutative diagram:

$$\begin{array}{ccccc} G_7(3) & \xrightarrow{d} & G_6(3) & \xrightarrow{g_2^3} & \mathcal{B}_3(F) \\ \downarrow p & & \downarrow p & & \downarrow \delta \\ G_6(2) & \xrightarrow{d} & G_5(2) & \xrightarrow{g_1^3} & \mathcal{B}_2(F)\otimes F^\times \\ \downarrow p & & \downarrow p & & \downarrow \delta \\ G_5(1) & \xrightarrow{d} & G_4(1) & \xrightarrow{g_0^3} & \wedge^3 F^\times \end{array} \tag{D}$$

where,

$$g_2^3(v_0,...,v_5) = \frac{1}{15}Alt_6\Big[\frac{\triangle(v_0,v_1,v_5)\triangle(v_1,v_2,v_3)\triangle(v_2,v_0,v_4)}{\triangle(v_0,v_1,v_3)\triangle(v_1,v_2,v_4)\triangle(v_2,v_0,v_5)}\Big]_3. \tag{3.11}$$

Morphism g_2^3 is well defined because if the length of vectors or volume formation is changed, then, due to homogeneity property, $\triangle(\alpha v_0) = \alpha\triangle(v_0)$ and no effect is observed.

Lemma 3.5. $g_1^3 \circ p = \delta \circ g_2^3.$

Proof. Let $(v_0,...,v_5)$ be 6 points $\in G_6(3)$ such that the differential map p is a applied to the 6 points as

$$p(v_0,...,v_5) = \sum_{i=0}^{5}(-1)^i(v_i|v_0,...,\hat{v}_i,...,v_5). \tag{3.12}$$

Now by applying morphism g_1^3 on Eq.(3.12):

$$g_1^3\circ p(v_0,...,v_5) = -\frac{1}{3}\sum_{j\neq i}^{5}(-1)^j\sum_{i\neq j}^{5}(-1)^i[r(v_i|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2\otimes \prod_{r\neq j}^{5}\triangle(v_i|v_r,v_j). \tag{3.13}$$

Let us consider $(v_0,...,v_5)\in G_6(3)$ again and apply morphism g_2^3

$$g_2^3(v_0,...,v_5) = \frac{1}{15}Alt_6\Big[\frac{\triangle(v_0,v_1,v_5)\triangle(v_1,v_2,v_3)\triangle(v_2,v_0,v_4)}{\triangle(v_0,v_1,v_3)\triangle(v_1,v_2,v_4)\triangle(v_2,v_0,v_5)}\Big]_3. \tag{3.14}$$

Now by applying homomorphism δ

$$\delta\circ g_2^3 = \frac{1}{15}Alt_6\Big[\frac{\triangle(v_0,v_1,v_5)\triangle(v_1,v_2,v_3)\triangle(v_2,v_0,v_4)}{\triangle(v_0,v_1,v_3)\triangle(v_1,v_2,v_4)\triangle(v_2,v_0,v_5)}\Big]_2\otimes \frac{\triangle(v_0,v_1,v_5)\triangle(v_1,v_2,v_3)\triangle(v_2,v_0,v_4)}{\triangle(v_0,v_1,v_3)\triangle(v_1,v_2,v_4)\triangle(v_2,v_0,v_5)}. \tag{3.15}$$

Simplify Eq.(3.15) using Siegel properties of sections (2.3.1, 2.3.2), Theorem (2.4), tensor and odd cycle properties, to obtain

$$\delta \circ g_2^3(v_0, ..., v_5) = -\frac{1}{3}\sum_{j\neq i}^{5}(-1)^j \sum_{i\neq j}^{5}(-1)^i[r(v_i|v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_5)]_2 \otimes \prod_{r\neq j}^{5} \triangle(v_i|v_r, v_j). \tag{3.16}$$

From Eq.(3.13) and Eq.(3.16), it is observed that, $g_1^3 \circ p = \delta \circ g_2^3$. □

3.6. **Geometry for Weight 4.** Geometry for weight 4 is defined in [13] as follows

$$\begin{array}{ccccc}
G_8(3) & \xrightarrow{d} & G_7(3) & & \\
\downarrow p & & \downarrow p & & \\
G_7(2) & \xrightarrow{d} & G_6(2) & \xrightarrow{g_1^4} & \mathcal{B}_2(F) \otimes \wedge^2 F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_6(1) & \xrightarrow{d} & G_5(1) & \xrightarrow{g_0^4} & \wedge^4 F^\times
\end{array} \tag{E}$$

where

$$g_0^4(v_0, ..., v_4) \to \sum_{i=j+1}^{4} (-1)^{i+1} \bigwedge_{j\neq i}^{4} \triangle(v_j)\ (mod\ 5) \tag{3.17}$$

and

$$g_1^4(v_0, ..., v_5) \to \frac{1}{6}\sum_{i\neq j}^{5}(-1)^i[r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_5)]_2 \otimes \prod_{r\neq i}^{5} \triangle(v_i, v_r) \wedge \prod_{r\neq j}^{5} \triangle(v_j, v_r)\ (mod\ 6). \tag{3.18}$$

Lemma 3.7. *The diagram* E *is commutative* (see [13]).

3.7.1. *Extension in Geometry for Weight 4.* For this extension, two morphisms g_2^4 and g_3^4 are introduced

$$\begin{array}{ccccc}
G_9(4) & \xrightarrow{d} & G_8(4) & \xrightarrow{g_3^4} & \mathcal{B}_4(F) \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_8(3) & \xrightarrow{d} & G_7(3) & \xrightarrow{g_2^4} & \mathcal{B}_3(F) \otimes F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_7(2) & \xrightarrow{d} & G_6(2) & \xrightarrow{g_1^4} & \mathcal{B}_2(F) \otimes \wedge^2 F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_6(1) & \xrightarrow{d} & G_5(1) & \xrightarrow{g_0^4} & \wedge^4 F^\times
\end{array} \tag{F}$$

$$g_2^4(v_0,...,v_6) = -\frac{1}{28}\sum_{i=0}^{6}(-1)^i Alt_6\Big[r(v_0,...,\hat{v}_i,...,v_6)\Big]_3 \otimes \prod_{r\neq i\neq j}^{6} \triangle(v_r,v_i,v_j), \quad (3.19)$$

$$g_3^4(v_0,...,v_7) = \frac{1}{66}Alt_8[r(v_0,...,v_7)]_4. \quad (3.20)$$

Following lemmas (3.8) and (3.9), shows that the extended diagram F is commutative.

Lemma 3.8. $g_1^4 \circ p = \delta \circ g_2^4.$

Proof. Let $(v_0,...,v_6)$ be 7 points $\in G_7(3)$, applying the differential map p yields

$$p(v_0,...,v_6) = \sum_{i=0}^{6}(-1)^i(v_i|v_0,...,\hat{v}_i,...,v_6). \quad (3.21)$$

Now applying morphism g_1^4

$$g_1^4 \circ p(v_0,...,v_6) = \frac{1}{6}\sum_{j\neq i}^{6}(-1)^j\sum_{i\neq j}^{6}(-1)^i[r(v_i|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6)]_2 \otimes \prod_{r\neq j}^{6}\triangle(v_i|v_j,v_r)\wedge\prod_{r\neq k}^{6}\triangle(v_i|v_k,v_r). \quad (3.22)$$

Taking $(v_0,...,v_6) \in G_7(3)$ again and apply morphism g_2^4 followed by the composition with morphism δ, yields

$$g_2^4(v_0,...,v_6) = -\frac{1}{28}\sum_{i=0}^{6}(-1)^i Alt_6\Big[r(v_0,...,\hat{v}_i,...,v_6)\Big]_3 \otimes \prod_{r\neq i\neq j}^{6} \triangle(v_r,v_i,v_j). \quad (3.23)$$

$$\delta\circ g_2^4 = -\frac{1}{28}\sum_{i=0}^{6}(-1)^i Alt_6\Big[r(v_0,...,\hat{v}_i,...,v_6)\Big]_2 \otimes r(v_0,...,\hat{v}_i,...,v_6)\wedge\prod_{r\neq i}^{6}\triangle(v_r,v_i,v_j). \quad (3.24)$$

After simplifying Eq. (3.24), by using Siegel properties (2.3.1, 2.3.2), Theorem (2.4), tensor and odd cycle properties:

$$\delta\circ g_2^4(v_0,...,v_6) = \frac{1}{6}\sum_{j\neq i}^{6}(-1)^j\sum_{i\neq j}^{6}(-1)^i[r(v_i|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6)]_2 \otimes \prod_{r\neq j}^{6}\triangle(v_i|v_j,v_r)\wedge\prod_{r\neq k}^{6}\triangle(v_i|v_k,v_r). \quad (3.25)$$

From Eq. (3.22) and Eq. (3.25), it is observed that, $g_1^3\circ p = \delta\circ g_2^3$. □

Lemma 3.9. $g_2^4 \circ p = \delta \circ g_3^4.$

Proof. Let $(v_0,...,v_7)$ be 8 points $\in G_8(4)$ and by applying differential map p followed by composition with morphism g_2^4, which yield,

$$p(v_0,...,v_7)=\sum_{i=0}^{7}(-1)^i(v_i|v_0,...,\hat{v}_i,...,v_7). \tag{3.26}$$

$$g_2^4\circ p(v_0,...,v_7)=-\frac{1}{28}\sum_{j\neq i}^{7}(-1)^j\sum_{i\neq j}^{7}(-1)^i Alt_6\Big[r(v_i|v_0,...,\hat{v}_i,\hat{v}_j,...,v_7)\Big]_3\otimes \prod_{r\neq j\neq k}^{7}\triangle(v_i|v_r,v_j,v_k). \tag{3.27}$$

Let us take $(v_0,...,v_7)\in G_8(4)$ again, applying morphism g_3^4

$$g_3^4(v_0,...,v_7)=\frac{1}{66}Alt_8[r(v_0,...,v_7)]_4. \tag{3.28}$$

Now by applying differential morphism δ, the above Eq. (3.7.1) becomes

$$\delta\circ g_3^4(v_0,...,v_7)=\frac{1}{66}Alt_8[r(v_0,...,v_7)]_3\otimes r(v_0,...,v_7). \tag{3.29}$$

By using Siegel properties (2.3.1, 2.3.2), Theorem (2.4), tensor and odd cycle properties, Eq. (3.29) becomes

$$\delta\circ g_3^4(v_0,...,v_7)=-\frac{1}{28}\sum_{j\neq i}^{7}(-1)^j\sum_{i\neq j}^{7}(-1)^i Alt_6\Big[r(v_i|v_0,...,\hat{v}_i,\hat{v}_j,...,v_7)\Big]_3\otimes \prod_{r\neq j\neq k}^{7}\triangle(v_i|v_r,v_j,v_k). \tag{3.30}$$

From Eq. (3.27) and Eq. (3.30), it is observed that, $g_2^4\circ p=\delta\circ g_3^4$. □

3.10. **Geometry for Weight 5.** As defined in [13], the following commutative diagram is obtained

$$\begin{array}{ccccc}
G_9(3) & \xrightarrow{d} & G_8(3) & & \\
\downarrow p & & \downarrow p & & \\
G_8(2) & \xrightarrow{d} & G_7(2) & \xrightarrow{g_1^5} & \mathcal{B}_2(F)\otimes\wedge^3F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_7(1) & \xrightarrow{d} & G_6(1) & \xrightarrow{g_0^5} & \wedge^5F^\times
\end{array} \tag{G}$$

where,

$$g_0^5(v_0,...,v_5)\rightarrow\sum_{i=j+1}^{5}(-1)^i\bigwedge_{j\neq i}^{5}\triangle(v_j)\ (\mod 6) \tag{3.31}$$

and

$$g_1^5(v_0,...,v_6) \to -\frac{1}{10}\sum_{i\neq j\neq k}^{6}(-1)^i[r(v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6)]_2 \otimes \prod_{r\neq i}^{6}\triangle(v_i,v_r)$$
$$\wedge \prod_{l\neq j}^{6}\triangle(v_j,v_r)\wedge\prod_{r\neq k}^{6}\triangle(v_k,v_r)\ (mod\ 7). \tag{3.32}$$

Lemma 3.11. *The diagram* G *is commutative* (see [13]).

3.11.1. *Extension in Geometry for Weight 5.* For this extension in geometry, three new morphisms are introduced, namely g_2^5, g_3^5 and g_4^5:

$$\begin{array}{ccccc}
G_{11}(5) & \xrightarrow{d} & G_{10}(5) & \xrightarrow{g_4^5} & \mathcal{B}_5(F) \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_{10}(4) & \xrightarrow{d} & G_9(4) & \xrightarrow{g_3^5} & \mathcal{B}_4(F)\otimes F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_9(3) & \xrightarrow{d} & G_8(3) & \xrightarrow{g_2^5} & \mathcal{B}_3(F)\otimes \wedge^2 F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_8(2) & \xrightarrow{d} & G_7(2) & \xrightarrow{g_1^5} & \mathcal{B}_2(F)\otimes \wedge^3 F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_7(1) & \xrightarrow{d} & G_6(1) & \xrightarrow{g_0^5} & \wedge^5 F^\times
\end{array} \tag{H}$$

Morphisms g_2^5, g_3^5 and g_4^5 are respectively defined as

$$g_2^5(v_0,...,v_7) = \frac{1}{45}\sum_{i\neq j}^{7}(-1)^i Alt_6\Big[r(v_0,...,\hat{v}_i,\hat{v}_j,...,v_7)\Big]_3 \otimes$$
$$\prod_{r\neq ineqk}^{7}\triangle(v_r,v_i,v_k)\wedge\prod_{r\neq j}^{7}\triangle(v_r,v_j,v_k). \tag{3.33}$$

$$g_3^5(v_0,...,v_8) = -\frac{1}{105}\sum_{i=0}^{8}(-1)^i Alt_8\Big[r(v_0,...,\hat{v}_i,...,v_8)\Big]_4 \otimes \prod_{r\neq i\neq j\neq k}^{8}\triangle(v_r,v_i,v_j,v_k). \tag{3.34}$$

$$g_4^5(v_0,...,v_9) = \frac{1}{190}Alt_{10}\Big[r(v_0,...,v_9)\Big]_5. \tag{3.35}$$

It is shown in lemmas (3.12), (3.13) and (3.14) that the extended diagram H is commutative.

Theorem 3.12. $g_1^5 \circ p = \delta \circ g_2^5.$

Proof. Suppose $(v_0, ..., v_7)$ 8 points $\in G_8(3)$. Applying differential map p followed by the morphism g_1^5,

$$p(v_0, ..., v_7) = \sum_{i=0}^{7} (-1)^i (v_i|v_0, ..., \hat{v}_i, ..., v_7), \tag{3.36}$$

$$g_1^5 \circ p(v_0, ..., v_7) = -\frac{1}{10} \sum_{j\neq i\neq k}^{7} (-1)^j \sum_{i=0}^{7} (-1)^i [r(v_i|v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, \hat{v}_l, ..., v_7)]_2 \otimes \prod_{r\neq j}^{7} \triangle(v_i|v_j, v_r) \wedge \prod_{r\neq k}^{7} \triangle(v_i|v_k, v_r) \wedge \prod_{r\neq l}^{7} \triangle(v_i|v_l, v_r). \tag{3.37}$$

Let us take $(v_0, ..., v_7) \in G_8(3)$ again, apply morphism g_2^5

$$g_2^5 \circ (v_0, ..., v_7) = \frac{1}{45} \sum_{i\neq j}^{7} (-1)^i Alt_6 \Big[r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_7) \Big]_3 \otimes \prod_{r\neq i\neq k}^{7} \triangle(v_r, v_i, v_k) \wedge \prod_{r\neq j\neq k}^{7} \triangle(v_r, v_j, v_k). \tag{3.38}$$

By applying differential morphism δ, the following is obtained.

$$\delta \circ g_2^5(v_0, ..., v_7) = \frac{1}{45} \sum_{i\neq j}^{7} (-1)^i Alt_6 \Big[r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_7) \Big]_2 \otimes r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_7) \wedge \prod_{r\neq i\neq k}^{7} \triangle(v_r, v_i, v_k) \wedge \prod_{r\neq j\neq k}^{7} \triangle(v_r, v_j, v_k). \tag{3.39}$$

Simplification yields

$$\delta \circ g_2^5(v_0, ..., v_7) = -\frac{1}{10} \sum_{j\neq i\neq k}^{7} (-1)^j \sum_{i=0}^{7} (-1)^i [r(v_i|v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, \hat{v}_l, ..., v_7)]_2 \otimes \prod_{r\neq j}^{7} \triangle(v_i|v_j, v_r) \wedge \prod_{r\neq k}^{7} \triangle(v_i|v_k, v_r) \wedge \prod_{r\neq l}^{7} \triangle(v_i|v_l, v_r). \tag{3.40}$$

From Eq. (3.37) and Eq. (3.40), it is observed, $g_1^5 \circ p = \delta \circ g_2^5$. □

Theorem 3.13. $g_2^5 \circ p = \delta \circ g_3^5.$

Proof. Let $(v_0, ..., v_8)$ be 9 points $\in G_9(4)$ and apply the differential map p followed by morphism g_2^5,

$$p(v_0, ..., v_8) = \sum_{i=0}^{8} (-1)^i (v_i|v_0, ..., \hat{v}_i, ..., v_8) \tag{3.41}$$

$$g_2^5 \circ p(v_0, ..., v_8) = \frac{1}{45} \sum_{i \neq j}^{8} (-1)^i Alt_6 \Big[r(v_i | v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, ..., v_8) \Big]_3 \otimes$$
$$\prod_{r \neq j \neq l}^{8} \triangle(v_i | v_r, v_j, v_l) \wedge \prod_{r \neq k \neq l}^{7} \triangle(v_i | v_r, v_k, v_l). \quad (3.42)$$

Now take $(v_0, ..., v_8) \in G_9(4)$ again and applying morphism g_3^5 followed by δ,

$$g_3^5(v_0, ..., v_8) = -\frac{1}{105} \sum_{i=0}^{8} (-1)^i Alt_8 \Big[r(v_0, ..., \hat{v}_i, ..., v_8) \Big]_4 \otimes$$
$$\prod_{r \neq i \neq j \neq k}^{8} \triangle(v_r, v_i, v_j, v_k). \quad (3.43)$$

$$\delta \circ g_3^5(v_0, ..., v_8) = -\frac{1}{105} \sum_{i=0}^{8} (-1)^i Alt_8 \Big[r(v_0, ..., \hat{v}_i, ..., v_8) \Big]_3 \otimes$$
$$r(v_0, ..., \hat{v}_i, ..., v_8) \wedge \prod_{r \neq i \neq j \neq k}^{8} \triangle(v_r, v_i, v_j, v_k). \quad (3.44)$$

After simplifying by using Siegel, wedge, tensor and odd cycle properties, Eq. (3.44) becomes

$$\delta \circ g_3^5(v_0, ..., v_8) = \frac{1}{45} \sum_{i \neq j}^{8} (-1)^i Alt_6 \Big[r(v_i | v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, ..., v_8) \Big]_3 \otimes$$
$$\prod_{r \neq j \neq l}^{8} \triangle(v_i | v_r, v_j, v_l) \wedge \prod_{r \neq k \neq l}^{7} \triangle(v_i | v_r, v_k, v_l). \quad (3.45)$$

Then using Eq. (3.42) and Eq. (3.45), it is observed that $g_2^5 \circ p = \delta \circ g_3^5$. □

Theorem 3.14. $g_3^5 \circ p = \delta \circ g_4^5.$

Proof. Let $(v_0, ..., v_9)$ be 10 points $\in G_{10}(5)$ and apply the projection map p

$$p(v_0, ..., v_9) = \sum_{i=0}^{9} (-1)^i (v_i | v_0, ..., \hat{v}_i, ..., v_9), \quad (3.46)$$

now by applying morphism g_3^5

$$g_3^5 \circ p(v_0, ..., v_9) = -\frac{1}{105} \sum_{i=0}^{9} (-1)^i Alt_8 \Big[r(v_i | v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_9) \Big]_4 \otimes$$
$$\prod_{r \neq j}^{9} \triangle(v_i | v_r, v_j, v_k, v_l). \quad (3.47)$$

Let us take $(v_0, ..., v_9) \in G_{10}(5)$ again and by applying morphism g_4^5 followed by δ

$$g_4^5(v_0, ..., v_9) = \frac{1}{90} Alt_{10}\Big[r(v_0, ..., v_9)\Big]_5, \tag{3.48}$$

$$\delta \circ g_4^5 : (v_0, ..., v_9) = \frac{1}{90} Alt_{10}\Big[r(v_0, ..., v_9)\Big]_4 \otimes r(v_0, ..., v_9). \tag{3.49}$$

After simplification, we obtain

$$\delta \circ g_4^5(v_0, ..., v_9) = -\frac{1}{105} \sum_{i=0}^{9} (-1)^i Alt_8 \Big[r(v_i | v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_9)\Big]_4 \otimes \prod_{r \neq j}^{9} \triangle(v_i | v_r, v_j, v_k, v_l). \tag{3.50}$$

Then Eq.(3.47) and Eq.(3.50) shows that, $g_3^5 \circ p = \delta \circ g_4^5$. □

4. Conclusion

In this paper, the extension of homomorphisms in geometry of configuration and Goncharov motivic polylogarithmic chain complexes has been proposed to produce commutative diagrams. Initially, for weight 3 single homomorphism g_2^3 is defined to extend commutative diagram. For the geometry of weight 4, two morphisms g_2^4 and g_3^4 have been presented. Eventually, this work has been extended up to weight 5 by introducing three morphisms g_2^5, g_3^5 and g_4^5 for extending commutative diagram. This work, will be very helpful in understanding the generalized extension in geometry of configuration and Goncharov motivic polylogarithmic chain complexes.

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References

[1] J.L. Cathelineau, *Remarques sur les différentielles des polylogarithmes uniformes*, Ann. Inst. Fourier, Grenoble, **46,** No.5 (1996) 1327-1347. doi:10.5802/aif.1551
[2] J.L. Cathelineau, *Infinitesimal polylogarithms, multiplicative presentation of Kähler Differential and Goncharov Complexes*, Talk at the workshop on polylogarthms, Essen, Germany, May 1-4, 1997.
[3] J.L. Dupont and C.H. Sah, *Homology of Euclidean groups of motion made discrete and Euclidean scissors congruences*, Acta Math., **164** (1990) 1-24.
[4] A.B. Goncharov, *The classical trilogarithm, algebraic K-theory of field and Dedekind zeta functions*, Bull. of AMS., **24,** No.1 (1991) 155-162.
[5] A.B. Goncharov, *Geometry of configuration, polylogarithms and motivic cohomology*, Adv. Math., **114,** No.2 (1995) 197-318. doi:10.1006/aima.1995.1045.
[6] A.B. Goncharov, *Geometry of the trilogarithm and the motivic Lie algebra of a field*, Regulators in Analysis, Geometry and Number Theory, **171** (2000) 127-165.
[7] A.B. Goncharov and J. Zhao, *Grassmannian trilogarithm*, Compositio Mathematica, **127,** No.1 (2001) 93-108. doi:10.1023/A:1017504115184.

[8] A.B. Goncharov, *Euclidean scissor congruence groups and mixed Tate motives over dual numbers*, Mathematical Research Letters, **11,** No.6 (2004) 771-784. doi:10.4310/MRL.2004.v11.n6.a5.

[9] M. Khalid, J. Khan and A. Iqbal, *New homomorphism between Grassmannian and infinitesimal complexes*, International Journal of Algebra, **10,** No.3 (2016) 97-112. doi:10.12988/ija.2016.6213.

[10] M. Khalid, J. Khan and A. Iqbal, *Generalization of Grassmannian and polylogarithmic groups complexes*, International Journal of Algebra, **10,** No.5 (2016) 221-237. doi:10.12988/ija.2016.6323.

[11] M. Khalid, J. Khan and A. Iqbal, *Higher order Grassmannian complexes*, International Journal of Algebra, **10,** No.9 (2016) 405-413. doi:10.12988/ija.2016.6640

[12] M. Khalid, J. Khan and A. Iqbal, *Generalization of higher order homomorphism in configuration complexes*, Punjab Univ. j. math., **49,** No.2 (2017) 37-49.

[13] M. Khalid, J. Khan and A. Iqbal, *Generalized geometry of Goncharov and Configuration Complexes*, Turk. J. Math., **42,**No.3 (2018) 1509-1527. doi:10.3906/mat-1702-25

[14] M. Khalid, A. Iqbal and J. Khan, *Extension of morphisms in geometry of chain complexes*, Punjab Univ. j. math., **51,** No.1 (2019) 29-49.

[15] M. Khalid and A. Iqbal, *Generalized Extension of Morphisms in Geometry of Configuration and Infinitesimal Polylogarithmic Groups Complexes*, Punjab Univ. j. math., **51,** No.8 (2019) 111-127.

[16] C.L. Siegel, *Approximation algebraischer zahlen*, Mathem.Ze/tschr, **10** (1921) 173-213 (in German).

[17] A.A. Suslin, *Homology of* GL_n, *characteristic classes and Milnor's K-theory*, In Proceedings of the Steklov Institute of Mathematics 1985, Lecture Notes in Mathematics (1046), New York, USA, Springer-Verlag; pp. 207-226, 1989.