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Analysis of the small oscillations of a heavy almost homogeneous inviscid liquid partially filling an elastic body with negligible density

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Abstract.: In this paper, we study the small oscillations of a system formed by an elastic container with negligible density and a heavy heterogeneous inviscid liquid filling partially the container, in the particular case of an alomost homogeneous liquid, i.e a liquid whose the density in the equilibrium position is practically a linear function of the depth, that differs very little from a constant. By means of an auxiliary problem, that requires a careful study, we reduce the problem to a problem for a liquid only. From the variational formulation of the problem, we obtain its operatorial equations in a suitable Hilbert space. From these, we prove the existence of a spectrum formed by a point spectrum constituted by a countable set of positive real eigenvalues, whose the point of accumulation is the infinity and an essential spectrum filling an interval, that is physically a domain of resonance. Finally, we prove the existence and the unicity of the solution of the associated evolution problem.

AMS (MOS) Subject Classification Codes: 76B03; 74B05; 47A50; 49R50; 47A75
Key Words: Small oscillations; Mixed boundary conditions problems; Variational; Operatorial and spectral methods.

1. Introduction

The problem of the small oscillations of a heavy homogeneous inviscid liquid in an open rigid container has been the subject, from the pionering work by Moiseyev [10], of numerous papers that are analyzed in the books [11, 7, 12]. The same problem in the case of an elastic container is studied in the book [13]. In the works [1, 2], the second author has considered the problem of the small oscillations of a heavy heterogeneous liquid and has proved that it was not a classical vibration problem. These works have been carried on in our papers [4, 5], where the liquid is almost-homogeneous, i.e has a density in the equilibrium postion that is practically a linear function of the depth, that differs very little from a constant. Recently [6], we have solved the problem of the small oscillations of an almost-homogeneous liquid in an elastic container.

In this work, we study the case where the elastic body containing an almost homogeneous liquid has a negligible density, circumtance that can happen in the transport of liquids. At first, we establish the equations of motion of the system body-liquid and the boundary conditions. Afterwards, introducing an auxiliary problem, that requires a careful discussion, and that is the problem of the motion of the body when the motion of the liquid is known, we show a linear operator depending on the elasticity of the body, that permits us to reduce the problem for the liquid only. From the variational equation of this last problem, we deduce its operatorial equations in a suitable Hilbert space. From these, we prove the existence of a spectrum formed by a point spectrum constituted by a countable set of positive real eigenvalues, whose point of accumulation is the infinity, and an essential spectrum filling an interval, that is physically a domain of resonance. Finally, we prove the existence and the unicity of the solution of the associated evolution problem.

2. POSITION OF THE PROBLEM

We consider, in the field of the gravity, an elastic body with negligible density, that occupies in the equilibrium position a domain Ω' bounded by a fixed external surface S and an internal surface. The interior of this surface is partially filled by a heavy inviscid liquid that occupies a domain Ω bounded by a part Σ of the internal surface and the horyzontal free boundary Γ. We denote by σ the part of the internal surface wetted by the air with constant pressure p_c.

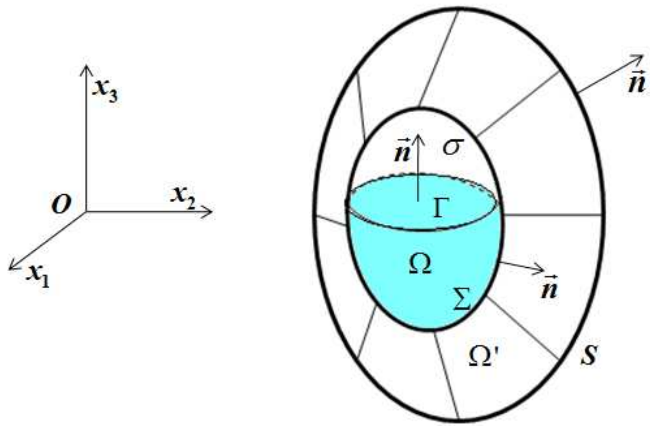

FIGURE 1. Model of the system.

We choose orthogonal axes $Ox_1x_2x_3$, Ox_3 vertical directed upwards. We denote by $\vec{n}$ the unit vector normal to the surfaces [Figure 1].

We are going to study the small oscillations of the system elastic body-liquid about its equilibrium position, in the framework of the linear theory.

3. The equations of the problem

3.1. The equations of the elastic body with negligible density.
Let $\vec{\hat{u}}'(x_1, x_2, x_3)$ the (small) displacement of the particle of the body from the natural state to the equilibrium state.
The equilibrium equations are:

$$0 = \frac{\partial \sigma'_{ij}(\vec{\hat{u}}')}{\partial x_j} \qquad \text{in} \qquad \Omega' \quad (i, j = 1, 2, 3) \tag{3. 1}$$

and the boundary conditions are

$$\vec{\hat{u}}'_{|S} = 0 \quad ; \quad \sigma'_{ij}(\vec{\hat{u}}')n_j = -p_{0|\Sigma} n_i \quad \text{on} \quad \Sigma \quad ; \quad \sigma'_{ij}(\vec{\hat{u}}')n_j = -p_c n_i \quad \text{on} \quad \sigma \; , \tag{3. 2}$$

where p_0 is the pressure of the liquid in the equilibrium position and we have set:

$$\sigma'_{ij}(\vec{\hat{u}}') = \lambda' \text{div} \vec{\hat{u}}' \, \delta_{ij} + 2\mu' \epsilon'_{ij}(\vec{\hat{u}}') \qquad ; \qquad \epsilon'_{ij}(\vec{\hat{u}}') = \frac{1}{2} \left(\frac{\partial \hat{u}'_i}{\partial x_j} + \frac{\partial \hat{u}'_j}{\partial x_i} \right) \; ;$$

λ' and μ' are the Lame's coefficients; $\sigma'_{ij}(\vec{\hat{u}}')$ and $\epsilon'_{ij}(\vec{\hat{u}}')$ are the components of the stress tensor and the strain tensor respectively.
Let $\vec{u}'(x_1, x_2, x_3, t)$ the displacement of a particle from its equilibrium position to its position at the instant t.
We have

$$0 = \frac{\partial \sigma'_{ij}(\vec{\hat{u}}' + \vec{u}')}{\partial x_j} \qquad \text{in} \qquad \Omega' \; ,$$

then, taking into account (3. 1)

$$0 = \frac{\partial \sigma'_{ij}(\vec{u}')}{\partial x_j} \qquad \text{in} \qquad \Omega' \quad (i, j = 1, 2, 3) \tag{3. 3}$$

Using the first and the third conditions (3. 2), we have

$$\vec{u}'_{|S} = 0 \qquad \sigma'_{ij}(\vec{u}')n_j = 0 \quad \text{on} \quad \sigma \; , \tag{3. 4}$$

We remark that we can write the second condition of (3. 4)

$$\overrightarrow{T}\,(\vec{u}') = 0 \quad \text{on} \quad \sigma \; , \tag{3. 5}$$

$\overrightarrow{T}\,(\vec{u}')$ being the stress vector for the direction $\vec{n}$ on σ.
We will write the conditions on Σ in the following.

3.2. The equations of the liquid.

3.2.1. *Equations of heterogeneous liquid.*
Let $p(x_i)$, $\rho_0(x_i)$, the pressure and the density of the liquid in the equilibrium position. We have

$$\overrightarrow{\text{grad}}\, p_0 = -\rho_0 g \vec{x}_3, \tag{3. 6}$$

so that, p_0 and ρ_0 are functions of x_3 verifying

$$\frac{\mathrm{d}p_0(x_3)}{\mathrm{d}x_3} = -\rho_0(x_3) g$$

In the following, we suppose that $\rho_0(x_3)$ grows with the depth, so that

$$\rho_0'(x_3) < 0$$

Let $\vec{u}(x_i, t)$ the dispalcement of a particle of the liquid from its equilibrium position, $p^*(x_i, t)$, $\rho^*(x_i, t)$ the pressure and the density of the liquid.
The equations of the motion are

$$\rho^* \ddot{\vec{u}} = -\overrightarrow{\text{grad}}\, p^* - \rho^* g \vec{x}_3 \qquad \text{(Euler's equation)}, \tag{3. 7}$$

$$\text{div}\, \vec{u} = 0 \qquad \text{(incompressibility) in } \Omega, \tag{3. 8}$$

$$\frac{\partial \rho^*}{\partial t} + \text{div}(\rho^* \dot{\vec{u}}) = 0 \qquad \text{(continuity equation)}, \tag{3. 9}$$

the second being obtained by integrating div $\dot{\vec{u}}$ between the time of equilibrium and the instant t.
Taking into account of (3. 8), the equation (3. 9) becomes

$$\frac{\partial \rho^*}{\partial t} = -\dot{\vec{u}} \cdot \overrightarrow{\text{grad}}\, \rho^* \ .$$

We set

$$\rho^* = \rho_0(x_3) + \hat{\rho}(x_i, t) + \cdots \ ,$$

$$p^* = p_0(x_3) + p(x_i, t) + \cdots \ .$$

$\hat{\rho}$, p being of the first order with respect to the amplitude of the oscillations.
The linearization of the continuity equation gives

$$\frac{\partial}{\partial t} \left(\hat{\rho} + u_3\ \rho_0'(x_3) \right) = 0$$

and then, integrating like above

$$\hat{\rho} = -u_3(x_i, t)\ \rho_0'(x_3).$$

Therefore, the linearized Euler's equation becomes, using (3. 6):

$$\rho_0(x_3) \ddot{\vec{u}} = -\overrightarrow{\text{grad}} p + \rho_0'(x_3) g u_3(x_i, t) \vec{x}_3 \qquad \text{in } \Omega. \tag{3. 10}$$

The kinematic condition on Σ is:

$$u_{n|\Sigma} = u'_{n|\Sigma} \, . \tag{3. 11}$$

3.2.2. *The particular case of an almost homogeneous liquid.*
Let h ($h > 0$) the height of the lowest point of Σ. In Ω, we have $|x_3| \leq h$.
We suppose that the density of the liquid in the equilibrium position can be written

$$\rho_0(x_3) = f(\beta x_3)\ ,$$

with $f(0) > 0$, $f'(0) < 0$, β being a positive constant such that βh is sufficiently small in order that $(\beta h)^2$, $(\beta h)^3$, $\cdots$ are negligible with respect to βh.
Since $|\beta x_3| \leq \beta h$ in Ω, we have

$$\rho_0(x_3) = f(0) + \beta x_3 f'(0) + \mathrm{o}(\beta h)$$

and the liquid is called <u>*almost-homogeneous in* Ω</u>.
Changing the notations, we write

$$\rho_0(x_3) = \rho\,(1 - \beta x_3) + \mathrm{o}(\beta h),$$

where ρ is a positive constant.
Then, in the equation (3. 10), we replace $\rho_0(x_3)$ by ρ and $\rho_0'(x_3)$ by $-\rho\ \beta$ and we obtain an approximation equation analalogous to the boussinesq equation of the theory of the convective omotion of a fluid:

$$\rho\,\ddot{\vec{u}} = -\overrightarrow{\text{grad}}p - \rho\beta g u_3\ \vec{x}_3. \tag{3. 12}$$

On the other hand, we have

$$p_0(x_3) = -\rho\ g\ x_3 + p_c \tag{3. 13}$$

3.2.3. *The dynamic conditions.*
a) On Γ_t, postion of the free surface at the instant t, the equation of which being $x_3 = u_{n|\Gamma} + \cdots$, where the dots indicate terms of order greater than one, we must have

$$p^* = p_c$$

and, consequently, in linear theory

$$p_0(u_{n|\Gamma}) + p(x_1, x_2, 0, t) = p_c$$

or

$$p_{|\Gamma} = \rho\ g u_{n|\Gamma} \tag{3. 14}$$

b) Let us write the dynamic conditions on Σ_t, position of Σ at the instant t:
Let M a point of Σ. We denote by M_ℓ (resp. M_s) the particle of the liquid (resp. the body) that occupies the position M in the equilibrium position. If M'_ℓ and M'_s are the positions of M_ℓ and M_s at the instant t we have

$$\overrightarrow{MM'_\ell} = \vec{u} \qquad ; \qquad \overrightarrow{MM'_s} = \vec{u}'.$$

In linear theory, we can admit that the unit vectors normal to Σ_t in M'_ℓ and M'_s are equal to the unit vector $\vec{n}$ normal to Σ in M and that the pressure p^* of the liquid in M'_ℓ is equal to the pressure of the liquid $p^*(M', t)$ in M', intersection of Σ_t and the normal in M to Σ.
Then, the dynamic conditions on Σ_t are

$$\sigma'_{ij}(\vec{\tilde{u}}' + \vec{u}')n_j = -p^*(M', t)n_i \quad (i, j = 1, 2, 3)$$

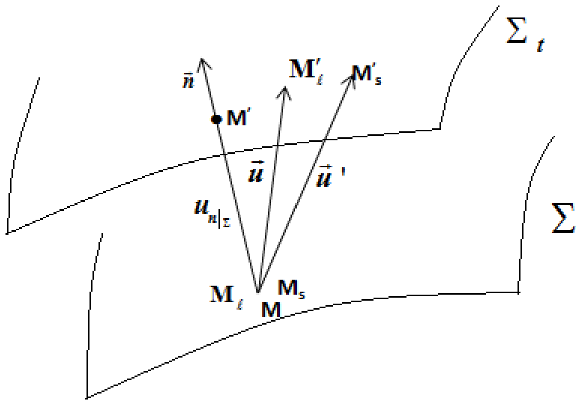

FIGURE 2. Configurations of Σ and Σ_t.

The second condition (3. 2) can be written:

$$\sigma'_{ij}(\vec{\tilde{u}}')n_j = -p_0(M)\; n_i,$$

so that

$$\sigma'_{ij}(\vec{u}')n_j = -\left[p^*(M',t) - p_0(M)\right] n_i \qquad \text{on } \; \Sigma\;.$$

We have

$$p^*(M',t) = p^*(M + u_{n|\Sigma}\vec{n}, t) = p^*(M,t) + \overrightarrow{\text{grad}} p^*(M) \cdot u_{n|\Sigma}\vec{n} + \cdots$$

Since $u_{n|\Sigma}$ is of the first order, we can, in linear theory, replace $\overrightarrow{\text{grad}} P^*(M,t)$ by

$$\overrightarrow{\text{grad}} P_0 = -\rho g \vec{x}_3\;,$$

so that we can write

$$p^*(M',t) = p^*(M,t) - \rho\; g\, u_{n|\Sigma}\; n_{3|\Sigma} + \cdots$$

Finally, the dinamic conditions on Σ are

$$\sigma'_{ij}(\vec{u}')n_j = \left[p(M,t) + \rho\; g n_{3|\Sigma}\; u_{n|\Sigma}\right] n_i \qquad \text{on } \; \Sigma\;. \tag{3. 15}$$

If $\overrightarrow{T}_t\,(\vec{u}')$ and $T_n\,(\vec{u}')$ are respectively the tangential stress vector and the normal stress for the direction $\vec{n}$; the conditions (3. 15) can be written

$$\overrightarrow{T}_t\,(\vec{u}')_{|\Sigma} = 0 \quad ; \quad T_n\,(\vec{u}')_{|\Sigma} = -p_{|\Sigma} + \rho\; g n_{3|\Sigma}\; u_{n|\Sigma}\;. \tag{3. 16}$$

Finally, the volume of the liquid remaining constant, we have

$$\int_\Sigma u_{n|\Sigma} \mathrm{d}\Sigma + \int_\Gamma u_{n|\Gamma} \mathrm{d}\Gamma = 0, \tag{3. 17}$$

that is equivalent to $\int_\Omega \text{div}\vec{u}\; \mathrm{d}\Omega = 0$.
The equations (3. 3), (3. 4), (3. 5), (3. 8), (3. 11), (3. 12), (3. 14), (3. 15), (3. 16), (3. 17) are the equations of the problem.

4. THE AUXILIARY PROBLEM

4.1. We introduce the auxiliary problem:

$$\begin{cases} -\dfrac{\partial \sigma'_{ij}(\vec{u}')}{\partial x_j} = 0 \quad \text{in} \quad \Omega' \;\; ; \;\; \vec{u}'_{|S} = 0 \\ \sigma'_{ij}(\vec{u}')n_{j|\sigma} = 0 \;\; ; \;\; u'_{n|\Sigma} = u_{n|\Sigma} \;\; ; \;\; \overrightarrow{T}_t(\vec{u}')_{|\Sigma} = 0 \, , \end{cases} \tag{4. 18}$$

where $u_{n|\Sigma}$ is considered as a datum.
It is the problem of the motion of the elastic body when the motion of the liquid is known.
We are going to seek $\vec{u}'$ in the space

$$\widehat{\Xi}^1(\Omega') \stackrel{\text{def}}{=} \left\{ \vec{u}' \in \Xi^1(\Omega') \stackrel{\text{def}}{=} \left[H^1(\Omega')\right]^3 ; \;\; \vec{u}'_{|S} = 0 \right\} \; .$$

Then, $u'_{n|\Sigma} \in H^{1/2}(\Sigma)$, so that we suppose that $u_{n|\Sigma} \in H^{1/2}(\Sigma)$.

4.2. Let $\overrightarrow{\Phi}$ an element of $\widehat{\Xi}^1(\Omega')$ such that $\Phi_{n|\Sigma} = u_{n|\Sigma} \in H^{1/2}(\Sigma)$.
We will construct $\overrightarrow{\Phi}$ in the sequel.
We introduce the space V_0, subspace of $\hat{\Xi}^1(\Omega')$, defined by

$$V_0 = \left\{ \vec{v}_0 \in \widehat{\Xi}^1(\Omega') \quad ; \quad v_{0n|\Sigma} = 0 \right\}$$

and we seek $\vec{u}'$ in the form

$$\vec{u}' = \overrightarrow{\Phi} + \vec{u}_0 \;\; ; \;\; \vec{u}_0 \in V_0 .$$

The problem (4. 18) becomes a problem for $\vec{u}_0$:

$$\begin{cases} -\dfrac{\partial \sigma'_{ij}(\vec{u}_0)}{\partial x_j} = \dfrac{\partial \sigma'_{ij}(\overrightarrow{\Phi})}{\partial x_j} \quad \text{in} \quad \Omega' \;\; ; \;\; u_{0n|\Sigma} = 0; \\ \sigma'_{ij}(\vec{u}_0)n_j = -\sigma'_{ij}(\overrightarrow{\Phi})n_j \;\; \text{on} \;\; \sigma \;\; ; \;\; \overrightarrow{T}_t(\vec{u}'_0)_{|\Sigma} = -\overrightarrow{T}_t(\overrightarrow{\Phi})_{|\Sigma} \; . \end{cases} \tag{4. 19}$$

We are going to seek a variational formulation of this problem.
We have:

$$-\int_{\Omega'} \frac{\partial \sigma'_{ij}(\vec{u}_0)}{\partial x_j} \cdot \bar{v}_{0i} \, \mathrm{d}\Omega' = \int_{\Omega'} \frac{\partial \sigma'_{ij}(\overrightarrow{\Phi})}{\partial x_j} \cdot \bar{v}_{0i} \, \mathrm{d}\Omega' \quad \forall \vec{v}_0 \in V_0 .$$

Using the Green's formula and denoting by $\vec{n}_{\text{e}}$ the external normal unit vector to the boundary of Ω', we obtain easily:

$$\begin{cases} -\displaystyle\int_\Sigma \sigma'_{ij}(\vec{u}_0)n_{\text{ej}}\bar{v}_{0i} \, \mathrm{d}\Sigma + \int_{\Omega'} \sigma'_{ij}(\vec{u}_0)\epsilon'_{ij}(\bar{\vec{v}}_0) \, \mathrm{d}\Omega' \\ = \displaystyle\int_\Sigma \sigma'_{ij}(\overrightarrow{\Phi})n_{\text{ej}}\bar{v}_{0i} \, \mathrm{d}\Sigma - \int_{\Omega'} \sigma'_{ij}(\overrightarrow{\Phi})\epsilon'_{ij}(\bar{\vec{v}}_0) \, \mathrm{d}\Omega' \; . \end{cases}$$

Taking into account of $v_{0n|\Sigma} = 0$ and denoting by $\vec{v}_{0t|\Sigma}$ the tangential component of $\vec{v}_0$, we have

$$\int_\Sigma \sigma'_{ij}(\vec{u}_0)n_{\text{ej}}\bar{v}_{0i} \, \mathrm{d}\Sigma = -\int_\Sigma \overrightarrow{T}_t(\vec{u}_0) \cdot \vec{v}_{0t|\Sigma} \, \mathrm{d}\Sigma.$$

We obtain analogous formula by replacing $\vec{u}_0$ by $\overrightarrow{\Phi}$.

Carrying out in the precedent equation, we obtain the variational formulation of the problem (4. 19):

To find $\vec{u}_0 \in V_0$ such that

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}_0)\epsilon'_{ij}(\bar{\vec{v}}_0)\,\mathrm{d}\Omega' = -\int_{\Omega'} \sigma'_{ij}(\overrightarrow{\Phi})\epsilon'_{ij}(\bar{\vec{v}}_0)\,\mathrm{d}\Omega' \quad \forall \vec{v}_0 \in V_0\ . \tag{4. 20}$$

Reciprocally, let $\vec{u}_0$ a function of t with values in V_0 and verifying (4. 20).
We have, using Green's formula and taking into account of $\vec{v}_{0|S} = 0$:

$$\int_{\Omega'} \frac{\partial \sigma'_{ij}(\vec{u}_0)}{\partial x_j} \cdot \bar{v}_{0i}\,\mathrm{d}\Omega' = \int_{\Sigma+\sigma} \sigma'_{ij}(\vec{u}_0)n_{\mathrm{ej}}\bar{v}_{0i}\,\mathrm{d}(\Sigma+\sigma) - \int_{\Omega'} \sigma'_{ij}(\vec{u}_0)\epsilon'_{ij}(\bar{\vec{v}}_0)\,\mathrm{d}\Omega'\ ,$$

and an analogous formula by replacing $\vec{u}_0$ by $\overrightarrow{\Phi}$.

Since $\vec{u}_0$ and $\overrightarrow{\Phi}$ verify (4. 20), we obtain

$$\begin{cases} -\displaystyle\int_{\Omega'} \frac{\partial \sigma'_{ij}(\vec{u}_0)}{\partial x_j} \cdot \bar{v}_{0i}\,\mathrm{d}\Omega' + \int_{\Sigma+\sigma} \sigma'_{ij}(\vec{u}_0)n_{\mathrm{ej}}\bar{v}_{0i}\,\mathrm{d}(\Sigma+\sigma) \\ = \displaystyle\int_{\Omega'} \frac{\partial \sigma'_{ij}(\overrightarrow{\Phi})}{\partial x_j} \cdot \bar{v}_{0i}\,\mathrm{d}\Omega' - \int_{\Sigma+\sigma} \sigma'_{ij}(\overrightarrow{\Phi})n_{\mathrm{ej}}\bar{v}_{0i}\,\mathrm{d}(\Sigma+\sigma)\ . \end{cases}$$

Taking $\vec{v}_0 \in [\mathscr{D}(\Omega')]^3$, we have

$$-\frac{\partial \sigma'_{ij}(\vec{u}_0)}{\partial x_j} = \frac{\partial \sigma'_{ij}(\overrightarrow{\Phi})}{\partial x_j} \quad \text{in} \quad \left[(\mathscr{D}(\Omega'))^3\right]'\ .$$

Taking into account of $v_{0n|\Sigma} = 0$, we have

$$\begin{cases} \displaystyle\int_{\Sigma} \overrightarrow{T}_t(\vec{u}_0)\cdot \vec{v}_{0t|\Sigma}\,\mathrm{d}\Sigma + \int_{\sigma} \sigma'_{ij}(\vec{u}_0)n_{\mathrm{ej}}\bar{v}_{0i}\,\mathrm{d}\sigma \\ = -\displaystyle\int_{\Sigma} \overrightarrow{T}_t(\overrightarrow{\Phi})\cdot \vec{v}_{0t|\Sigma}\,\mathrm{d}\Sigma - \int_{\sigma} \sigma'_{ij}(\overrightarrow{\Phi})n_{\mathrm{ej}}\bar{v}_{0i}\,\mathrm{d}\sigma\ . \end{cases}$$

Taking $\vec{v}_{0|\sigma} = 0$, we have, since $\vec{v}_{0t|\Sigma}$ is arbitrary

$$\overrightarrow{T}_t(\vec{u}_0)_{|\Sigma} = -\overrightarrow{T}_t(\overrightarrow{\Phi})_{|\Sigma}$$

and, finally, taking $\vec{v}_{0|\sigma}$ is arbitrary

$$\sigma'_{ij}(\vec{u}_0)n_j = -\sigma'_{ij}(\overrightarrow{\Phi})n_j \quad \text{on} \quad \sigma,$$

and we find the problem (4. 19).

The left-hand side of (4. 20) can be considered as a scalar product in V_0

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}_0)\epsilon'_{ij}(\bar{\vec{v}}_0)\,\mathrm{d}\Omega' = (\vec{u}_0, \vec{v}_0)_{V_0}\ .$$

It is well-known that the associated norm $\|\vec{u}_0\|_{V_0}$ is equivalent in V_0 to the classical norm $\|\vec{u}_0\|_1$ of $\Xi^1(\Omega')$ [7: Section 2.2.4].
In the same manner, the right-hand side of (4. 20) can be considered as a scalar product in $\widehat{\Xi}^1(\Omega')$:

$$\int_{\Omega'} \sigma'_{ij}(\overrightarrow{\Phi})\epsilon'_{ij}(\bar{\vec{v}}_0)\,\mathrm{d}\Omega' = \left(\overrightarrow{\Phi}, \vec{v}_0\right)_{\widehat{\Xi}^1(\Omega')} .$$

Obviously we have

$$\|\vec{u}_0\|_{V_0} = \|\vec{u}_0\|_{\widehat{\Xi}^1(\Omega')} \quad \forall \vec{u}_0 \in V_0 .$$

Then, we can write the equation (4. 20) in the form

$$(\vec{u}_0, \vec{v}_0)_{V_0} = \left(-\overrightarrow{\Phi}, \vec{v}_0\right)_{\widehat{\Xi}^1(\Omega')} \quad \forall \vec{v}_0 \in V_0 . \tag{4. 21}$$

Using the precedent result, we see that the right hand-side of (4. 21) is a continuous antilinear form in V_0.
Therefore, the problem (4. 21) has one and only one solution in V_0 according to the Lax-Milgram theorem. It is the same thing for the problem (4. 20) in V_0 and (4. 18) in $\widehat{\Xi}^1(\Omega')$
.
We notice that the equation (4. 21) can be written

$$(\vec{u}', \vec{v}_0)_{\widehat{\Xi}^1(\Omega')} = 0 \quad \forall \vec{v}_0 \in V_0 .$$

The solution $\vec{u}'$ of the auxiliary problem (4. 18) belongs to the orthogonal of V_0 in $\widehat{\Xi}^1(\Omega')$.

4.3. This solution $\vec{u}'$ doesnt depend on the choice of $\overrightarrow{\Phi}$, since $\overrightarrow{\Phi}$ is not in the terms of the problem (4. 18). We are going to take advantage of this remark for estimating $\|\vec{u}'\|_{\widehat{\Xi}^1(\Omega')}$.

The datum $u_{n|\Sigma}$ of the auxiliary problem (4. 18) belongs to $H^{1/2}(\Sigma)$ (and we have $\Phi_{n|\Sigma} = u_{n|\Sigma}$).
There exists an extension operator $\mathscr{P}$ continuous from $H^{1/2}(\Sigma)$ into $H^{1/2}(\Sigma+\sigma)$ [15: Chapter 1, section 5.2]; we write

$$u''_n = \mathscr{P}\, u_{n|\Sigma} \quad ; \quad \|u''_n\|_{H^{1/2}(\Sigma+\sigma)} \leq c \left\|u_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} \quad (c > 0) .$$

Since the solution of (4. 18) doesnt depend on $\overrightarrow{\Phi}$, we take for $\overrightarrow{\Phi}$ a continuous lifting of $u''_n \vec{n}$ in $\widehat{\Xi}^1(\Omega')$ (so that $\Phi_{n|\Sigma+\sigma} = u''_n$ and $\Phi_{n|\Sigma} = u_{n|\Sigma}$); then, we have

$$\left\|\overrightarrow{\Phi}\right\|_{\widehat{\Xi}^1(\Omega')} \leq c' \left\|u''_n\right\|_{H^{1/2}(\Sigma+\sigma)} \quad (c' > 0)$$

and consequently

$$\left\|\overrightarrow{\Phi}\right\|_{\widehat{\Xi}^1(\Omega')} \leq c\,c' \left\|u_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} .$$

Using (4. 21), we obtain

$$\|\vec{u}_0\|_{V_0} \leq \left\|\overrightarrow{\Phi}\right\|_{\widehat{\Xi}^1(\Omega')} \leq c\,c' \left\|u_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)}$$

and, finally, the estimate of the solution of the auxiliary problem (4. 18)

$$\|\vec{u}'\|_{\widehat{\Xi}^1(\Omega')} \leq 2c\,c' \left\|u_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} . \tag{4. 22}$$

4.4. Now, we are going to study $T_n(\vec{u}')_{|\Sigma}$ that appears in the second dynamic condition (3. 16) and calculate it by means of $u_{n|\Sigma}$.
Let $\vec{\bar{w}}' \in \widehat{\Xi}^1(\Omega')$. For the solution of the problem, we have

$$0 = -\int_{\Omega'} \frac{\partial \sigma'_{ij}(\vec{u}')}{\partial x_j} \cdot \bar{\bar{w}}_i \, \mathrm{d}\Omega' = -\int_{(S+\sigma+\Sigma)} \sigma'_{ij}(\vec{u}') n_{\mathrm{ej}} \bar{\bar{w}}'_i \, \mathrm{d}(\partial\Omega') + \int_{\Omega'} \sigma'_{ij}(\vec{u}') \epsilon'_{ij}(\vec{\bar{w}}') \, \mathrm{d}\Omega' \, .$$

Since

$$\vec{\bar{w}}'_{|S} = 0 \;\;,\;\; \vec{T}_t(\vec{u}')_{|\sigma} = 0 \;\;,\;\; \vec{T}_t(\vec{u}')_{|\Sigma} = 0 \;\text{ and }\; \vec{n}_e = -\vec{n}\,,$$

we have

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}') \epsilon'_{ij}(\vec{\bar{w}}') \, \mathrm{d}\Omega' = -\int_{\Sigma} T_n(\vec{u}')_{|\Sigma} \, .\bar{\bar{w}}'_{n|\Sigma} \, \mathrm{d}\Sigma \; , \quad \forall \vec{\bar{w}}' \in \widehat{\Xi}^1(\Omega') \, . \tag{4. 23}$$

On the other hand, if $\vec{v}' \in [\mathscr{D}(\Omega')]^3$, we have in accordance with the definition of the derivatives in the sens of the distributions

$$0 = \left\langle -\frac{\partial \sigma'_{ij}(\vec{u}')}{\partial x_j}, v'_i \right\rangle = \int_{\Omega'} \sigma'_{ij}(\vec{u}') \frac{\partial \bar{v}'_i}{\partial x_j} \, \mathrm{d}\Omega'$$

Then, we have

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}') \epsilon'_{ij}(\bar{\vec{v}}') \, \mathrm{d}\Omega' = 0 \quad \forall \vec{v}' \in [\mathscr{D}(\Omega')]^3$$

and by density

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}') \epsilon'_{ij}(\bar{\vec{v}}') \, \mathrm{d}\Omega' = 0 \quad \forall \vec{v}' \in \Xi^1_0(\Omega') \, .$$

We particularize $\vec{\bar{w}}'$. Let $w'_{n|\Sigma}$ a function belonging to $H^{1/2}(\Sigma)$.
We introduce the extension operator $\mathscr{P}$ and we set

$$w''_n = \mathscr{P} w'_{n|\Sigma}.$$

We take for $\vec{\bar{w}}'$ a lifting of $w''_{n|\Sigma}\vec{n}$ into $\widehat{\Xi}^1(\Omega')$ and we set

$$\ell(\vec{\bar{w}}') = \int_{\Omega'} \sigma'_{ij}(\vec{u}') \epsilon'_{ij}(\vec{\bar{w}}') \, \mathrm{d}\Omega' \, .$$

Since the difference between two liftings belongs obviously to $\Xi^1_0(\Omega')$, the right-hand side doesnt depend on the lifting $\vec{\bar{w}}'$. Then, ℓ depends on $w'_{n|\Sigma}$.
Let us take for $\vec{\bar{w}}'$ a continuous lifting of $w''_{n|\Sigma}\vec{n}$ into $\hat{\Xi}^1(\Omega')$. For such a lifting, we have

$$\left\|\vec{\bar{w}}'\right\|_{\widehat{\Xi}^1(\Omega')} \leq \alpha \left\|w''_n\right\|_{H^{1/2}(\Sigma+\sigma)} \quad (\alpha > 0)\,,$$

and, $\mathscr{P}$ being continuous

$$\left\|\vec{\bar{w}}'\right\|_{\widehat{\Xi}^1(\Omega')} \leq \beta \left\|w'_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} \quad (\beta > 0)\,.$$

Since we have

$$\left|\ell(\vec{\tilde{w}}')\right| \leq \|\vec{u}'\|_{\widehat{\Xi}^1(\Omega')} \cdot \left\|\vec{\tilde{w}}'\right\|_{\widehat{\Xi}^1(\Omega')} \ ,$$

we obtain

$$\left|\ell(\vec{\tilde{w}}')\right| \leq \beta \|\vec{u}'\|_{\widehat{\Xi}^1(\Omega')} \cdot \left\|w'_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} .$$

Since ℓ depends on $w'_{n|\Sigma}$, it is an element of $\left[H^{1/2}(\Sigma)\right]'$.
The relation (4. 23) can be written, since $\tilde{w}'_{n|\Sigma} = w'_{n|\Sigma}$:

$$\int_\Sigma T_n(\vec{u}')_{|\Sigma} \cdot \bar{w}'_{n|\Sigma} \, \mathrm{d}\Sigma = -\ell(\vec{w}') \,,$$

so that the normal stress $T_n(\vec{u}')_{|\Sigma}$ can be considered as an element of $\left(H^{1/2}(\Sigma)\right)'$ and we have

$$\left\|T_n(\vec{u}')_{|\Sigma}\right\|_{\left(H^{1/2}(\Sigma)\right)'} \leq \beta \|\vec{u}'\|_{\widehat{\Xi}^1(\Omega')} \ ,$$

then, using (4. 22):

$$\left\|T_n(\vec{u}')_{|\Sigma}\right\|_{\left(H^{1/2}(\Sigma)\right)'} \leq \delta \left\|u_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} \quad (\delta = 2c\,c'\beta) \ .$$

Consequently, there exists a continuous linear operator $\widehat{T}$ from $H^{1/2}(\Sigma)$ into $\left(H^{1/2}(\Sigma)\right)'$ such that

$$\widehat{T} u_{n|\Sigma} = -T_n(\vec{u}')_{|\Sigma} \ . \tag{4. 24}$$

This operator depends obviously on the elasticity of the body; it has properties of symmetry and positivity.
Indeed, let us consider, beside the problem (4. 18) the same problem for $\vec{\tilde{u}}' \in \widehat{\Xi}^1(\Omega')$.
We have, using (4. 23) and (4. 24)

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}')\epsilon'_{ij}(\bar{\vec{\tilde{u}}}') \, \mathrm{d}\Omega' = \left\langle \widehat{T} u_{n|\Sigma}, \tilde{u}_{n|\Sigma} \right\rangle \ .$$

Inverting the roles of $\vec{u}'$ and $\vec{\tilde{u}}'$, we have

$$\int_{\Omega'} \sigma'_{ij}(\vec{\tilde{u}}')\epsilon'_{ij}(\bar{\vec{u}}') \, \mathrm{d}\Omega' = \left\langle \widehat{T} \tilde{u}_{n|\Sigma}, u_{n|\Sigma} \right\rangle \ .$$

The hermitian symmetry of $\widehat{T}$ follows from the classical symmetry of the left-hand side.
Setting $\vec{\tilde{u}}' = \vec{u}'$, we obtain

$$\left\langle \widehat{T} u_{n|\Sigma}, u_{n|\Sigma} \right\rangle = \int_{\Omega'} \sigma'_{ij}(\vec{u}')\epsilon'_{ij}(\bar{\vec{u}}') \, \mathrm{d}\Omega' = \|\vec{u}'\|^2_{\widehat{\Xi}^1(\Omega')} \ .$$

Using a trace theorem, we have

$$\left\|u'_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} \leq C \|\vec{u}'\|_{\widehat{\Xi}^1(\Omega')} \quad \forall \vec{u}' \in \Xi^1(\Omega') \quad (C > 0) \ .$$

so that, since $u'_{n|\Sigma} = u_{n|\Sigma}$, we obtain

$$\left\langle \widehat{T} u_{n|\Sigma}, u_{n|\Sigma} \right\rangle \geq C^{-2} \left\| u_{n|\Sigma} \right\|^2_{H^{1/2}(\Sigma)} .$$

4.5. The second dynamic condition (3. 16) can be written

$$p_{|\Sigma} = \widehat{T} u_{n|\Sigma} + \rho\, g\, n_{3|\Sigma}\ u_{n|\Sigma} . \tag{4. 25}$$

So, we obtain a problem for the liquid only:

$$\begin{cases} \rho\, \ddot{\vec{u}} = -\overrightarrow{\text{grad}} p - \rho\, \beta g u_3\, \vec{x}_3 \quad \text{in} \quad \Omega \\ \text{div}\, \vec{u} = 0 \quad \text{in} \quad \Omega \\ p_{|\Gamma} = \rho\, g u_{n|\Gamma} \\ p_{|\Sigma} = \widehat{T} u_{n|\Sigma} + \rho\, g\, n_{3|\Sigma}\ u_{n|\Sigma} \end{cases} \tag{4. 26}$$

This problem being solved, the auxiliary problem (4. 18) -that we solved-gives $\vec{u}'$, i.e the motion of the elastic body.

5. Variational formulation of the problem of the motion of the liquid

5.1. We introduce the field of the kinematically admissible displacements $\bar{\vec{u}}(x_i)$, $\bar{\vec{u}}$ sufficiently smooth in Ω, div $\bar{\vec{u}} = 0$ in Ω.

We have

$$\int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\vec{u}}\, \mathrm{d}\Omega = -\int_\Omega \overrightarrow{\text{grad}} p \cdot \bar{\vec{u}}\, \mathrm{d}\Omega - \rho\beta g \int_\Omega u_3 \bar{u}_3\, \mathrm{d}\Omega.$$

The Green's formula gives

$$\begin{aligned} \int_\Omega \overrightarrow{\text{grad}} p \cdot \bar{\vec{u}}\, \mathrm{d}\Omega &= \int_\Omega [\text{div}(p\bar{\vec{u}}) - p\text{div}(\bar{\vec{u}})]\, \mathrm{d}\Omega \\ &= \int_\Sigma p_{|\Sigma} \bar{u}_{n|\Sigma}\, \mathrm{d}\Sigma + \int_\Gamma p_{|\Gamma} \bar{u}_{n|\Gamma}\, \mathrm{d}\Gamma, \end{aligned}$$

Using the last condition (4. 26), we obtain the variational equation:

$$\begin{cases} \displaystyle\int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\vec{u}}\, \mathrm{d}\Omega + \int_\Gamma \rho g u_{n|\Gamma} \bar{u}_{n|\Gamma}\, \mathrm{d}\Gamma + \int_\Sigma \left(\widehat{T} u_{n|\Sigma} + \rho\, g\, n_{3|\Sigma}\ u_{n|\Sigma} \right) \bar{u}_{n|\Sigma}\, \mathrm{d}\Sigma \\ \displaystyle + \rho\beta g \int_\Omega u_3 \bar{u}_3\, \mathrm{d}\Omega = 0 \end{cases} \tag{5. 27}$$

for each admissible $\bar{\vec{u}}$.

Reciprocally, let $\vec{u}$ a function of t with values in the set of admissible displacements and verifying (5. 27). Let us prove that $\vec{u}$ is solution of the problem (4. 26).

We take a virtual displacement, still denoted by $\bar{\vec{u}}$, but does not verify div $\bar{\vec{u}} = 0$. Introducing the associated multiplier ν_0, we replace the equation (5. 27) by the equivalent equation

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\tilde{\vec{u}}} \,\mathrm{d}\Omega + \int_\Gamma \rho g u_{n|\Gamma} \bar{\tilde{u}}_{n|\Gamma} \,\mathrm{d}\Gamma + \int_\Sigma \left(\widehat{T} u_{n|\Sigma} + \rho\, g\, n_{3|\Sigma}\, u_{n|\Sigma} \right) \bar{\tilde{u}}_{n|\Sigma} \,\mathrm{d}\Sigma \\ \displaystyle + \rho\beta g \int_\Omega u_3 \bar{\tilde{u}}_3 \,\mathrm{d}\Omega + \int_\Omega \nu_0 \,\mathrm{div}\, \bar{\tilde{\vec{u}}} \,\mathrm{d}\Omega = 0 \end{array} \right. \tag{5. 28}$$

for all new admissible $\tilde{\vec{u}}$.
We have

$$\begin{aligned} \int_\Omega \nu_0 \,\mathrm{div} \bar{\tilde{\vec{u}}} \,\mathrm{d}\Omega &= \int_\Omega \left[\mathrm{div}\,(\nu_0 \bar{\tilde{\vec{u}}}) - \overrightarrow{\mathrm{grad}}\, \nu_0 \cdot \bar{\tilde{\vec{u}}} \right] \mathrm{d}\Omega \\ &= \int_\Gamma \nu_{0|\Gamma} \bar{\tilde{u}}_{n|\Gamma} \,\mathrm{d}\Gamma + \int_\Sigma \nu_{0|\Sigma} \bar{\tilde{u}}_{n|\Sigma} \,\mathrm{d}\Sigma - \int_\Omega \overrightarrow{\mathrm{grad}}\, \nu_0 \cdot \bar{\tilde{\vec{u}}} \,\mathrm{d}\Omega \end{aligned}$$

The variational equation (5. 28) becomes

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \left(\rho \ddot{\vec{u}} - \overrightarrow{\mathrm{grad}}\, \nu_0 + \rho\beta g u_3 \vec{x}_3 \right) \cdot \bar{\tilde{\vec{u}}} \,\mathrm{d}\Omega + \int_\Gamma \left(\nu_{0|\Gamma} + \rho g u_{n|\Gamma} \right) \cdot \bar{\tilde{u}}_{n|\Gamma} \,\mathrm{d}\Gamma \\ \displaystyle + \int_\Sigma \left(\nu_{0|\Gamma} + \widehat{T} u_{n|\Sigma} + \rho\, g\, n_{3|\Sigma}\, u_{n|\Sigma} \right) \bar{\tilde{u}}_{n|\Sigma} \,\mathrm{d}\Sigma = 0 \end{array} \right.$$

Taking $\tilde{\vec{u}} \in [\mathscr{D}(\Omega)]^3$, we obtain

$$\int_\Omega \left(\rho \ddot{\vec{u}} - \overrightarrow{\mathrm{grad}}\, \nu_0 + \rho\beta g u_3 \vec{x}_3 \right) \cdot \bar{\tilde{\vec{u}}} \,\mathrm{d}\Omega = 0 \quad , \quad \forall\, \tilde{\vec{u}} \in [\mathscr{D}(\Omega)]^3$$

and then

$$\rho \ddot{\vec{u}} - \overrightarrow{\mathrm{grad}}\, \nu_0 + \rho\beta g u_3 \vec{x}_3 = 0 \qquad \text{in} \quad \left([\mathscr{D}(\Omega)]^3 \right)'.$$

Taking $\tilde{u}_{n|\Gamma}$ and $\tilde{u}_{n|\Sigma}$ arbitrary, we have

$$\nu_{0|\Gamma} + \rho g u_{n|\Gamma} = 0 \quad ; \quad \nu_{0|\Gamma} + \widehat{T} u_{n|\Sigma} + \rho\, g\, n_{3|\Sigma}\, u_{n|\Sigma} = 0.$$

Setting $\nu_0 = -p$ we find again the equations (4. 26).

5.2. We are going to seek $\vec{u}$ in the the space

$$\vec{u} \in J(\Omega) \stackrel{\text{def}}{=} \left\{ \vec{u} \in \mathscr{L}^2(\Omega) \stackrel{\text{def}}{=} \left[L^2(\Omega) \right]^3 ;\ \mathrm{div} \vec{u} = 0 \quad \text{in} \quad \Omega \right\}$$

closed subspace of $\mathscr{L}^2(\Omega)$, equipped with the norm of $\mathscr{L}^2(\Omega)$, in the form

$$\vec{u} = \vec{v} + \vec{U}$$

with

$$\vec{v} \in J_0(\Omega) \stackrel{\text{def}}{=} \left\{ \vec{v} \in \mathscr{L}^2(\Omega);\ \ \mathrm{div} \vec{v} = 0;\ \ v_{n|\partial\Omega} = 0 \right\}$$

$$\vec{U} \in G_h(\Omega) \stackrel{\text{def}}{=} \left\{ \vec{U} = \overrightarrow{\mathrm{grad}} \Phi;\ \ \Phi \in H^1(\Omega);\ \ \int_\Omega \Phi \,\mathrm{d}\Omega = 0;\ \ \Delta\Phi = 0 \right\}$$

By virtue of the orthogonal decomposition in $\mathscr{L}^2(\Omega)$ [7: Section 2.1.10]

$$J(\Omega) = J_0(\Omega) \oplus G_h(\Omega).$$

In the following, we will use the Weyl's decompositions [7: Section 2.1.10]

$$\mathscr{L}^2(\Omega) = J_0(\Omega) \oplus G(\Omega),$$

where $G(\Omega)$ is the space of the potential fields and

$$G(\Omega) = G_h(\Omega) \oplus G_0(\Omega),$$

where

$$G_0(\Omega) = \left\{ \overrightarrow{\text{grad}} q; \quad q \in H_0^1(\Omega) \right\}$$

We write the Euler's equation (3. 12) in the form

$$\ddot{\vec{v}} + \ddot{\vec{U}} = -\frac{1}{\rho} \overrightarrow{\text{grad}} p - \beta g v_3 \vec{x}_3 - \beta g U_3 \vec{x}_3.$$

Let P_0 the orthogonal projector from $\mathscr{L}^2(\Omega)$ into $J_0(\Omega)$; we have

$$\ddot{\vec{v}} = -\beta g P_0(v_3 \vec{x}_3) - \beta g P_0(U_3 \vec{x}_3) \tag{5. 29}$$

Now, we set

$$\bar{\vec{u}} = \bar{\vec{v}} + \bar{\vec{U}},$$

with

$$\bar{\vec{v}} \in J_0(\Omega) \quad ; \quad \bar{\vec{U}} \in G_h(\Omega).$$

Since $J_0(\Omega)$ and $G_h(\Omega)$ are orthogobal in $\mathscr{L}^2(\Omega)$, we have

$$\int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\vec{u}} \, d\Omega = \int_\Omega \rho \left(\ddot{\vec{v}} \cdot \bar{\vec{v}} + \ddot{\vec{U}} \cdot \bar{\vec{U}} \right) d\Omega.$$

On the other hand, since $v_{n|\partial\Omega} = 0$, we have $u_{n|\partial\Omega} = U_{n|\partial\Omega}$ too $\tilde{u}_{n|\partial\Omega} = \tilde{U}_{n|\partial\Omega}$; so that the variational equation (5. 27) takes the form

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \rho \left(\ddot{\vec{v}} \cdot \bar{\vec{v}} + \ddot{\vec{U}} \cdot \bar{\vec{U}} \right) d\Omega + \rho g \int_\Gamma U_{n|\Gamma} \bar{\tilde{U}}_{n|\Gamma} \, d\Gamma + \left\langle \widehat{T} U_{n|\Sigma}, \tilde{U}_{n|\Sigma} \right\rangle \\ \displaystyle + \rho g \int_\Sigma n_{3|\Sigma} U_{n|\Sigma} \bar{\tilde{U}}_{n|\Sigma} \, d\Sigma + \rho\beta g \int_\Omega (v_3 + U_3) \left(\bar{\tilde{v}}_3 + \bar{\tilde{U}}_3 \right) d\Omega = 0 \end{array} \right. \tag{5. 30}$$

But, we have

$$\beta g \int_\Omega (v_3 + U_3) \, \bar{\tilde{v}}_3 \, d\Omega = \int_\Omega \left[\beta g P_0(v_3 \vec{x}_3) + \beta g P_0(U_3 \vec{x}_3) \right] \cdot \bar{\vec{v}} \, d\Omega.$$

Using (5. 29) , we obtain

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \rho \ddot{\vec{U}} \cdot \bar{\vec{U}} \, d\Omega + \rho g \int_\Gamma U_{n|\Gamma} \bar{\tilde{U}}_{n|\Gamma} \, d\Gamma + \left\langle \widehat{T} U_{n|\Sigma}, \tilde{U}_{n|\Sigma} \right\rangle \\ \displaystyle + \rho g \int_\Sigma n_{3|\Sigma} U_{n|\Sigma} \bar{\tilde{U}}_{n|\Sigma} \, d\Sigma + \rho\beta g \int_\Omega (v_3 + U_3) \cdot \bar{\tilde{U}}_3 \, d\Omega = 0. \end{array} \right. \tag{5. 31}$$

At least if ρ is sufficiently small, $\widehat{T}$ is strongly positive, so that

$$\left(\left\langle \widehat{T} U_{n|\Sigma}, U_{n|\Sigma}\right\rangle + \rho g \int_{\Sigma} n_{3|\Sigma} \left|U_{n|\Sigma}\right|^2 \, \mathrm{d}\Sigma\right)^{1/2}$$

defines a norm that is equivalent to $\left\|U_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)}$.

We denote this norm by $\left|\left\|U_{n|\Sigma}\right\|\right|$ and the associated scalar product by $\left[U_{n|\Sigma}, \tilde{U}_{n|\Sigma}\right]$.

Then, we introduce the space

$$V = \left\{ \begin{array}{l} \vec{U} : \vec{U} = \overrightarrow{\mathrm{grad}}\Phi, \ \Phi \in \widetilde{H}^1(\Omega) = \left\{\Phi \in H^1(\Omega); \ \int_{\Sigma \cup \Gamma} \Phi \mathrm{d}(\partial\Omega) = 0\right\}; \\ \mathrm{div}\, \vec{U} = \Delta\Phi = 0; \ U_{n|\Gamma} \in L^2(\Gamma); \ U_{n|\Sigma} \in H^{1/2}(\Sigma) \end{array} \right\}.$$

equipped with the hilbertian norm defined by

$$\left\|\vec{U}\right\|_V^2 = \int_{\Omega} \left|\vec{U}\right|^2 d\Omega + \left\|U_{n|\Gamma}\right\|^2_{L^2(\Gamma)} + \left|\left\|U_{n|\Sigma}\right\|\right|^2_{H^{1/2}(\Sigma)},$$

and the space χ, completion of V for the norm associated to the scalar product

$$\left(\vec{U}, \tilde{\vec{U}}\right)_\chi = \int_{\Omega} \rho \vec{U} \cdot \bar{\tilde{\vec{U}}} \, \mathrm{d}\Omega$$

The "intermediate" variational equation (5. 31) that contains the unknown v_3, can be written:

$$\left(\ddot{\vec{U}}, \tilde{\vec{U}}\right)_\chi + \left[U_{n|\Sigma}, \tilde{U}_{n|\Sigma}\right] + \rho g \int_{\Gamma} U_{n|\Gamma} \bar{\tilde{U}}_{n|\Gamma} \, \mathrm{d}\Gamma + \rho\beta g \int_{\Omega} (v_3 + U_3) \bar{\tilde{U}}_3 \, \mathrm{d}\Omega = 0; \ \ \forall \ \tilde{\vec{U}} \in V. \tag{5. 32}$$

The equations (5. 29) and (5. 32) are the the equations of the motion of the liquid .

5.3. We are going to introduce a few operators.
We set

$$\beta g P_0 \left(v_3 \vec{x}_3\right) = A_{11} \vec{v},$$

where A_{11} is a non negative, selfadjoint, bounded operator from $J_0(\Omega)$ into $J_0(\Omega)$. It is known [2, 5] that this operator has a spectrum that coincides with its essential spectrum, that is the clowsed interval $[0, \beta g]$, so that $\|A_{11}\| = \beta g$.
We set still

$$\beta g P_0 \left(U_3 \vec{x}_3\right) = A_{12} \vec{U},$$

A_{12} being a bounded operator from χ into $J_0(\Omega)$.
Then, the equation (5. 29) can be written

$$\ddot{\vec{v}} + A_{11}\vec{v} + A_{12}\vec{U} = 0. \tag{5. 33}$$

Now, we have

$$\left|\int_{\Omega} \beta g v_3 \bar{\tilde{U}}_3 \, \mathrm{d}\Omega\right| \leq c_1 \left\|\vec{v}\right\|_{J_0(\Omega)} \left\|\tilde{\vec{U}}\right\|_\chi \quad , \quad (c_1 > 0),$$

so that we can set

$$\int_{\Omega} \beta g v_3 \bar{\tilde{U}}_3 \, \mathrm{d}\Omega = \left(A_{21}\vec{v}, \tilde{\vec{U}}\right)_\chi$$

where A_{21} is a bounded operator from $J_0(\Omega)$ into χ.
It is easy to see that A_{21} and A_{12} are mutually adjoint; indeed, we have

$$\left(A_{21}\vec{v},\vec{U}\right)_\chi = \int_\Omega \beta g v_3 \bar{U}_3 \, \mathrm{d}\Omega = \left(\vec{v}, A_{12}\vec{U}\right)_{J_0(\Omega)}.$$

Finally, we can set

$$\beta g \int_\Omega U_3 \bar{\tilde{U}}_3 \mathrm{d}\Omega = \left(A_{22}\vec{U},\vec{\tilde{U}}\right)_\chi,$$

A_{22} being a non negative, self-adjoint, bounded operator, from χ into χ.
Then, the variational equation (5. 32) takes the form

$$\left(\ddot{\vec{U}},\vec{\tilde{U}}\right)_\chi + \left[U_{n|\Sigma},\tilde{U}_{n|\Sigma}\right] + \rho g \int_\Gamma U_{n|\Gamma}\,\bar{\tilde{U}}_{n|\Gamma}\ \mathrm{d}\Gamma + \rho\left(A_{21}\vec{v}+A_{22}\vec{U},\vec{\tilde{U}}\right)_\chi = 0 \ \ \forall\, \vec{\tilde{U}} \in V. \tag{5. 34}$$

6. The operatorial equations of the problem

At first, we have the equation (5. 33)

$$\ddot{\vec{v}} + A_{11}\vec{v} + A_{12}\vec{U} = 0.$$

We are going to obtain another operatorial equation from the variational equation (5. 34).

6.1. We set

$$a\left(\vec{U},\vec{\tilde{U}}\right) = \left[U_{n|\Sigma},\tilde{U}_{n|\Sigma}\right] + \rho g \int_\Gamma U_{n|\Gamma}\,\bar{\tilde{U}}_{n|\Gamma}\ \mathrm{d}\Gamma.$$

It is an hermitian sesquilinear form on $V \times V$.
We can prove that

1) $a(\cdot,\cdot)$ is continuous and coercive in $V \times V$.
2) the embedding $V \in \chi$, obviously dense and continuous, is compact.

We omit the proof that is strictly identical to the proof of the Lemma 8.3 in [14: Section 2.8].
We denote by A the unbounded operator of χ associated to the form $a(\cdot,\cdot)$ and to the pair (V,χ).

6.2. The variational equation (5. 34) can be written

$$\left(\ddot{\vec{U}},\vec{\tilde{U}}\right)_\chi + a\left(\vec{U},\vec{\tilde{U}}\right) + \rho\left(A_{21}\vec{v}+A_{22}\vec{U},\vec{\tilde{U}}\right)_\chi = 0 \ \ \forall\, \vec{\tilde{U}} \in V.$$

It is well-known [9] that this equation is equivalent to the operatorial equation

$$\ddot{\vec{U}} + A\vec{U} + \rho\left(A_{21}\vec{v}+A_{22}\vec{U}\right) = 0, \ \ \forall\, \vec{U} \in V. \tag{6. 35}$$

The equation (5. 33) and (6. 35) are the operatorial equations of the problem.
We can eliminate the unbounded operator A by setting

$$A^{1/2}\vec{U} = \vec{U}_0 \in \chi.$$

We obtain the operatorial equations with bounded coefficients

$$\ddot{\vec{v}} + A_{11}\vec{v} + A_{12}A^{-1/2}\vec{U}_0 = 0 \qquad (6.\ 36)$$

$$A^{-1}\ddot{\vec{U}}_0 + \rho A^{-1/2}A_{21}\vec{v} + \left(I_\chi + \rho A^{-1/2}A_{22}A^{-1/2}\right)\vec{U}_0 = 0 \qquad (6.\ 37)$$

$$\vec{v} \in J_0(\Omega),\ \vec{U}_0 \in \chi.$$

A_{11} and I_χ are not compact , but $A^{-1},\ A_{12}A^{-1/2},\ A^{-1/2}A_{21}$ and $A^{-1/2}A_{22}A^{-1/2}$ are compact.

7. EXISTENCE OF THE SPECTRUM

We consider the precedent equations, (6. 36) being multiplied by ρ.
We set

$$X = \left(\vec{v}, \vec{U}_0\right)^t \in H_0 \overset{\text{def}}{=} J_0(\Omega) \oplus \chi,$$

$$Q = \begin{pmatrix} \rho I_{J_0(\Omega)} & 0 \\ 0 & A^{-1} \end{pmatrix} \quad ; \quad B = \begin{pmatrix} \rho A_{11} & \rho A_{12}A^{-1/2} \\ \rho A^{-1/2}A_{21} & I_\chi + \rho A^{-1/2}A_{22}A^{-1/2} \end{pmatrix}$$

The equations can be written

$$Q\ddot{X} + BX = 0; \qquad (7.\ 38)$$

Q and B are bounded and self-adjoint; Q is positive definite.
Using the definition of the A_{ij}; we have easily

$$(BX, X)_{H_0} = \rho\, \beta\, g \int_\Omega |v_3 + U_3|^2\, \mathrm{d}\Omega\ + a\left(\vec{U}, \vec{U}\right) \geq 0$$

Ker$B$ is the set of the $X = (\vec{v}, 0)^t$, with $\vec{v} \in J_0(\Omega)$, $v_3 = 0$, so that B is not negative.
By direct calculations, we obtain

$$((Q+B)X, X)_{H_0} = \rho\|\vec{v}\|^2_{J_0(\Omega)} + \left(A^{-1}\vec{U}_0, \vec{U}_0\right)_\chi + \rho\beta g\int_\Omega |v_3 + U_3|^2\, \mathrm{d}\Omega + a\left(\vec{U}, \vec{U}\right).$$

But we have

$$\|\vec{U}_0\|^2_\chi = \|A^{1/2}\vec{U}\|^2_\chi = \left(A\vec{U}, \vec{U}\right)_\chi = a\left(\vec{U}, \vec{U}\right),$$

so that

$$((Q+B)X, X)_{H_0} \geq \min(\rho, 1)\|X\|^2_{H_0}.$$

$Q + B$, being selfadjoint and strongly positive, has an inverse having the same properties.
We seek the solutions of the equation (7. 38) depending on t according to the law $e^{-\lambda t}$, $\lambda \in \mathbb{C}$. We have

$$\left(\lambda^2 Q + B\right) X = 0 \quad , \quad X \in H_0.$$

$\lambda = 1$, $\lambda = -1$ are not eigenvalues, because $Q + B$ is strongly positive.
$\lambda = 0$ is an eigenvalue with infinity multiplicity, the eigenspace being Ker$B$.
We write the equation in the form

$$\left(\lambda^2(Q+B) + (1-\lambda^2)B\right) X = 0.$$

Setting

$$(Q+B)^{1/2}X=\kappa\in H_0 \quad ; \quad \mathscr{C}=(Q+B)^{-1/2}B(Q+B)^{-1/2},$$

we obtain the equation

$$\mathscr{C}\kappa=\frac{\lambda^2}{\lambda^2-1}\kappa \quad , \quad \kappa\in H_0. \tag{7. 39}$$

$\mathscr{C}$, being a not negative, selfadjoint operator, has a real spectrum that is located on the positive real half-axis.
Therefore, we must have

$$\frac{\lambda^2}{\lambda^2-1} \quad \text{real positive},$$

and then, λ^2 real with $\lambda^2>1$ or $\lambda^2<0$.
We must dismiss $\lambda^2>1$ since $\left(\lambda^2Q+B\right)$ strongly positive.
We must have λ^2 real negative, or $\lambda=\pm i\omega$, ω real.
The spectrum of the problem exists and is located on the imaginary axis, symmetrical with respec to the origin.

8. Study of the spectrum of the problem

Using the precedent result, we seek the solutions of the equations (5. 33), (6. 35) and (6. 36), (6. 37) depending on t to the law $e^{i\omega t}$, ω real.
We obtain:

$$\omega^2\vec{v}=A_{11}\vec{v}+A_{12}A^{-1/2}\vec{U}_0 \tag{8. 40}$$

$$\omega^2A^{-1}\vec{U}_0=\rho A^{-1/2}A_{21}\vec{v}+\left(I_\chi+\rho A^{-1/2}A_{22}A^{-1/2}\right)\vec{U}_0 \tag{8. 41}$$

or, setting $\nu=\omega^{-2}$

$$\vec{v}=\nu A_{11}\vec{v}+\nu A_{12}A^{-1/2}\vec{U}_0 \tag{8. 42}$$

$$A^{-1}\vec{U}_0=\nu\rho A^{-1/2}A_{21}\vec{v}+\nu\left(I_\chi+\rho A^{-1/2}A_{22}A^{-1/2}\right)\vec{U}_0 \tag{8. 43}$$

8.1. The spectrum in the domain: $\omega^2>\beta g$
We have

$$|\nu|<(\beta g)^{-1}$$

Since $\|A_{11}\|=\beta g$, the operator $I_{J_0(\Omega)}-\nu A_{11}$ has an inverse $\mathscr{R}(\nu)$ that is holomorphic in the domain $|\nu|<(\beta g)^{-1}$ and the equation (8. 42) can be written

$$\vec{v}=\nu\mathscr{R}(\nu)A_{12}A^{-1/2}\vec{U}_0.$$

Carrying out in the equation (8. 43), we obtain

$$Q(\nu)\vec{U}_0\stackrel{\text{def}}{=}\left[\nu^2\rho A^{-1/2}A_{21}\mathscr{R}(\nu)A_{12}A^{-1/2}+\nu\left(I_\chi+\rho A^{-1/2}A_{22}A^{-1/2}\right)-A^{-1}\right]\vec{U}_0=0. \tag{8. 44}$$

$Q(\nu)$ is a holomorphic self-adjoint operatorial function in the domain $|\nu| < (\beta g)^{-1}$; we have:

$$Q(0) = -A^{-1} \text{ compact, negative definite},$$

$$Q'(0) = I_\chi + \rho A^{-1/2} A_{22} A^{-1/2} \text{ strongly positive.}$$

Consequently [7: Section 1.6.10], for each ϵ, $0 < \epsilon < (\beta g)^{-1}$, in the interval $]0, \epsilon[$, there exists a denumerable infinity of eigenvalues v_k, $v_k \to 0$ when $k \to +\infty$. The corresponding eigenelements $\left\{\vec{U}_{0k}\right\}$ form a Riesz basis in a subspace of χ with finite defect.

For our problem, there is a denumerable infinity of positive real eigenvalues $\omega_k = \nu_k^{-1/2}$ having the infinity as point of accumulation.

8.2. The spectrum in the domain: $0 \le \omega^2 \le \beta g$

The equation (8. 41) can be written

$$\left(I_\chi + \rho A^{-1/2} A_{22} A^{-1/2} - \omega^2 A^{-1}\right) \vec{U}_0 = -\rho A^{-1/2} A_{21} \vec{v}.$$

Since $\omega^2 \le \beta g$ and $\|A_{22}\| \le \beta g$, the coefficient of $\vec{U}_0$ is a strongly positive, a self-adjoint, bounded operator if βg is sufficiently small. Then it has an inverse with the same properties and we have

$$\vec{U}_0 = -\rho \left(I_\chi + \rho A^{-1/2} A_{22} A^{-1/2} - \omega^2 A^{-1}\right)^{-1} A^{-1/2} A_{21} \vec{v}.$$

Carrying out in the equation (8. 40), we obtain

$$A_{11}\vec{v} - \mathscr{N}(\omega^2)\vec{v} = \omega^2 \vec{v}, \qquad \vec{v} \in J_0(\Omega)$$

with

$$\mathscr{N}(\omega^2) = \rho A_{12} A^{-1/2} \left(I_\chi + \rho A^{-1/2} A_{22} A^{-1/2} - \omega^2 A^{-1}\right)^{-1} A^{-1/2} A_{21}$$

$\mathscr{N}(\omega^2)$ is an analytical function of ω^2 in the domain $\omega^2 < \beta g$ and, for each ω^2, it is a compact selfadjoint operator.

We are going tu use a methode indicated in [7: Section 6.5.7].

Setting

$$\mathscr{M}(\omega^2) = A_{11} - \mathscr{N}(\omega^2), \tag{8. 45}$$

we have the equation

$$\left(\mathscr{M}(\omega^2) - \omega^2 I_{J_0(\Omega)}\right) \vec{v} = 0, \qquad \vec{v} \in J_0(\Omega)$$

Let ω_1^2 in $[0, \beta g]$. By virtue of a well-known Weyl's theorem [7: Section 1.1.19, p 24], we have

$$\sigma_{ess}\left(\mathscr{M}(\omega_1^2)\right) = \sigma_{ess}(A_{11}) = [0, \beta g].$$

Let ω_2^2 an arbitrary element of $\sigma_{ess}\left(\mathscr{M}(\omega_1^2)\right)$. By virtue of another Weyl's theorem, there exists a sequence $\{\vec{v}_n\}$, depending on ω_1^2 and ω_2^2 such that $\vec{v}_n \to 0$ weakly in $J_0(\Omega)$;

$\inf\limits_{J_0(\Omega)} \|\vec{v}_n\|_{J_0(\Omega)} > 0$; $\left(\mathscr{M}(\omega_1^2) - \omega_2^2 I_{J_0(\Omega)}\right) \vec{v}_n \to 0$ in $J_0(\Omega)$.
Choosing $\omega_2^2 = \omega_1^2$, there exists a sequence $\left\{\vec{\tilde{v}}_n\right\}$, depending on ω_1^2 only, such that

$$\vec{\tilde{v}}_n \to 0 \text{weakly in} J_0(\Omega);\ \inf_{J_0(\Omega)} \left\|\vec{\tilde{v}}_n\right\|_{J_0(\Omega)} > 0;\ \left(\mathscr{M}(\omega_1^2) - \omega_1^2 I_{J_0(\Omega)}\right) \vec{\tilde{v}}_n \to 0 \text{ in } J_0(\Omega).$$

Therefore, ω_1^2 belongs to the essential spectrum of the problem (8. 45).
ω_1^2 being arbitrary in $[0, \beta g]$, this interval is the essential spectrum of the problem.
Physically the interval $[0, \beta g]$ is a ”domain of resonance”.

9. Existence and unicity of the solution of the associated evolution problem

From the equation (5. 33), we deduce

$$\rho\left(\ddot{\vec{v}}, \vec{\tilde{v}}\right)_{J_0(\Omega)} + \rho\left(A_{11}\vec{v} + A_{12}\vec{U}, \vec{\tilde{v}}\right)_{J_0(\Omega)} = 0 \quad \forall \vec{\tilde{v}} \in J_0(\Omega).$$

Adding to the variational equation (5. 34), we obtain

$$\rho\left(\ddot{\vec{v}}, \vec{\tilde{v}}\right)_{J_0(\Omega)} + \left(\ddot{\vec{U}}, \vec{\tilde{U}}\right)_{\chi} + a\left(\vec{U}, \vec{\tilde{U}}\right) + \rho\left(A_{11}\vec{v} + A_{12}\vec{U}, \vec{\tilde{v}}\right)_{J_0(\Omega)} + \rho\left(A_{21}\vec{v} + A_{22}\vec{U}, \vec{\tilde{U}}\right)_{\chi} = 0. \tag{9. 46}$$

for each $\vec{\tilde{v}} \in J_0(\Omega)$ and each $\vec{\tilde{U}} \in V$.
We introduce the spaces

$$V_0 = J_0(\Omega) \oplus V \quad ; \quad H_0 = J_0(\Omega) \oplus \chi.$$

The imbeding $V_0 \subset H_0$ is dense and continuous, but not compact.
Let $\hat{C}$ the operator from H_0 onto H_0 defined by

$$\hat{C} = \begin{pmatrix} \rho I_{J_0(\Omega)} & 0 \\ 0 & I_\chi \end{pmatrix}.$$

We set

$$\zeta = \left(\vec{v}, \vec{U}\right)^t \in V_0 \quad ; \quad \tilde{\zeta} = \left(\vec{\tilde{v}}, \vec{\tilde{U}}\right)^t \in V_0,$$

$$\hat{a}\left(\zeta, \tilde{\zeta}\right) = \rho\left[\left(A_{11}\vec{v} + A_{12}\vec{U}, \vec{\tilde{v}}\right)_{J_0(\Omega)} + \left(A_{21}\vec{v} + A_{22}\vec{U}, \vec{\tilde{U}}\right)_{\chi}\right] + a\left(\vec{U}, \vec{\tilde{U}}\right).$$

Then, the equation (9. 46) can be written

$$(\hat{C}\ddot{X}, \tilde{\zeta})_{H_0} + \hat{a}(\zeta, \tilde{\zeta}) = 0, \quad \forall \hat{\zeta} \in V_0. \tag{9. 47}$$

Let λ_0 a positive real number; we have

$$\hat{a}(\zeta, \zeta) + \lambda_0 \|\zeta\|_{H_0}^2 = \rho\beta g \int_\Omega |v_3 + U_3|^2 \, d\Omega \ + a\left(\vec{U}, \vec{U}\right) + \lambda_0 \left[\|\vec{v}\|_{J_0(\Omega)}^2 + \|\vec{U}\|_\chi^2\right].$$

Since $a(\cdot, \cdot)$ is coercive in $V \times V$, we have

$$\hat{a}(\zeta, \zeta) \geq \lambda_0 \|\vec{v}\|_{J_0(\Omega)}^2 + C_0 \|\vec{U}\|_V^2 - \lambda_0 \|\zeta\|_{H_0}^2 \quad , \quad (C_0 > 0),$$

so that $\hat{a}(.,.)$ is V_0- coercive with respect to H_0.
Consequently, we can apply a known theorem [3; pp 664-670]:
If the initial data verify

$$\zeta_0 = \left((\vec{v})^0, (\vec{U})^0\right)^t \in V_0 \quad ; \quad \dot{\zeta}_0 = \left((\dot{\vec{v}})^0, (\dot{\vec{U}})^0\right)^t \in H_0,$$

the evolution problem has one and only one solution $\zeta(.)$ suth that

$$\zeta(t) \in L^2\left((0,T); V_0\right) \quad ; \quad \dot{\zeta}(t) \in L^2\left((0,T); V_0\right),$$

where T is an arbitrary positive constant.

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