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Hyper-Zagreb index of graphs based on generalized subdivision related operations

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Abstract. Mathematical modeling of the molecular graphs plays a fundamental part in the analysis of the quantitative structures activity relationship (QSAR) and quantitative structures property relationship models (QSPR models). In 2013, Shirdel et. al. [IJMC; 4(2013); 213-220] defined the new topological index of a graph (Γ) named as hyper-Zagreb index [HM(Γ)] is $HM(\Gamma) = \sum_{yz\epsilon E(\Gamma)} [d_\Gamma(y) + d_\Gamma(z)]^2$ Liu et al. [IEEE Access; 7(2019); 105479-105488] defined the concept of the generalized subdivision operations on graphs and obtained the generalized F-sum graphs. In this paper, the hyper-Zagreb index is calculated for the generalized F-sum graphs in terms of their factor graphs. In fact, the obtained results are the general extension of the results Anandkumar et al. [IJPAM; 112(2017); 239-252]

AMS (MOS) Subject Classification Codes: 05C78.
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1. Introduction

The structural study of the relations between the vertices (nodes) and edges (bonds) comes along with a very large variety of applications in various discipline of science. In chemistry, this study is used to characterize the organic molecules (carbon-based or biological) with respect to the certain physical, chemical and biological properties. Topological index (TI) is a function that associate a graph with a real number and remains invariant for isomorphic graphs.

TI's of a molecular graphs predicts the different physical attributes, biological activities or properties and chemical reactivities such as heat of evaporation, heat of formation, chromatographic retention times, viscosity, flash point, boiling point, freezing point, melting point, vapor pressure, surface tension, octanol-water partition coefficient, stability, critical temperature, weight, density, connectivity, solubility and polarizability. Several crystalline materials, nano-materials and drugs which are used in many industries are analyzed with the help of TI's, see [15, 1, 3, 5, 4, 13, 20, 24]. As far as the organic compounds are concerned, the physicochemical characteristics of organic compounds are frequently demonstrated in terms of molecular graphs depend on its structural descriptors, which are also mentioned as TI's [19, 7].

In the last few decades, many TI's are introduced but degree based TI's got much more attention of the researchers, see the latest survey [2]. In 1947, the very first TI is introduced by Winer to compute the boiling point of the paraffin [12]. After that, Trinajsti and Gutman (1972) [6] defined the first and second Zagreb indices that are used to compute the different structure base characteristics of the molecular graphs. In 2013, Shirdel et al. [8] defined the concept of hyper-Zagreb index HM(Γ) as a new degree-based TI. In addition, the results for the HM(Γ) of the cartesian product, composition, join and disjunction of graphs can be found in [25, 8]. Wei Gao et al. [26] calculated HM(Γ) for titania nanotubes. For further different results, we refer to [13, 3, 13, 5, 4, 20, 24].

The different operations on graphs perform a fundamental role in the formation of new classes of graphs, see [14]. Yan et al. [15] introduced the four subdivided operations S_1, R_1, Q_1 and T_1 and formulated the Wiener index of the graphs which are obtained by using these operations. Taeri and Eliasi [16] gave the concept of F-sum graphs by using the above four subdivided operations. Moreover, Deng et al. [21] computed the first and second Zagreb indices, Imran and Shehnaz [22] calculated the forgotten index and Liu et al. [29] obtained the first general Zagreb index of F-sum graphs. Recentaly, Liu et al. [30] defined the generalized version of these subdivided operations of graphs which are denoted by S_k, R_k, Q_k and T_k, where $k \geq 1$ is some integer. They also defined the generalized F-sum graphs $(\Gamma_1 \ {+_{F_k}} \ \Gamma_2)$ by using these generalized operations and computed the first and second Zagreb indices of newly obtained generalized F-sum graphs. Furthermore, Awais et al. [28] computed the forgotten index and Xiujun Zhang et al. [27] calculated the multiplicative Zagreb indices of generalized F-sum graphs.

In this paper, the Hyper-Zagreb index is computed for the generalized F-sum graphs. The remaining portion is managed as; Section II covers the very elementary definitions, Section III includes the results and Section IV contains the applications and conclusions. In fact, the obtained results are the general extension of the results Anandkumar et al. [9] who worked only for k=1.

2. Notations and Preliminaries

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a simple graph having vertex set $V(\Gamma)$ and edge set $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$. Every vertex $y \in V(\Gamma)$ in graph theory is consider as atom and link within the two atoms is known as edge. The cardinality of vertex set $(V(\Gamma))$ and edge set $(E(\Gamma))$ recognize as order and size of the graph Γ. The number of edges which are incident on a vertex is known as degree of $(d_\Gamma(y))$. Now, we discus some famous TI's are given:

Definition 2.1. Let Γ be a simple graph, the first and second Zagreb indices are defined as

$$M_1(\Gamma) = \sum_{y \epsilon V(\Gamma)} [d_\Gamma(y)]^2 = \sum_{yz \epsilon E(\Gamma)} [d_\Gamma(y) + d_\Gamma(z)],$$

$$M_2(\Gamma) = \sum_{yz \epsilon E(\Gamma)} [d_\Gamma(y) \times d_\Gamma(z)].$$

In 1972, Trinajsti and Gutman [6] defined the first and second Zagreb indices that are used to compute the different structures based characteristics of molecular graphs like energy, connectivity, complexity, chirality, branching and hetero-systems, see [17, 18].
However, the Hyper-Zagreb index of a graph (Γ) is represented as HM(Γ) and defined by G.H Shirdel et. al. in 2013 [8].

$$HM = HM(\Gamma) = \sum_{yz \epsilon E(\Gamma)} [d_\Gamma(y) + d_\Gamma(z)]^2$$

The generalized F-sum graphs defined in [30] are given as follows:
- $S_k(\Gamma)$ is the generalized sub-division of graph (Γ).
- $R_k(\Gamma)$ is generalized semi-total (point) graph.
- $Q_k(\Gamma)$ is generalized semi-total (line) graph.
- Finally, $T_k(\Gamma)$ is the generalized total graph.

For further details, see Figure 1.

Throughout this paper, we are studying the different types of graph parameters and just want to discuss finite, undirected, simple (multiple edges or without loops) and connected graphs.

Definition 2.2. Assume that Γ_1 and Γ_2 be two simple graphs, $F_k \in \{S_k, R_k, Q_k, T_k\}$ and $F_k(\Gamma_1)$ be a new graph obtained after using F_k on Γ_1 with edge set $E(F_k(\Gamma_1))$ as well as vertex set $V(F_k(\Gamma_1))$. Then, the generalized F-sum graph ($\Gamma_1 \ {}_{+F_k} \ \Gamma_2$) is a molecular graph with vertex set

$$V(\Gamma_1 \ {}_{+F_k} \ \Gamma_2) = V(F_k(\Gamma_1)) \times V(\Gamma_2) = (V(\Gamma_1) \cup E(\Gamma_1)) \times V(\Gamma_2)$$

such that two vertices (y_1, z_1) and (y_2, z_2) of $V(\Gamma_1 \ {}_{+F_k} \ \Gamma_2)$ are adjacent iff [$y_1 = y_2 \in V(\Gamma_1)$ and $(z_1, z_2) \in E(\Gamma_2)$] or [$z_1 = z_2 \in V(\Gamma_2)$ and $(y_1, y_2) \in E(F_k(\Gamma_1))$], where $k \geq 1$ is consider a counting number. For detail clarification Figure 2 and 3.

3. Main Results

Now, we compute the key outcomes of HM(Γ) for the above classes of F-sum graphs.

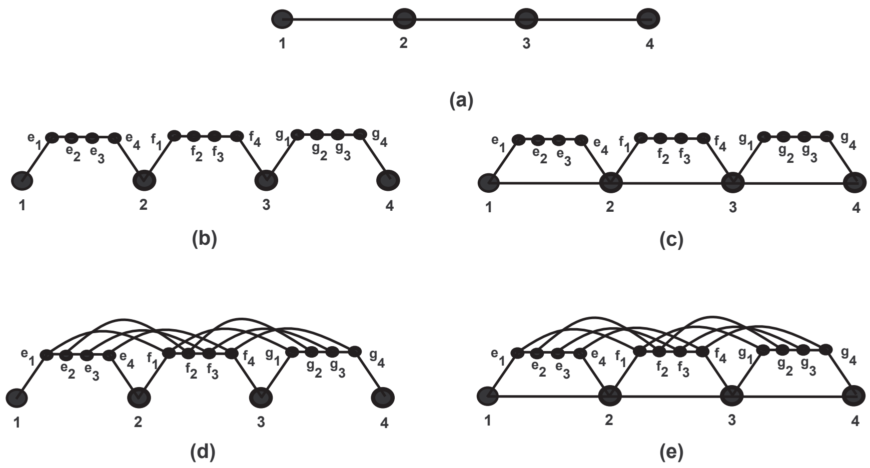

FIGURE 1. (a)Γ, (b)$S_4(\Gamma)$, (c)$R_4(\Gamma)$, (d)$Q_4(\Gamma)$ and (e)$T_4(\Gamma)$

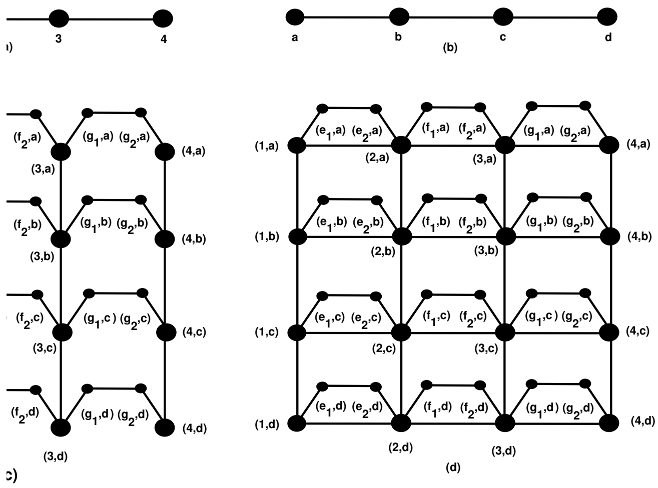

FIGURE 2. (a)$\Gamma_1 \cong P_4$, (b)$\Gamma_2 \cong P_4$, (c)$\Gamma_{1+S_2}\Gamma_2$, (d)$\Gamma_{1+R_2}\Gamma_2$

Theorem 3.1. Let Γ_1 and Γ_2 be two molecular graphs having $\mid V(\Gamma_1) \mid, \mid V(\Gamma_2) \mid \geq 4$. For $k \geq 1$, we have

$$HM(\Gamma_1 +_{S_k} \Gamma_2) = 4 \mid E(\Gamma_2) \mid M_1(\Gamma_1) + \mid V(\Gamma_1) \mid HM(\Gamma_2) + 10 \mid E(\Gamma_1) \mid M_1(\Gamma_2)$$

$$+ \mid V(\Gamma_2) \mid HM(S_1(\Gamma_1)) + 4 \mid E(\Gamma_2) \mid M_1(S_1(\Gamma_1)) + 16(k-1) \mid V(\Gamma_2) \mid\mid E(\Gamma_1) \cdot \mid$$

Proof. Let $d(y,z) = d_{\Gamma_1 +_{S_k} \Gamma_2}(y,z)$ be the degree of a vertex set (y,z) in the graph $\Gamma_1 +_{S_k} \Gamma_2$.

$$HM(\Gamma_1 +_{S_k} \Gamma_2) = \sum_{(y,z)\epsilon E(\Gamma_1)} (d(y) + d(z))^2$$

$$= \sum_{(y_1,z_1)(y_2,z_2)\epsilon E(\Gamma_1 +_{S_k} \Gamma_2)} [d(y_1,z_1) + d(y_2,z_2)]^2$$

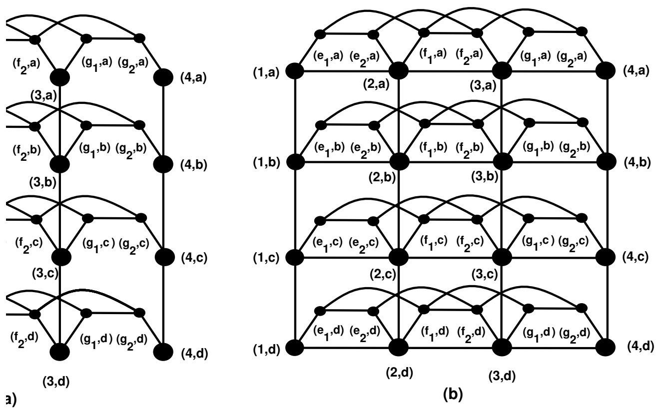

FIGURE 3. (a)$H_{1+Q_2}H_2$, (b)$\Gamma_{1+T_2}\Gamma_2$

$$= \sum_{y\epsilon V(\Gamma_1)}\sum_{z_1z_2\epsilon E(\Gamma_2)}[d(y,z_1)+d(y,z_2)]^2+\sum_{z\epsilon V(\Gamma_2)}\sum_{y_1y_2\epsilon E(S_k(\Gamma_1))}[d(y_1,z)+d(y_2,z)]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)}\sum_{z_1z_2\epsilon E(\Gamma_2)}[d(y,z_1)+d(y,z_2)]^2$$

$$+ \sum_{z\epsilon V(\Gamma_2)}\sum_{\substack{y_1y_2\epsilon E(S_k(\Gamma_1))\\ y_1\epsilon V(\Gamma_1),\ y_2\epsilon V(S_k(\Gamma_1))-V(\Gamma_1)}}[d(y_1,z)+d(y_2,z)]^2$$

$$+ \sum_{z\epsilon V(\Gamma_2)}\sum_{\substack{y_1y_2\epsilon E(S_k(\Gamma_1))\\ y_1,\ y_2\epsilon V(S_k(\Gamma_1))-V(\Gamma_1)}}[d(y_1,z)+d(y_2,z)]^2$$

$$= \sum 1+\sum 2+\sum 3.$$

Consider,

$$\sum 1 = \sum_{y\epsilon V(\Gamma_1)}\sum_{z_1z_2\epsilon E(\Gamma_2)}[d(y,z_1)+d(y,z_2)]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)}\sum_{z_1z_2\epsilon E(\Gamma_2)}[(d_{\Gamma_1}(y)+d_{\Gamma_2}(z_1))+(d_{\Gamma_1}(y)+d_{\Gamma_2}(z_2))]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)}\sum_{z_1z_2\epsilon E(\Gamma_2)}[2d_{\Gamma_1}(y)+d_{\Gamma_2}(z_1)+d_{\Gamma_2}(z_2)]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)}\sum_{z_1z_2\epsilon E(\Gamma_2)}[4(d_{\Gamma_1}(y))^2+(d_{\Gamma_2}(z_1))^2+(d_{\Gamma_2}(z_2))^2+4d_{\Gamma_1}(y)d_{\Gamma_2}(z_1)$$

$$+4d_{\Gamma_1}(y)d_{\Gamma_2}(z_2)+2d_{\Gamma_2}(z_1)d_{\Gamma_2}(z_2)]$$

$$= 4\mid E(\Gamma_2)\mid M_1(\Gamma_1)+\mid V(\Gamma_1)\mid HM(\Gamma_2)+\sum_{y\epsilon V(\Gamma_1)}\sum_{z_1z_2\epsilon E(\Gamma_2)}4d_{\Gamma_1}(y)[d_{\Gamma_2}(z_1)+d_{\Gamma_2}(z_2)]$$

$$= 4\mid E(\Gamma_2)\mid M_1(\Gamma_1)+\mid V(\Gamma_1)\mid HM(\Gamma_2)+8\mid E(\Gamma_1)\mid M_1(\Gamma_2),$$

$$\sum 2 = \sum_{z\epsilon V(\Gamma_2)}\sum_{\substack{y_1y_2\epsilon E(S_k(\Gamma_1))\\ y_1\epsilon V(\Gamma_1),\ y_2\epsilon V(S_k(\Gamma_1))-V(\Gamma_1)}}[d(y_1,z)+d(y_2,z)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(S_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1),\ y_2\epsilon V(S_k(\Gamma_1))-V(\Gamma_1)}} [(d_{S_k(\Gamma_1)}(y_1)+d_{\Gamma_2}(z))+d_{S_k(\Gamma_1)}(y_2)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(S_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1),\ y_2\epsilon V(S_k(\Gamma_1))-V(\Gamma_1)}} [(d_{S_k(\Gamma_1)}(y_1))^2+(d_{\Gamma_2}(z))^2+(d_{S_k(\Gamma_1)}(y_2))^2$$

$$+2d_{S_k(\Gamma_1)}(y_1)d_{\Gamma_2}(z)+2d_{S_k(\Gamma_1)}(y_1)d_{S_k(\Gamma_1)}(y_2)+2d_{\Gamma_2}(z)d_{S_k(\Gamma_1)}(y_2)]$$

Since $y_1\epsilon V(\Gamma_1)$ and $y_2\epsilon V(S_k(\Gamma_1)) - V(\Gamma_1)$, therefore

$$=\mid V(\Gamma_2)\mid HM(S_1(\Gamma_1))+\sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(S_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1),\ y_2\epsilon V(S_k(\Gamma_1)-\Gamma_1)}} [(d_{\Gamma_2}(z))^2+2d_{\Gamma_2}(z)$$

$$[d_{S_1(\Gamma_1)}(y_1)+d_{S_1(\Gamma_1)}(y_2)]]$$

$$=\mid V(\Gamma_2)\mid HM(S_1(\Gamma_1))+4\mid E(\Gamma_2)\mid M_1(S_1(\Gamma_1))+2\mid E(\Gamma_1)\mid M_1(\Gamma_2),$$

and

$$\sum 3 = \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(S_k(\Gamma_1)) \\ y_1,\ y_2\epsilon V(S_k(\Gamma_1))-V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(S_k(\Gamma_1)) \\ y_1,\ y_2\epsilon V(S_k(\Gamma_1))-V(\Gamma_1)}} [2+2]^2$$

Since in this case $|E(S_k(\Gamma_1))| = (k-1)|E(\Gamma_1)|$, so

$$= (k-1)|E(\Gamma_1)| \sum_{z\epsilon V(\Gamma_2)} (16) = 16(k-1)\mid V(\Gamma_2)\mid\mid E(\Gamma_1)\mid .$$

Consequently,

$$HM(\Gamma_1+_{S_k}\Gamma_2) = 4\mid E(\Gamma_2)\mid M_1(\Gamma_1)+\mid V(\Gamma_1)\mid HM(\Gamma_2)+10\mid E(\Gamma_1)\mid M_1(\Gamma_2)$$
$$+\mid V(\Gamma_2)\mid HM(S_1(\Gamma_1))+4\mid E(\Gamma_2)\mid M_1(S_1(\Gamma_1))+16(k-1)\mid V(\Gamma_2)\mid\mid E(\Gamma_1)\mid \cdot$$

Theorem 3.2.
Let Γ_1 and Γ_2 be two molecular graphs having $\mid V(\Gamma_1)\mid, \mid V(\Gamma_2)\mid \geq 4$. For $k \geq 1$, so

$$HM(\Gamma_1+_{R_k}\Gamma_2) = 8\mid V(\Gamma_2)\mid [F(\Gamma_1)+M_1(\Gamma_1)+M_2(\Gamma_1)]+2\mid E(\Gamma_1)\mid$$
$$[11M_1(\Gamma_2)+4\mid V(\Gamma_2)\mid]+8\mid E(\Gamma_2)\mid [5M_1(\Gamma_1)+2\mid E(\Gamma_1)\mid]+\mid V(\Gamma_1)\mid HM(\Gamma_2)$$
$$+16(k-1)\mid V(\Gamma_2)\mid\mid E(\Gamma_1)\mid .$$

Proof.
Let $d(y,z) = d_{(\Gamma_1+_{R_k}\Gamma_2)}(y,z)$ be the degree of a node (y,z) in the molecular graph $\Gamma_1 +_{R_k} \Gamma_2$.

$$HM(\Gamma_1+_{R_k}\Gamma_2) = \sum_{(y_1,z_1)(y_2,z_2)\epsilon E(\Gamma_1+_{R_k}\Gamma_2)} [d(y_1,z_1)+d(y_2,z_2)]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [d(y,z_1)+d(y,z_2)]^2+ \sum_{z\epsilon V(\Gamma_2)} \sum_{y_1y_2\epsilon E(R_k(\Gamma_1))} [d(y_1,z)+d(y_2,z)]^2$$

$$= \sum 1+\sum 2.$$

Consider,

$$\sum 1 = \sum_{y \epsilon V(\Gamma_1)} \sum_{z_1 z_2 \epsilon E(\Gamma_2)} [d(y,z_1) + d(y,z_2)]^2$$
$$= \sum_{y \epsilon V(\Gamma_1)} \sum_{z_1 z_2 \epsilon E(\Gamma_2)} [(d_{R_k(\Gamma_1)}(y)+d_{\Gamma_2}(z_1))+(d_{R_k(\Gamma_1)}(y)+d_{\Gamma_2}(z_2))]^2$$
$$= \sum_{y \epsilon V(\Gamma_1)} \sum_{z_1 z_2 \epsilon E(\Gamma_2)} [2d_{R_k(\Gamma_1)}(y)+d_{\Gamma_2}(z_1)+d_{\Gamma_2}(z_2)]^2$$
$$= \sum_{y \epsilon V(\Gamma_1)} \sum_{z_1 z_2 \epsilon E(\Gamma_2)} [4(d_{R_k(\Gamma_1)}(y))^2+(d_{\Gamma_2}(z_1))^2+(d_{\Gamma_2}(z_2))^2+4d_{R_k(\Gamma_1)}(y)d_{\Gamma_2}(z_1)$$
$$+4d_{R_k(\Gamma_1)}(y)d_{\Gamma_2}(z_2)+2d_{\Gamma_2}(z_1)d_{\Gamma_2}(z_2)]$$
$$= \sum_{y \epsilon V(\Gamma_1)} \sum_{z_1 z_2 \epsilon E(\Gamma_2)} [16(d_{\Gamma_1}(y))^2+(d_{\Gamma_2}(z_1))^2+(d_{\Gamma_2}(z_2))^2+8d_{\Gamma_1}(y)d_{\Gamma_2}(z_1)$$
$$+8d_{\Gamma_1}(y)d_{\Gamma_2}(z_2)+2d_{\Gamma_2}(z_1)d_{\Gamma_2}(z_2)]$$
$$= 16 \mid E(\Gamma_2) \mid M_1(\Gamma_1)+ \mid V(\Gamma_1) \mid F(\Gamma_2)+16 \mid E(\Gamma_1) \mid M_1(\Gamma_2)+2 \mid V(\Gamma_1) \mid M_2(\Gamma_2)$$
$$= 16 \mid E(\Gamma_2) \mid M_1(\Gamma_1)+ \mid V(\Gamma_1) \mid HM(\Gamma_2)+16 \mid E(\Gamma_1) \mid M_1(\Gamma_2),$$

and

$$\sum 2 = \sum_{z \epsilon V(\Gamma_2)} \sum_{y_1 y_2 \epsilon E(R_k(\Gamma_1))} [d(y_1,z)+d(y_2,z)]^2$$
$$= \sum_{z \epsilon V(\Gamma_2)} \sum_{\substack{y_1 y_2 \epsilon E(R_k(\Gamma_1)) \\ y_1, y_2 \epsilon V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2 + \sum_{z \epsilon V(\Gamma_2)} \sum_{\substack{y_1 y_2 \epsilon E(R_k(\Gamma_1)) \\ y_1 \epsilon V(\Gamma_1) \\ y_2 \epsilon V(R_k(\Gamma_1))-V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2$$
$$+ \sum_{z \epsilon V(\Gamma_2)} \sum_{\substack{y_1 y_2 \epsilon E(R_k(\Gamma_1)) \\ y_1, y_2 \epsilon V(R_k(\Gamma_1))-V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2$$
$$= \sum^{'} 2 + \sum^{''} 2 + \sum^{'''} 2.$$

For $y_1 y_2 \epsilon V(\Gamma_1)$, we have $y_1 y_2 \epsilon E(R_k(\Gamma_1))$ if and only if $y_1 y_2 \epsilon E(\Gamma_1)$; for $y_1 \epsilon V(\Gamma_1)$, we obtain $d_{R_k(\Gamma_1)}(y_1) = 2d_{\Gamma_1}(y_1)$ and for $y_2 \epsilon V(R_k(\Gamma_1))-V(\Gamma_1)$, we have $d_{R_k(\Gamma_1)}(y_2) = 2$. Now, consider

$$\sum^{'} 2 = \sum_{z \epsilon V(\Gamma_2)} \sum_{\substack{y_1 y_2 \epsilon E(R_k(\Gamma_1)) \\ y_1 y_2 \epsilon V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2 = \sum_{z \epsilon V(\Gamma_2)} \sum_{y_1 y_2 \epsilon V(\Gamma_1)} [d(y_1,z)+d(y_2,z)]^2$$
$$= \sum_{z \epsilon V(\Gamma_2)} \sum_{y_1, y_2 \epsilon V(\Gamma_1)} [(d_{R_k(\Gamma_1)}(y_1)+d_{\Gamma_2}(z))+(d_{R_k(\Gamma_1)}(y_2)+d_{\Gamma_2}(z))]^2$$
$$= \sum_{z \epsilon V(\Gamma_2)} \sum_{y_1, y_2 \epsilon V(\Gamma_1)} [d_{R_k(\Gamma_1)}(y_1)+2d_{\Gamma_2}(z)+d_{R_k(\Gamma_1)}(y_2)]^2$$
$$= \sum_{z \epsilon V(\Gamma_2)} \sum_{y_1, y_2 \epsilon V(\Gamma_1)} [(d_{R_k(\Gamma_1)}(y_1))^2+4(d_{\Gamma_2}(z))^2+(d_{R_k(\Gamma_1)}(y_2))^2+4d_{R_k(\Gamma_1)}(y_1)$$
$$d_{\Gamma_2}(z)+2d_{R_k(\Gamma_1)}(y_1)d_{R_k(\Gamma_1)}(y_2)+4d_{\Gamma_2}(z)d_{R_k(\Gamma_1)}(y_2)]$$
$$\sum_{z \epsilon V(\Gamma_2)} \sum_{y_1, y_2 \epsilon V(\Gamma_1)} [4(d_{(\Gamma_1)}(y_1))^2+4(d_{\Gamma_2}(z))^2+4(d_{(\Gamma_1)}(y_2))^2+8d_{(\Gamma_1)}(y_1)d_{\Gamma_2}(z)$$

$+8d_{(\Gamma_1)}(y_1)d_{(\Gamma_1)}(y_2)+8d_{\Gamma_2}(z)d_{(\Gamma_1)}(y_2)]$
$= 4 \mid E(\Gamma_1) \mid M_1(\Gamma_2))+4 \mid V(\Gamma_2) \mid HM(\Gamma_1)+16 \mid E(\Gamma_2) \mid M_1(\Gamma_1),$

$$\sum^{''} 2 = \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(R_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(R_k(\Gamma_1))-V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(R_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(R_k(\Gamma_1))-V(\Gamma_1)}} [(d_{R_k(\Gamma_1)}(y_1)+d_{\Gamma_2}(z))+d_{R_k(\Gamma_1)}(y_2)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(R_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(R_k(\Gamma_1))-V(\Gamma_1)}} [2d_{\Gamma_1}(y_1)+d_{\Gamma_2}(z)+2]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(R_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(R_k(\Gamma_1))-V(\Gamma_1)}} [4d_{\Gamma_1}(y_1)^2+d_{\Gamma_2}(z)^2+4+4d_{\Gamma_1}(y_1)d_{\Gamma_2}(z)+4d_{\Gamma_2}(z)+8d_{\Gamma_1}(y_1)]$$

$= 4 \mid V(\Gamma_2) \mid F(\Gamma_1)+2 \mid E(\Gamma_1) \mid M_1(\Gamma_2)+8 \mid E(\Gamma_2) \mid M_1(\Gamma_1)+8 \mid V(\Gamma_2) \mid\mid E(\Gamma_1) \mid$
$+8 \mid V(\Gamma_2) \mid M_1(\Gamma_1)+16 \mid E(\Gamma_1) \mid\mid E(\Gamma_2) \mid,$

and

$$\sum^{'''} 2 = \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(R_k(\Gamma_1)) \\ y_1,y_2\epsilon V(R_k(\Gamma_1))-V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2$$

$$= (k-1) \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(R_k(\Gamma_1)) \\ y_1,y_2\epsilon V(R_k(\Gamma_1))-V(\Gamma_1)}} [d_{R_k(\Gamma_1)}(y_1)+d_{R_k(\Gamma_1)}(y_2)]^2$$

$= 16(k-1) \mid V(\Gamma_2) \mid\mid E(\Gamma_1) \mid .$

Hence

$HM(\Gamma_1+_{R_k}\Gamma_2) = 8 \mid V(\Gamma_2) \mid [F(\Gamma_1)+M_1(\Gamma_1)+M_2(\Gamma_1)]+2 \mid E(\Gamma_1) \mid$
$[11M_1(\Gamma_2)+4 \mid V(\Gamma_2) \mid]+8 \mid E(\Gamma_2) \mid [5M_1(\Gamma_1)+2 \mid E(\Gamma_1) \mid]+ \mid V(\Gamma_1) \mid HM(\Gamma_2)$
$+16(k-1) \mid V(\Gamma_2) \mid\mid E(\Gamma_1) \mid .$

Theorem 3.3.
Let Γ_1 and Γ_2 be two molecular graphs having $\mid V(\Gamma_1) \mid, \mid V(\Gamma_2) \mid \geq 4.$ For $k \geq 1$, we have

$HM(\Gamma_1+_{Q_k}\Gamma_2) =\mid V(\Gamma_2) \mid [k\{HM(L(\Gamma_1))+8M_1(L(\Gamma_1)+8M_1(\Gamma_1)-16 \mid E(\Gamma_1) \mid\}+5F(\Gamma_1)$
$+8M_2(\Gamma_1)]+10 \mid E(\Gamma_1) \mid M_1(\Gamma_2)+16 \mid E(\Gamma_2) \mid M_1(\Gamma_1)+ \mid V(\Gamma_1) \mid HM(\Gamma_2)$
$+4(k-1) \mid V(\Gamma_2) \mid HM(\Gamma_1).$

Proof.
Let $d(y,z) = d_{\Gamma_1+_{S_k}\Gamma_2}(y,z)$ be the degree of a vertex set (y,z) in the graph $\Gamma_1 +_{S_k} \Gamma_2$.

$$HM(\Gamma_1+_{Q_k}\Gamma_2) = \sum_{(y_1,z_1)(y_2,z_2)\epsilon E(\Gamma_1+_{Q_k}\Gamma_2)} [d(y_1,z_1)+d(y_2,z_2)]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [d(y,z_1)+d(y,z_2)]^2 + \sum_{z\epsilon V(\Gamma_2)} \sum_{y_1y_2\epsilon E(Q_k(\Gamma_2))} [d(y_1,z)+d(y_2,z)]^2$$

$$= \sum 1 + \sum 2.$$

Now

$$\sum 1 = \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [d(y,z_1)+d(y,z_2)]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [(d_{Q_k(\Gamma_1)}(y)+d_{\Gamma_2}(z_1))+(d_{Q_k(\Gamma_1)}(y)+d_{\Gamma_2}(z_2))]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [2d_{Q_k(\Gamma_1)}(y)+d_{\Gamma_2}(z_1)+d_{\Gamma_2}(z_2)]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [4(d_{Q_k(\Gamma_1)}(y))^2+(d_{\Gamma_2}(z_1))^2+(d_{\Gamma_2}(z_2))^2+4d_{Q_k(\Gamma_1)}(y)d_{\Gamma_2}(z_1)$$

$$+4d_{Q_k(\Gamma_1)}(y)d_{\Gamma_2}(z_2)+2d_{\Gamma_2}(z_1)d_{\Gamma_2}(z_2)]$$

$$= 4 \mid E(\Gamma_2) \mid M_1(\Gamma_1) + \mid V(\Gamma_1) \mid HM(\Gamma_2) + \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} 4d_{\Gamma_1}(y)[d_{\Gamma_2}(z_1)+d_{\Gamma_2}(z_2)]$$

$$= 4 \mid E(\Gamma_2) \mid M_1(\Gamma_1) + \mid V(\Gamma_1) \mid HM(\Gamma_2) + 8 \mid E(\Gamma_1) \mid M_1(\Gamma_2),$$

and

$$\sum 2 = \sum_{z\epsilon V(\Gamma_2)} \sum_{y_1y_2\epsilon E(Q_k(\Gamma_1))} [d(y_1,z)+d(y_2,z)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(Q_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(Q_k(\Gamma_1))-V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2$$

$$+ \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(Q_k(\Gamma_1)) \\ y_1y_2\epsilon V(Q_k(\Gamma_1))-V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2$$

$$= \sum{}' 2 + \sum{}'' 2.$$

Now we talk,

$$\sum{}' 2 = \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(Q_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(Q_k(\Gamma_1))-V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(Q_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(Q_k(\Gamma_1))-V(\Gamma_1)}} [d_{Q_k(\Gamma_1)}(y_1)+d_{\Gamma_2}(z)+d_{Q_k(\Gamma_1)}(y_2)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(Q_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(Q_k(\Gamma_1))-V(\Gamma_1)}} [d_{Q_k(\Gamma_1)}(y_1)^2+d_{\Gamma_2}(z)^2+d_{Q_k(\Gamma_1)}(y_2)^2+2d_{Q_k(\Gamma_1)}(y_1)d_{\Gamma_2}(z)$$

$$+2d_{\Gamma_2}(z)d_{Q_k(\Gamma_1)}(y_2)+2d_{Q_k(\Gamma_1)}(y_1)d_{Q_k(\Gamma_1)}(y_2)]$$

$=\mid V(\Gamma_2)\mid F(\Gamma_1)+2\mid E(\Gamma_1)\mid M_1(\Gamma_2)+4\mid E(\Gamma_2)\mid M_1(\Gamma_1)+2\mid V(\Gamma_2)\mid F(\Gamma_1)$

$+4\mid V(\Gamma_2)\mid M_2(\Gamma_1)+2\mid V(\Gamma_2)\mid (F(\Gamma_1)+2M_2(\Gamma_1))+8\mid E(\Gamma_2)\mid M_1(\Gamma_1)$

so,

$$\sum{}' 2=\mid V(\Gamma_2)\mid (5F(\Gamma_1)+8M_2(\Gamma_1))+2\mid E(\Gamma_1)\mid M_1(\Gamma_2)+12\mid E(\Gamma_2)\mid M_1(\Gamma_1),$$

and

$$\sum{}'' 2=\sum_{z\epsilon V(\Gamma_2)}\sum_{\substack{y_1y_2\epsilon E(Q_k(\Gamma_1))\\ y_1,y_2\epsilon V(Q_k(\Gamma_1))-V(\Gamma_1)}}[d(y_1,z)+d(y_2,z)]^2,$$

Now we divide $\sum{}'' 2$ into $\sum 3$ and $\sum 4$ for the nodes, y_1 and y_2, where $y_1y_2\epsilon V(Q_k(\Gamma_1))-V(\Gamma_1)$. So $\sum{}'' 2=\sum 3+\sum 4$, where $\sum 3$ include the edges of $Q_k(\Gamma_1)$ having the similar edges of Γ_1 and $\sum 4$ of $Q_k(\Gamma_1)$ in two dissimilar adjacent edges of Γ_1.

$$\sum 3=\sum_{z\epsilon V(\Gamma_2)}\sum_{\substack{y_1y_2\epsilon E(Q_k(\Gamma_1))\\ y_1,y_2\epsilon V(Q_k(\Gamma_1))-V(\Gamma_1)}}[d_{Q_k(\Gamma_1)}(y_1)+d_{Q_k(\Gamma_1)}(y_2)]^2$$

$$=4(k-1)\sum_{z\epsilon V(\Gamma_2)}\sum_{uv\epsilon E(\Gamma_1)}[d_{\Gamma_1}(u)+d_{\Gamma_1}(v)]^2=4(k-1)\mid V(\Gamma_2)\mid HM(\Gamma_1),$$

and

$$\sum 4=\sum_{z\epsilon V(\Gamma_2)}\sum_{\substack{y_1y_2\epsilon E(Q_k(\Gamma_1))\\ y_1y_2\epsilon V(Q_k(\Gamma_1))-V(\Gamma_1)}}[d_{Q_k(\Gamma_1)}(y_1)+d_{Q_k(\Gamma_1)}(y_2)]^2$$

$$=(k)\sum_{z\epsilon V(\Gamma_2)}\sum_{\substack{uv\epsilon E(\Gamma_1)\\ vw\epsilon E(\Gamma_2)}}[d_{\Gamma_1}(u)+d_{\Gamma_1}(v)+d_{\Gamma_1}(v)+d_{\Gamma_1}(w)]^2$$

W and X are the nodes of $L(\Gamma_1)$, so we have

$$=(k)\sum_{WX\epsilon E(L(\Gamma_1))}[d_{L(\Gamma_1)}(W)+d_{L(\Gamma_1)}(X)+4]^2$$

$$=(k)\sum_{WX\epsilon E(L(\Gamma_1))}[d_{L(\Gamma_1)}(W)^2+d_{L(\Gamma_1)}(X)^2$$

$$+16+2d_{L(\Gamma_1)}(W)d_{L(\Gamma_1)}(X)+8d_{L(\Gamma_1)}(W)+8d_{L(\Gamma_1)}(X)]$$

$$=k[HM(L(\Gamma_1))+8M_1(L(\Gamma_1))+16(\frac{1}{2}M_1(\Gamma_1)-E(\Gamma_1))],$$

$$\sum 4=k[\mid V(\Gamma_2)\mid (HM(L(\Gamma_1))+8M_1(L(\Gamma_1))+8M_1(\Gamma_1)-16E(\Gamma_1))].$$

Consequently, we have

$HM(\Gamma_1+_{Q_k}\Gamma_2)=\mid V(\Gamma_2)\mid [k\{HM(L(\Gamma_1))+8M_1(L(\Gamma_1)+8M_1(\Gamma_1)-16\mid E(\Gamma_1)\mid\}+5F(\Gamma_1)$

$+8M_2(\Gamma_1)]+10\mid E(\Gamma_1)\mid M_1(\Gamma_2)+16\mid E(\Gamma_2)\mid M_1(\Gamma_1)+\mid V(\Gamma_1)\mid HM(\Gamma_2)$

$+4(k-1)\mid V(\Gamma_2)\mid HM(\Gamma_1).$

Theorem 3.4.
Let Γ_1 and Γ_2 be two molecular graphs having $\mid V(\Gamma_1)\mid, \mid V(\Gamma_2)\mid \geq 4$. For $k \geq 1$,
$HM(\Gamma_1 +_{T_k} \Gamma_2) = \mid V(\Gamma_2)\mid [(k)HM(L(\Gamma_1)) + 10HM(\Gamma_1) + 4F(\Gamma_1) + 8(k)M_1(L(\Gamma_1) + 8(k)M_1(\Gamma_1)$
$-16(k)\mid E(\Gamma_1)\mid] + 22\mid E(\Gamma_1)\mid M_1(\Gamma_2) + 48\mid E(\Gamma_2)\mid M_1(\Gamma_1) + \mid V(\Gamma_1)\mid HM(\Gamma_2)$
$+4(k-1)\mid V(\Gamma_2)\mid HM(\Gamma_1)$.

Proof.
Let $d(y,z) = d_{(\Gamma_1 +_{T_k} \Gamma_2)}(y,z)$ be the degree of a node (y,z) in the molecular graph $\Gamma_1 +_{T_k} \Gamma_2$.

$$HM(\Gamma_1 +_{T_k} \Gamma_2) = \sum_{(y_1,z_1)(y_2,z_2)\epsilon E(\Gamma_1 +_{T_k} \Gamma_2)} [d(y_1,z_1) + d(y_2,z_2)]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [d(y,z_1) + d(y,z_2)]^2 + \sum_{z\epsilon V(\Gamma_2)} \sum_{y_1y_2\epsilon E(T_k(\Gamma_1))} [d(y_1,z) + d(y_2,z)]^2$$

$$= \sum 1 + \sum 2.$$

Consider,

$$\sum 1 = \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [d(y,z_1) + d(y,z_2)]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [(d_{T_k(\Gamma_1)}(y) + d_{\Gamma_2}(z_1)) + (d_{T_k(\Gamma_1)}(y) + d_{\Gamma_2}(z_2))]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [2d_{T_k(\Gamma_1)}(y) + d_{\Gamma_2}(z_1) + d_{\Gamma_2}(z_2)]^2$$

$$= \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [4(d_{T_k(\Gamma_1)}(y))^2 + (d_{\Gamma_2}(z_1))^2 + (d_{\Gamma_2}(z_2))^2 + 4d_{T_k(\Gamma_1)}(y)d_{\Gamma_2}(z_1)$$
$$+4d_{T_k(\Gamma_1)}(y)d_{\Gamma_2}(z_2) + 2d_{\Gamma_2}(z_1)d_{\Gamma_2}(z_2)]$$

$$= \sum_{y\epsilon V(\Gamma_1)} \sum_{z_1z_2\epsilon E(\Gamma_2)} [16(d_{\Gamma_1}(y))^2 + (d_{\Gamma_2}(z_1))^2 + (d_{\Gamma_2}(z_2))^2 + 8d_{\Gamma_1}(y)d_{\Gamma_2}(z_1)$$
$$+8d_{\Gamma_1}(y)d_{\Gamma_2}(z_2) + 2d_{\Gamma_2}(z_1)d_{\Gamma_2}(z_2)]$$

$$= 16\mid E(\Gamma_2)\mid M_1(\Gamma_1) + \mid V(\Gamma_1)\mid F(\Gamma_2) + 16\mid E(\Gamma_1)\mid M_1(\Gamma_2) + 2\mid V(\Gamma_1)\mid M_2(\Gamma_2)$$

$$= 16\mid E(\Gamma_2)\mid M_1(\Gamma_1) + \mid V(\Gamma_1)\mid HM(\Gamma_2) + 16\mid E(\Gamma_1)\mid M_1(\Gamma_2),$$

and

$$\sum 2 = \sum_{z\epsilon V(\Gamma_2)} \sum_{y_1y_2\epsilon E(T_k(\Gamma_1))} [d(y_1,z) + d(y_2,z)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(T_k(\Gamma_1)) \\ y_1,y_2\epsilon V(\Gamma_1)}} [d(y_1,z) + d(y_2,z)]^2$$

$$+ \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(T_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(T_k(\Gamma_1)) - V(\Gamma_1)}} [d(y_1,z) + d(y_2,z)]^2$$

$$+ \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(T_k(\Gamma_1)) \\ y_1,y_2\epsilon V(T_k(\Gamma_1)) - V(\Gamma_1)}} [d(y_1,z) + d(y_2,z)]^2$$

$= \sum{}^{'} 2 + \sum{}^{''} 2 + \sum{}^{'''} 2.$

For $y_1 y_2 \epsilon V(\Gamma_1)$, we have $y_1 y_2 \epsilon E(T_k(\Gamma_1))$ if and only if $y_1 y_2 \epsilon E(\Gamma_1)$; for $y_1 \epsilon V(\Gamma_1)$, we obtain $d_{T_k(\Gamma_1)}(y_1) = 2d_{\Gamma_1}(y_1)$ and for $y_2 \epsilon V(T_k(\Gamma_1)) - V(\Gamma_1)$, we have $d_{T_k(\Gamma_1)}(y_2) = 2$.

Now, consider

$$\sum{}^{'} 2 = \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(T_k(\Gamma_1)) \\ y_1y_2\epsilon V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2 = \sum_{z\epsilon V(\Gamma_2)} \sum_{y_1,y_2\epsilon V(\Gamma_1)} [d(y_1,z)+d(y_2,z)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{y_1,y_2\epsilon V(\Gamma_1)} [(d_{T_k(\Gamma_1)}(y_1)+d_{\Gamma_2}(z))+(d_{T_k(\Gamma_1)}(y_2)+d_{\Gamma_2}(z))]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{y_1,y_2\epsilon V(\Gamma_1)} [d_{T_k(\Gamma_1)}(y_1)+2d_{\Gamma_2}(z)+d_{T_k(\Gamma_1)}(y_2)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{y_1,y_2\epsilon V(\Gamma_1)} [(d_{T_k(\Gamma_1)}(y_1))^2+4(d_{\Gamma_2}(z))^2+(d_{T_k(\Gamma_1)}(y_2))^2+4d_{T_k(\Gamma_1)}(y_1)$$
$$d_{\Gamma_2}(z)+2d_{T_k(\Gamma_1)}(y_1)d_{T_k(\Gamma_1)}(y_2)+4d_{\Gamma_2}(z)d_{T_k(\Gamma_1)}(y_2)]$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{y_1,y_2\epsilon V(\Gamma_1)} [4(d_{(\Gamma_1)}(y_1))^2+4(d_{\Gamma_2}(z))^2+4(d_{(\Gamma_1)}(y_2))^2+8d_{(\Gamma_1)}(y_1)d_{\Gamma_2}(z)$$
$$+8d_{(\Gamma_1)}(y_1)d_{(\Gamma_1)}(y_2)+8d_{\Gamma_2}(z)d_{(\Gamma_1)}(y_2)]$$

$$= 4 \mid E(\Gamma_1) \mid M_1(\Gamma_2)+4 \mid V(\Gamma_2) \mid HM(\Gamma_1)+16 \mid E(\Gamma_2) \mid M_1(\Gamma_1),$$

$$\sum{}^{''} 2 = \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(T_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(T_k(\Gamma_1))-V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(T_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(T_k(\Gamma_1))-V(\Gamma_1)}} [(d_{T_k(\Gamma_1)}(y_1)+d_{\Gamma_2}(z))+d_{T_k(\Gamma_1)}(y_2)]^2$$

$$= \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(T_k(\Gamma_1)) \\ y_1\epsilon V(\Gamma_1) \\ y_2\epsilon V(T_k(\Gamma_1))-V(\Gamma_1)}} [4d_{\Gamma_1}(y_1)^2+d_{\Gamma_2}(z)^2+d_{T(\Gamma_1)}(y_2)^2+4d_{\Gamma_1}(y_1)d_{\Gamma_2}(z)$$
$$+4d_{\Gamma_1}(y_1)d_{\Gamma_1}(y_2)+2d_{\Gamma_2}(z)d_{\Gamma_1}(y_2)]$$

$$= 4 \mid V(\Gamma_2) \mid F(\Gamma_1)+2 \mid E(\Gamma_1) \mid M_1(\Gamma_2)+2 \mid V(\Gamma_2) \mid HM(\Gamma_1)+8 \mid E(\Gamma_2) \mid M_1(\Gamma_1)$$
$$+4 \mid V(\Gamma_2) \mid HM(\Gamma_1)+2.2 \mid E(\Gamma_2) \mid (2M_1(\Gamma_1))$$
$$= 4 \mid V(\Gamma_2) \mid F(\Gamma_1)+2 \mid E(\Gamma_1) \mid M_1(\Gamma_2)+6 \mid V(\Gamma_2) \mid HM(\Gamma_1)+16 \mid E(\Gamma_2) \mid M_1(\Gamma_1),$$

and

$$\sum{}^{'''} 3 = \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(T_k(\Gamma_1)) \\ y_1y_2\epsilon V(T_k(\Gamma_1))-V(\Gamma_1)}} [d(y_1,z)+d(y_2,z)]^2$$

Now we divide $\sum{}^{''} 2$ into $\sum 3$ and $\sum 4$ for the nodes, y_1 and y_2, where $y_1y_2\epsilon V(T_k(\Gamma_1)) - V(\Gamma_1)$. So $\sum{}^{''} 2 = \sum 3 + \sum 4$, where $\sum 3$ include the edges of $T_k(\Gamma_1)$ having the similar edges of Γ_1 and $\sum 4$ of $T_k(\Gamma_1)$ in two dissimilar adjacent edges of Γ_1.

$$\sum 3 = \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(T_k(\Gamma_1)) \\ y_1y_2\epsilon V(T_k(\Gamma_1))-V(\Gamma_1)}} [d_{T_k(\Gamma_1)}(y_1)+d_{T_k(\Gamma_1)}(y_2)]^2$$

$$= 4(k-1) \sum_{z\epsilon V(\Gamma_2)} \sum_{uv\epsilon E(\Gamma_1)} [d_{\Gamma_1}(u)+d_{\Gamma_1}(v)]^2 = 4(k-1) \mid V(\Gamma_2) \mid HM(\Gamma_1),$$

$$\sum 4 = \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{y_1y_2\epsilon E(T_k(\Gamma_1)) \\ y_1y_2\epsilon V(T_k(\Gamma_1))-V(\Gamma_1)}} [d_{T_k(\Gamma_1)}(y_1)+d_{T_k(\Gamma_1)}(y_2)]^2$$

$$= (k) \sum_{z\epsilon V(\Gamma_2)} \sum_{\substack{uv\epsilon E(\Gamma_1) \\ vw\epsilon E(\Gamma_2)}} [d_{\Gamma_1}(u)+d_{\Gamma_1}(v)+d_{\Gamma_1}(v)+d_{\Gamma_1}(w)]^2$$

W and X are the nodes of $L(\Gamma_1)$, so we have

$$= (k) \sum_{WX\epsilon E(L(\Gamma_1))} [d_{L(\Gamma_1)}(W)+d_{L(\Gamma_1)}(X)+4]^2$$

$$= (k) \sum_{WX\epsilon E(L(\Gamma_1))} [d_{L(\Gamma_1)}(W)^2 + d_{L(\Gamma_1)}(X)^2$$

$$+16+2d_{L(\Gamma_1)}(W)d_{L(\Gamma_1)}(X)+8d_{L(\Gamma_1)}(W)+8d_{L(\Gamma_1)}(X)]$$

$$= k[HM(L(\Gamma_1))+8M_1(L(\Gamma_1))+16(\frac{1}{2}M_1-E(\Gamma_1))],$$

so,

$$\sum 4 = k[\mid V(\Gamma_2) \mid (HM(L(\Gamma_1))+8M_1(L(\Gamma_1))+8M_1(\Gamma_1)-16E(\Gamma_1))],$$

Consequently, we have

$$HM(\Gamma_1 +_{T_k} \Gamma_2) =\mid V(\Gamma_2) \mid [(k)HM(L(\Gamma_1))+10HM(\Gamma_1)+4F(\Gamma_1)+8(k)M_1(L(\Gamma_1)+8(k)M_1(\Gamma_1)$$
$$-16(k) \mid E(\Gamma_1) \mid]+22 \mid E(\Gamma_1) \mid M_1(\Gamma_2)+48 \mid E(\Gamma_2) \mid M_1(\Gamma_1)+ \mid V(\Gamma_1) \mid HM(\Gamma_2)$$
$$+4(k-1) \mid V(\Gamma_2) \mid HM(\Gamma_1).$$

4. Conclusion

For $a_1, a_2 \geq 4$ and $k = 4$, we consider that $\Gamma_1 = P_{a_1}$ and $\Gamma_2 = P_{a_2}$ are specific examples of alkane known as paths having orders a_1 and a_2 respectively. Then the following results are the direct outcomes of the above four theorems.

- $HM(P_{a_1+S_4}P_{a_2}) = 184a_1a_2 - 198a_1 - 138a_2 + 124,$
- $HM(P_{a_1+R_4}P_{a_2}) = 464a_1a_2 - 624a_1 - 338a_2 + 388,$
- $HM(P_{a_1+Q_4}P_{a_2}) = 640a_1a_2 - 1262a_1 - 154a_2 + 156,$
- $HM(P_{a_1+T_4}P_{a_2}) = 936a_1a_2 - 1724a_1 - 354a_2 + 420,$

In this paper, we computed the HM(Γ) of the subdivision related generalized F-sum graphs. However the problem is still open for other topological indices on the generalized F-sum graphs.

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