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On the Oscillatory Nature of Some Generalized Emden-Fowler Equation

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Abstract.: In this paper, we study the oscillatory character of a generalized differential equation of order $\alpha + \alpha$ with $\alpha \in (0, 1]$. Generalized criteria of Kamenev type are obtained, which are extensions of several known in the literature both integer and fractional.

AMS (MOS) Subject Classification Codes: 34L30, 34C15
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1. INTRODUCTION

In this paper we will consider the following generalized differential equation

$$N_F^{\alpha}\left[p(t)g(N_F^{\alpha}y)\right] + q(t)f(y(t)) = R(t, y), \tag{1. 1}$$

where the differential operator N_F^{α}, $0 < \alpha \leq 1$, will be defined later and the functions considered satisfy $p \in C([t_0, \infty), (0, \infty))$, $q \in C([t_0, \infty), \mathbb{R}$, $f \in C(\mathbb{R}, \mathbb{R})$ such that $yf(y) > 0$ for $y \neq 0$, $g \in C(\mathbb{R}, \mathbb{R})$ with $(N_F^{\alpha}y)g(N_F^{\alpha}y) > 0$ for $N_F^{\alpha}y \neq 0$ and $R(t, y(t)) \in C([t_0, \infty)x\mathbb{R}, \mathbb{R})$ satisfying $\frac{R(t,y(t))}{f(y(t))} \leq v(t)$, $y \neq 0$ with $s \in C([t_0, \infty), (0, \infty))$.

We will say that a solution of (1. 1), not zero, is called oscillatory if it cuts the axis $x \equiv 0$ infinitely, that is, if it has infinite zeros, for t large enough. If all its solutions of equation (1. 1) are oscillatory, we will say that this equation is oscillatory.

The nonlinear fractional equations plays a crucial role in a wide variety of applied sciences problems, in all these areas, the study of oscillating solutions, that is, the existence of a certain “work regime of periodic, quasi-periodic or of variable sign but bounded, is of vital importance (the following works are illustrative [53, 42, 43, 5, 6]). It is noteworthy that, in general, they are centered on equations with the classical “global” fractional derivatives (see [44] and the references cited there), and qualitative research is almost non-existent using local fractional derivatives (see [28], [29], [30] for attempts in this direction, although the techniques used will be different from those used in this work).

The continuous extensions of various notions of derivatives in recent years, of non-integer and / or variable order, are extensions of the classical differential equations, and have different theoretical and applied fields. (cf. [4], [37], [36] and [19]). More than five decades ago, intensified in recent years, that different researchers have been studying various qualitative aspects of fractional equation solutions (local and global) (see [39], [40], [10], [25], [53], [11], [16], [55], [38], [46], [2], [27], [54], [57], [24] and [7] and references cited therein). However, we should note that, in general, little attention is paid to the study of the oscillatory nature of the solutions, being one of the central properties in mathematical modeling.

Although certain differential operators that are called local fractional derivatives have appeared since the 1960s, it is not until in 2014 when a local derivative (conformable) was formalized and in 2018 (non-conformable) with very good properties, what we highlight is that all these operators can be considered as particular cases (including the ordinary classic) of the following Definition of Generalized Derivative, as discussed below.

In [33] a generalized fractional derivative was defined in the following way (see also [8]).

Definition 1.1. *Let $f : [0,+\infty) \to \mathbb{R}$ be a continuous function. The generalized N-derivative of f of order α is defined by*

$$N_F^\alpha f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon F(t,\alpha)) - f(t)}{\varepsilon}, \tag{1. 2}$$

for all $t>0$, $\alpha\in(0,1)$ where $F(\alpha,t)$ is some function. Here we will use some cases of F defined in function of $E_{a,b}(.)$ the classic Definition of Mittag-Leffler function with $Re(a), Re(b) > 0$. Also we consider $E_{a,b}(t^{-\alpha})_k$ is the k-nth term of $E_{a,b}(.)$.

If f is $\alpha-$differentiable in some $(0,\alpha)$, and $\lim_{t\to 0^+} N_F^\alpha f(t)$ exists, then define $N_F^\alpha f(0) = \lim_{t\to 0^+} N_F^\alpha f(t)$, note that if f is differentiable, then $N_F^\alpha f(t) = F(t,\alpha)f'(t)$ where $f'(t)$ is the ordinary derivative.

The original function $E_{\alpha,1}(z) = E_\alpha(z)$ was defined and studied by Mittag-Leffler in the year 1903, that is, a uniparameter function (see [14, 15]). It is a direct generalization of the exponential function. Wiman proposed and studied a generalization of the role of Mittag-Leffler, who we'll call it the Mittag-Leffler function with two parameters $E_{\alpha,\beta}(z)$ (see [3]), Agarwal in 1953 and Humbert and Agarwal in 1953, also made contributions to the final formalization of this function. Since then the applications of the Mittag-Leffler Function in many areas of science and engineering have multiplied, which has caused the study to have attracted the attention of many researchers, in [17] the authors present, in a unified way, a detailed review of the Mittag-Leffler function, of generalized Mittag-Leffler functions, of Mittag-Leffler type functions, their interesting and useful properties and most of the publications dedicated to this function.

We consider the following examples:

I) $F(t,\alpha)\equiv 1$, in this case we have the ordinary derivative.

II) $F(t,\alpha) = E_{1,1}(t^{-\alpha})$. In this case we obtain, from Definition 1.1, the derivative $N_1^{\alpha} f(t)$ defined in [45] (see also [31]).

III) $F(t,\alpha) = E_{1,1}((1-\alpha)t) = e^{(\alpha-1)t}$, this kernel satisfies that $F(t,\alpha) \to 1$ as $\alpha \to 1$.

IV)$F(t,\alpha) = E_{1,1}(t^{1-\alpha})_1 = t^{1-\alpha}$ with this kernel we have $F(t,\alpha) \to 0$ as $\alpha \to 1$ (see [51]).

V)$F(t,\alpha) = E_{1,1}(t^{-\alpha})_1 = t^{\alpha}$ with this kernel we have $F(t,\alpha) \to t$ as $\alpha \to 1$ (see [34]).

VI) $F(t,\alpha) = E_{1,1}(t^{-\alpha})_1 = t^{-\alpha}$ with this kernel we have $F(t,\alpha) \to t^{-1}$ as $\alpha \to 1$ This is the derivative N_3^{α} studied in [13].

Remark 1.2. *After these clarifications, it is clear that if in* (1. 1) *we do* $F(t,\alpha) \equiv 1$, $p \equiv 1$, $g(N^{\alpha}y) = y'$, $q(t) = b(t) > 0$, $f(y) = |y|^{\beta}$ *with* $\beta \neq 1$, $\beta \geq 0$ *and* $R \equiv 0$ *we obtain the classic Emden-Fowler equation*

$$y'' + b(t)|y|^{\beta} sgny = 0, \tag{1. 3}$$

called superlinear if $\beta > 1$ *and sublinear if* $\beta < 1$.

This equation was first studied by Emden and Fowller (see [49] and [47, 48], respectivelly), you can consult for additional details about its qualitative properties in [22].

The Emden-Fowler equation (or Lane-Emden equation) appears in several problems in mathematics and physics, and it has been investigated from various points of view. and many different methods for its solution have been proposed; is of great importance in Newtonian astrophysics and by introducing a set of new variables, this equation can be written as an autonomous system of two ordinary differential equations that can be analyzed using linear and nonlinear stability analysis. The Emden-Fowler equation also can be related to the diffusion and reaction problem in a slab, and it can be considered a particular case of the semilinear elliptic equation $\Delta u + un = 0$ with $u(0) = 1$ and $u'(0) = 0$ on the positive real line, where the constant n is called the polytropic index.

For these and other reasons, this equation has been the subject of research in recent years (see [50, 9, 41, 35, 52, 20, 56, 1]), as additional data Wong, in his review of 1975 ([22]), contains over 100 references, although it does not cover all reported papers.

It is noted that research on the oscillation of solutions is very scarce, so this work comes to fill this gap, not only in the classic case, but also in the generalized one. Hence the importance of the results obtained, new in both the classical and generalized cases.

Now, we give the Definition of a general fractional integral (see [8] and [32]):

Definition 1.3. *Let* $\alpha \in (0,1]$ *and* $0 \leq u \leq v$. *We say that a function* $h : [u,v] \to \mathbb{R}$ *is* α*-fractional integrable on* $[u,v]$, *if the integral*

$$_{N}J_u^{\alpha} h(x) =_{N_F} J_u^{\alpha} h(x) = \int_u^x \frac{h(t)}{F(t,\alpha)} dt, \tag{1. 4}$$

exists and is finite.

Remark 1.4. *Taking into account the examples of kernels presented above, it is clear that we will have different integral operators. To name just one case, if* $F(t,\alpha) \equiv 1$ *we will have the classic Riemann integral.*

The following theorem is similar to a known result of classical calculus (see [32] and [8]).

Theorem 1.5. *Let f be N-differentiable function in (t_0, ∞) with $\alpha \in (0, 1]$. Then for all $t > t_0$ we have*

a) *If f is differentiable ${}_{N_F}J^{\alpha}_{t_0}\left(N^{\alpha}_F f(t)\right) = f(t) - f(t_0)$.*

b) $N^{\alpha}_F\left({}_{N_F}J^{\alpha}_{t_0} f(t)\right) = f(t)$.

As in the classic case, we have the well-known Integration by Parts property.

Theorem 1.6. *(Integration by parts) Let u and v be N-differentiable function in (t_0, ∞) with $\alpha \in (0, 1]$. Then for all $t > t_0$ we have*

$$ {}_{N_F}J^{\alpha}_{t_0}\left((uN^{\alpha}_F v)(t)\right) = [uv(t) - uv(t_0)] - {}_{N_F}J^{\alpha}_{t_0}\left((vN^{\alpha}_F u)(t)\right). \tag{1. 5} $$

In this paper we will study the equation (1. 1) using a Riccati Transformation and then formulate two general oscillation criteria of Kamenev type, when $q(t)$ is allowed to take negative values for sufficiently large values of t. For this we have divided the study into two parts, first a particular case of the Equation (1. 1) is studied and in the second section the general equation itself.

2. Main Results

For the linear version of equation (1. 3), i.e. with $\beta = 1$, Kamenev established a new oscillation criterion using an integral average method (cf. [21]), which generalizes some previous results, specifically states that if

$$ \lim_{t\to\infty} sup \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} b(s) ds = +\infty, \tag{2. 6} $$

for $n > 2$, then

$$ y'' + b(t)y = 0, $$

is oscillatory. Based on that seminal result, we will focus our work.

2.1. A particular case.

Instead of the (1. 1) consider the following particular case:

$$ N^{\alpha}_F\left[p(t)N^{\alpha}_F y\right] + q(t)f(y(t)) = R(t, y), \tag{2. 7} $$

subject to the conditions stated above (with $g(N^{\alpha}_F y) = N^{\alpha}_F y$). For all of the above, it is natural to ask under what conditions on the functions involved in (2. 7) we can state a condition similar to the condition of Kamenev (2. 6). Thus we have the following result.

Theorem 2.2. *In addition to the conditions on the functions p, q, f and R, suppose that the following conditions are satisfied*

$$ \lim_{t\to\infty} inf \quad {}_{N}J^{\alpha}_{0} \quad (q)(t) = -\lambda > -\infty, \quad \lambda > 0, \tag{2. 8} $$

$$ \lim_{t\to\infty} sup \frac{1}{t^{n-1}} \quad {}_{N}J^{\alpha}_{t_0}((t-s)^{n-1}u - \frac{(n-1)^2}{4M} F(s,\alpha)^2 (t-s)^{n-3})(t) = \infty, \tag{2. 9} $$

then (2. 7) *is oscillatory, with* $u(t) = q(t) - v(t)$.

Proof. Assume the contrary, then (2. 7) has a nonoscillatory solution $y(t)$, which may be assumed $y(t) \neq 0$ for $t \geq t_0$. Define the function $w(t) = \frac{p(t) N_F^\alpha y(t)}{y(t)}$, then it follows from (2. 7)

$$N_F^\alpha w + M(t) w^2 + u(t) \leq 0, \quad t \geq t_0,$$

with $M(t) = \frac{N_F^\alpha f(y(t))}{p(t)}$. Hence, multiplying the above inequality by $(t-s)^{n-1}$ and integrate by parts (Theorem 2.5) to find

$$\begin{aligned} &(n-1) \quad {}_N J_{t_0}^\alpha ((t-s)^{n-2} F(s,\alpha) w)(t) + \\ + &\quad {}_N J_{t_0}^\alpha ((t-s)^{n-1} M w^2)(t) + {}_N J_{t_0}^\alpha ((t-s)^{n-1} u)(t) \leq (t-t_0)^{n-1} w(t_0). \end{aligned}$$

Completing the square, combining the first two terms of this inequality, we get

$$\begin{aligned} &{}_N J_{t_0}^\alpha \left[\sqrt{M} (t-s)^{\frac{n-1}{2}} w + \frac{1}{2} \frac{(n-1)}{\sqrt{M}} F(s,\alpha) (t-s)^{\frac{n-3}{2}} \right]^2 (t) + \\ + &\quad {}_N J_{t_0}^\alpha ((t-s)^{n-1} u - \frac{(n-1)^2}{4M} F(s,\alpha)^2 (t-s)^{n-3})(t) \leq (t-t_0)^{n-1} w(t_0). \end{aligned}$$

Note that the first integral of the left hand side of the previous inequality is nonnegative, this allows us to obtain

$${}_N J_{t_0}^\alpha ((t-s)^{n-1} u - \frac{(n-1)^2}{4M} F(s,\alpha)^2 (t-s)^{n-3})(t) \leq (t-t_0)^{n-1} w(t_0). \qquad (2.\ 10)$$

Dividing by $(t-t_0)^{n-1}$ and taking limit in both sides as $t \to \infty$, we have that the left hand side of (2. 10) tends to a finite limit, which produce the desired contradiction with the requirement (2. 9). □

Remark 2.3. *Let us indicate that the result obtained is consistent with those reported in the literature for the ordinary case, for example, if we use the kernel* $F(t,\alpha) \equiv 1$, $R \equiv 0$, *and* $f(y) = |y|^\beta sgny$ *our equation* (2. 7) *is reduced to Emden-Fowler differential equation* $(p(t)y')' + b(t)|y|^\beta sgny = 0$, *a general case of* (1. 3), *that is, Theorem 2.2 generalizes the results obtained in* [23] *for the equation* (1. 3).

Remark 2.4. *The result obtained before, can be written in a more general way, if instead of* $(t-t_0)^{n-1}$ *we use a general function* $H(t,t_0)$ *defined as follows.*
So, if we use a continuous function $H : D = \{(t,s) : t \geq s \geq t_0\} \to \mathbb{R}$ *satisfying* $H(t,t) = 0$ *for* $t \geq t_0$, $H(t,s) > 0$ *for* $t > s \geq t_0$, *having a continuous and nonpositive partial derivative on* $D_0 = \{(t,s) : t > s \geq t_0\}$ *with respect to variable s. Moreover, let* $h : D_0 \to \mathbb{R}$ *be a continuous function for which we have*

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)},$$

for all $(t,s) \in D_0$. *Thus we can state the following result, whose proof is very similar and we leave it to the reader.*

Theorem 2.5. *If we take $H(t,s)$ and $h(t,s)$ as before, then the equation* (2. 7) *is oscillatory if in addition to* (2. 8) *the condition, being $u(t) = q(t) - v(t)$*

$$\lim_{t\to\infty} sup \frac{1}{H(t,s)} \quad {}_N J^{\alpha}_{t_0} \left\{ H(t,s)u(s) - \frac{1}{4M}(h(t,s)F(s,\alpha))^2 \right\} = \infty, \tag{2. 11}$$

is satisfied.

Remark 2.6. *Another variant in the condition of Kamenev type* (2. 9) *is to use instead of $(t-t_0)^{n-1}$ or the function $H(t,t_0)$ above, the product of functions $(t-s)^a s^b$, with $a \in (1,\infty)$ and $b \in [0,1)$. The formulation of this result would be as follows.*

Theorem 2.7. *The equation* (2. 7) *is oscillatory if the conditions*

$$\lim_{t\to\infty} sup \frac{1}{t^s} \quad {}_N J^{\alpha}_{t_0} (t-s)^a s^b u(s)(t) = \infty, \tag{2. 12}$$

$$\lim_{t\to\infty} sup \frac{1}{t^s} \quad {}_N J^{\alpha}_{t_0} \left\{ [asF(s,\alpha) - b(t-s)F(s,\alpha)]^2 (t-s)^{a-2} s^{b-2} \right\}(t) < \infty, \tag{2. 13}$$

are satisfied with $u(t) = q(t) - v(t)$.

Proof. Proceeding as in the previous proof, after defining w in the same way, integrating, rearranging terms and dividing by t^a the desired contradiction is obtained. □

2.8. **A general case.** Consider the equation (1. 1) under the condition that $g = f$, that is, in this section we will study the equation

$$N^{\alpha}_F [p(t)f(N^{\alpha}_F y)] + q(t)f(y(t)) = R(t,y). \tag{2. 14}$$

Let us observe as before that if $p \equiv 1$, f and g are the identity functions and $F \equiv 1$, this equation is reduced to the linear equation of the second order.

Our main result is as follows.

Theorem 2.9. *If in addition to the considerations on the functions p, q, f, R, $1 \le \frac{N^{\alpha}_F y}{f(N^{\alpha}_F y)}$ for $N^{\alpha}_F y \neq 0$ and* (2. 8)*, suppose the following assumptions are fulfilled*

$$\lim_{t\to\infty} sup \frac{1}{t^{n-1}} \quad {}_N J^{\alpha}_{t_0} ((t-s)^{n-1} u - \frac{(n-1)^2}{4M} F(s,\alpha)^2 (t-s)^{n-3})(t) = \infty, \tag{2. 15}$$

then (2. 14) *is oscillatory, bieng $u(t) = q(t) - v(t)$.*

Proof. As before, we assume the contrary, i.e. the (2. 14) has a nonoscillatory solution $y(t)$, which without loss of generality, we can assume that $y(t) > 0$ for $t \ge t_0$. Define the function $w(t) = \frac{p(t)f(N^{\alpha}_F y(t))}{f(y(t))}$, this and (2. 14) imply

$$N^{\alpha}_F w + M(t) \left(\frac{N^{\alpha}_F y}{f(N^{\alpha}_F y)} \right) w^2 + u(t) \le 0, \quad t \ge t_0, \tag{2. 16}$$

with $M(t) = \frac{N^{\alpha}_F f(y)}{p(t)}$. Now we have to consider two cases:

1) $N^{\alpha}_F y \neq 0$. If we work as in the Theorem 2.2, we get easily

$$
\begin{aligned}
& {}_{N}J_{t_0}^{\alpha}\left[\sqrt{M}(t-s)^{\frac{n-1}{2}}w+\frac{1}{2}\frac{(n-1)}{\sqrt{M}}F(s,\alpha)(t-s)^{\frac{n-3}{2}}\right]^2(t)+\\
+ & \quad {}_{N}J_{t_0}^{\alpha}((t-s)^{n-1}u-\frac{(n-1)^2}{4M}F(s,\alpha)^2(t-s)^{n-3})(t)\leq\\
\leq & \quad (t-t_0)^{n-1}w(t_0).
\end{aligned}
$$

From here we reached the contradiction necessary to prove this case.

2) $N_F^{\alpha}y$ is oscillatory. Hence $f(N_F^{\alpha}y)$ it is also oscillatory. Then, there exists an infinite sequence t_n such that $t_n\to\infty,\quad n\to\infty$ and $f(N_F^{\alpha}y(t_n))=0$. So, from (2. 16) we have

$$
\begin{aligned}
& {}_{N}J_{t_0}^{\alpha}\left[\sqrt{M}(t-s)^{\frac{n-1}{2}}w+\frac{1}{2}\frac{(n-1)}{\sqrt{M}}F(s,\alpha)(t-s)^{\frac{n-3}{2}}\right]^2(t_n)+\\
+ & \quad {}_{N}J_{t_0}^{\alpha}((t-s)^{n-1}u-\frac{(n-1)^2}{4M}F(s,\alpha)^2(t-s)^{n-3})(t_n)\leq\\
\leq & \quad (t-t_0)^{n-1}w(t_0).
\end{aligned}
$$

We must point out that the singularity of the first two integrals in t_n has no impact on the oscillation of the solutions, since the oscillation is a qualitative property to infinity, so the result remains true, taking into account the boundary of the right member. Thus we complete the proof of the theorem. □

Remark 2.10. *As we indicated before, our results are not contradictory with the ordinary case. Thus, in the case that $F\equiv 1$, $f(z)=g(z)=|z|^{p-2}z$ with $p>1$ and $R\equiv 0$, we obtain Theorem 2 of* [18]. *It is clear that Corollaries 3 and 4 of this work are still valid. Obviously in this case, the proof is much simpler.*

Remark 2.11. *Without much difficulty, results equivalent to Theorems 2.5 and 2.7 of the previous section can be stated.*

Remark 2.12. *Our results complete those obtained in* [26] *for the equation $N_F^{\alpha}(p(t)N_F^{\alpha}y)+b(t)y=0$, a particular case of* (1. 1) *with $g(z)=f(z)=z$, $R\equiv 0$ and the kernel III) $F(t,\alpha)=t^{1-\alpha}$.*

Remark 2.13. *Finally, we must point out that in the case of considering different kernels, not only the cases II) -VI) presented at the beginning of the work, the results obtained have not been reported in the literature.*

3. Conclusion

In this work we obtained, through a Riccati transformation, generalized oscillation criteria that contain as particular cases several reported in the literature, both integer and fractional.

In [12] the authors study the existence of oscillatory solutions of certain conformable fractional equations with damping, it is clear that using the Definition 1.1 and a method similar to that presented in this work, we can draw conclusions on the oscillatory nature of the solutions of the generalized differential equation

$$N_F^{\alpha}\left[p(t)f(N_F^{\beta}y)\right]+q(t)g(t,y(t))=R(t,y),\quad 0<\alpha,\beta\leq 1,$$

a result of undoubted value.

On the other hand, it is noteworthy that in this way we can directly study the oscillatory nature of generalized differential equations, without the need to use a geometric transformation to reduce it to an ordinary differential equation, a matter that in the case of fractional differential equations with global derivatives, classic, it is not possible because there is no Chain Rule.

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