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A Binomial Ideal on Triangulations of $2-$Manifolds

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Abstract. In this paper, we introduce a binomial ideal $P(\mathcal{T}_n)$ on a simple graph $\mathcal{T}_n$ obtained from $n-$triangulation of a $2-$manifols. It is discussed on two particular classes of $n-$triangulation graphs: the class $\mathcal{T}_{n_1}$ and the class $\mathcal{T}_{\overline{n_2}}$. The Gröbner bases of the binomial ideals $P(\mathcal{T}_{n_1})$ and $P(\mathcal{T}_{\overline{n_2}})$ on n_1-triangulation and n_2-triangulation graphs $\mathcal{T}_{n_1}$ and $\mathcal{T}_{n_2}$ respectively are computed using a systematic way and it is shown that degree of polynomials in the Gröbner bases of these $n-$triangulation graphs is at most 5. Also, it is shown that the binomial ideals $P(\mathcal{T}_{n_1})$ and $\mathcal{T}_{\overline{n_2}}$ are regular ideals.

AMS (MOS) Subject Classification Codes: Primary 13P10; Secondary13H10, 13F20, 13C14.
Key Words: $n-$Triangulation, Binomial Ideal, Gröbner basis.

1. Introduction and Preliminaries

The study of binomial ideals associated with edge set of a graph is initiated by the work of Herzog et al. [10] in 2010, as a natural generalization of the ideal of $2-$minors of a $2 \times n$ matrix of indeterminants. They characterize those graphs for which quadratic graphs have Gröbner basis and studied some algebraic properties associated with these binomial edge ideals. Later on many researchers studied these ideals and their algebraic realizations from the view point of regularities, betti numbers, filtration, resolution, depths and other combinatorial and algebraic characterizations. For details see ([3, 4, 6, 10, 14, 15, 16, 18]). Motivated from the study of these ideals and the graphs arising from the topological theory of triangulation of a surface ([1, 2, 7, 8, 11, 13]), the notion of binomial ideals on an

$n-$triangulation graph of a surface is introduced in this paper. Some important notions are defined here.

Definition 1.1. [13]***(Triangulation of a Closed Surface)***
*A **triangulation of a closed surface** is a simple graph embedded on the surface so that each face is a triangle and any two faces share at most one edge.*

It is well known from the study of topology of manifolds that every surface can be triangulated ([5, 12]). Let $\mathcal{T}$ be a simple finite graph obtained from triangulation of a given $2-$manifold S. Then an $n-$**triangulation graph** of the $2-$manifold S is a subgraph of $\mathcal{T}$, consisting of n connected triangles from the triangulation of the $2-$manifold S. It will be denoted by $\mathcal{T}_n$ and its vertex set and edge set by $V_{\mathcal{T}_n} = \{v_1, v_2, \ldots v_p\}$ and $E_{\mathcal{T}_n} = \{e_1, e_2, \ldots e_q\}$ respectively. If v_i, v_j, v_k are vertices of a triangle ijk from the $n-$triangulation $\mathcal{T}_n$, then vertex and edge sets of ijk triangle will be represented by $V^{ijk}_{\mathcal{T}_n}(\subseteq V_{\mathcal{T}_n})$ and $E^{ijk}_{\mathcal{T}_n}(\subseteq E_{\mathcal{T}_n})$.

Definition 1.2. ***(Binomial Ideal on the*** $n-$***Triangulation Graph)***
Let $\mathcal{T}_n$ be the $n-$triangulation graph on a $2-$manifold S such that $V_{\mathcal{T}_n}$ and $E_{\mathcal{T}_n}$ are its vertex and edge set. Then for a field K, a binomial ideal defined by

$$P(\mathcal{T}_n) = \langle f_{ijk} = v_i v_j v_k - e_l e_m e_n \mid v_i, v_j, v_k \in V^{ijk}_{\mathcal{T}_n} \ \&\ e_l, e_m, e_n \in E^{ijk}_{\mathcal{T}_n} \rangle$$

*in the polynomial ring $S = K[v_1, v_2, \ldots v_p, e_1, e_2, \ldots e_q]$ is called a **binomial ideal on the*** $n-$***triangulation graph*** $\mathcal{T}_n$.

Indeed, for any collection of connected triangles from the $n-$triangulations graph $\mathcal{T}_n$, the Definition 1.2 can be similarly deduced.
Here, we include some preliminaries from the theory of commutative algebra that are required to make this paper self-contained. Throughout this section, $S = K[\mathbf{u}]$ represents a ring of polynomials over a field K in n variables $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, unless stated otherwise.

Definition 1.3. [9]***(Monomial Ideal)***
*Let $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{N}^n$ be a vector of non-negative integers. Then a representation $\mathbf{u}^{\mathbf{s}} = u_1^{s_1} u_2^{s_2} \ldots, u_n^{s_n}$ is called a **monomial** in $K[\mathbf{u}]$ and an ideal $J \subseteq K[\mathbf{u}]$ spanned by monomials is said to be a **monomial ideal**.*

If $\mathbf{u}^{\mathbf{s}}$ and $\mathbf{u}^{\mathbf{t}}$ are any two monomials in S, then $\mathbf{u}^{\mathbf{s}}\mathbf{u}^{\mathbf{t}} = \mathbf{u}^{\mathbf{s+t}}$. The a collection $\mathrm{Mon}(S)$ consisting of each monomial in S is indeed a $K-$basis of S. Let $f_1 \in S$ be any polynomial. Then $f_1 = \sum\limits_{v \in \mathrm{Mon}(S)} a_v v$ for $a_v \in K$ is a unique representation of f_1. The set $\{v \in \mathrm{Mon}(S) : \ a_v \neq 0\}$ is called support of f_1 in S, denoted by $\sup(f_1)$.

Lemma 1.4. [9]**(Dikson's Lemma)**
Let $\mathcal{M} \subseteq \mathrm{Mon}(S)$ be a nonempty collection. Then $\mathcal{M}^{\min}$, the collection of all minimal elements in $\mathcal{M}$, is finite.

Definition 1.5. [9] ***(Monomial Order)***
*A total order "$<$" defined on $\mathrm{Mon}(S)$ is called a **monomial order** on S if it satisfies following:*

(1) $1 < u \,\forall\, (1 \neq)u \in \mathrm{Mon}(S)$;
(2) *if* $u, v \in \mathrm{Mon}(S)$ *and* $u < v$*, then* $uw < vw \,\forall\, w \in \mathrm{Mon}(S)$.

Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ and $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ be in $\mathbb{Z}^n$ and define a **monomial order** $<_{\text{lex}}$ on $\mathrm{Mon}(S)$ by $u^s <_{\text{lex}} u^t$ if either (i) $\sum_{i=1}^{n} s_i < \sum_{i=1}^{n} t_i$, or (ii) $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} t_i$ and the difference $\mathbf{s} - \mathbf{t}$ has a $-ve$ leftmost nonzero component. Then $<_{\text{lex}}$ is called the lexicographic order on S w.r.t. the ordering $u_1 > u_2 > \cdots > u_n$.

Definition 1.6. [9]***(Initial Monomial)***
For a polynomial f_1 in S, the largest monomial, $\mathrm{in}_<(f_1)$*, w.r.t. "$<$" in* $\sup(f_1)$ *is notioned as the* ***initial monomial*** *of f_1.*

The coefficient of $\mathrm{in}_<(f_1)$ is called the **leading coefficient** of f_1, generally notated by $\mathrm{ldc}_<(f_1)$, whereas the term of f_1 containing $\mathrm{in}_<(f_1)$ is called the **leading term** of f_1, denoted by $\mathrm{ldt}_<(f_1)$.

Definition 1.7. [9]***(Initial Ideal)***
Let $J \subseteq S$ be an ideal. Then a monomial ideal of S spanned by the set $\{\mathrm{in}_<(f_1) \mid 0 \neq f_1 \in J\}$ *is said to be the* ***initial ideal*** *of J w.r.t. "$<$". It is denoted by* $\mathrm{in}_<(J)$.

Definition 1.8. [9]***(Gröbner basis)***
Let $0 \neq I$ be an ideal in S and $G = \{b_1, b_2, \ldots, b_s | b_i \in I\}$ be a finite set of polynomials. If $\mathrm{in}_<(I) = \langle \mathrm{in}_<(b_1), \mathrm{in}_<(b_2), \ldots, \mathrm{in}_<(b_s)\rangle$*, then G is said to be a* ***Gröbner basis*** *of I w.r.t. "$<$"*

Proposition 1.9. [17]**(The Division Algorithm)**
If $f, f_1, f_2, \ldots, f_q$ are polynomials in the polynomial ring $S = K[u_1, \ldots, u_n]$, then f can be written as

$$f = a_1 f_1 + \cdots + a_q f_q + r$$

where $a_i, r \in S$ and either $r = 0$ or $r \neq 0$ and none of $\mathrm{ldt}(f_1), \ldots, \mathrm{ldt}(f_q)$ divides any term of r. Furthermore, if $a_i f_i \neq 0$, then $\mathrm{ldt}(f) \geq \mathrm{ldt}(a_i f_i)$.

The polynomial r in the division algorithm is called remainder of f w.r.t. $f_1, f_2, \ldots, f_q$. One also says that f reduces to r w.r.t. $f_1, f_2, \ldots, f_q$.

Definition 1.10. [9]***($S-$polynomial)***
*Let f and g be any two polynomials in the polynomial ring $S = K[u_1, \ldots, u_n]$. Then the $S-$**polynomial** of f and g is denoted by $S(f, g)$ and is defined as*

$$S(f, g) = \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{\mathrm{ldt}_<(f)} f - \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{\mathrm{ldt}_<(g)} g$$

Theorem 1.11. [9]**(Buchberger's criterion)**
Let $0 \neq I$ be an ideal in S and $\mathcal{G} = \{b_1, \ldots, b_s\}$ a system of generators of I. Then $\mathcal{G}$ is a Gröobner basis of I iff $\forall\, i \neq j,\ S(b_i, b_j)$ reduces to zero w.r.t. $b_1, \ldots, b_s$.

The Buchberger's criterion provides an algorithm to compute a Gröbner basis of an ideal starting from a system of generator of the ideal. The algorithm is discussed below.

Algorithm 1.12. [9]***(Buchberger's Algorithm)***
Let $I = \langle \mathcal{G} = \{b_1, b_2, \ldots, b_s\}\rangle$ be a nonzero ideal in S. Then following steps will compute a Gröbner basis of I.

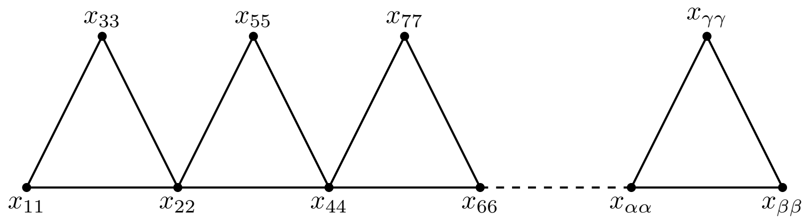

FIGURE 1. An n_1-triangulation graph $\mathcal{T}_{n_1}$

Step I: *compute the $S-$polynomials $S(b_i, b_j)\ \forall\ b_i, b_j \in \mathcal{G}$ with $i \neq j$*
Step II: *if all $S(b_i, b_j)$ reduce to zero w.r.t. $\{b_1, b_2, \ldots, b_s\}$ then by Buchberger's criterion $\mathcal{G}$ is a Gröbner basis of I and process stops, otherwise move to Step III*
Step III: *pick one of $S(b_i, b_j)$ that has nonzero remainder b_{s+1} w.r.t. $\{b_1, b_2, \ldots, b_s\}$, then none of the monomials $\text{in}_<(b_1), \text{in}_<(b_2), \ldots, \text{in}_<(b_s)$ divides $\text{in}_<(b_{s+1})$.*
Step IV: *include b_{s+1} in $\{b_1, b_2, \ldots, b_s\}$ i.e. replace $\mathcal{G} = \{b_1, b_2, \ldots, b_s\}$ by $\mathcal{G} = \{b_1, b_2, \ldots, b_s, b_{s+1}\}$ and move back to Step I.*

The process is guaranteed to terminate after finite number of steps by virtue of Dikson's lemma (Lemma 1.4) and Gröbner basis will be obtained.

In the following sections, the binomial ideal defined above is discussed for some particular classes of $n-$triangulation graphs.

2. BINOMIAL IDEAL ON n_1-TRIANGULATION GRAPH $\mathcal{T}_{n_1}$

In this section, a binomial ideal on a subclass $\mathcal{T}_{n_1}$ of the $n-$triangulation graph $\mathcal{T}_n$ is discussed. Here, $\mathcal{T}_{n_1}$ is a finite simple graph consisting of a connected chain of n_1 triangles from the graph $\mathcal{T}_n$, such that all consecutive triangles have exactly one vertex common between them. The vertex and edge sets of the graph $\mathcal{T}_{n_1}$ are fixed here as follows:

$$V_{\mathcal{T}_{n_1}} = \{x_{11}, x_{22}, \ldots, x_{\alpha\alpha}, x_{\beta\beta}, x_{\gamma\gamma}\} \text{ and } E_{\mathcal{T}_{n_1}} = \{x_{12}, x_{13}, x_{23}, \ldots, x_{\alpha\beta}, x_{\beta\gamma}, x_{\alpha\gamma}\} \quad (2.\ 1)$$

Here, $i \leq j$ for all x_{ij} either in $V_{\mathcal{T}_{n_1}}$ or in $E_{\mathcal{T}_{n_1}}$. The graph of n_1-triangulation graph $\mathcal{T}_{n_1}$ is shown in Figure 1.

Definition 2.1. ***(A Lexicographic Order on the Graph $\mathcal{T}_{n_1}$)***
Let $\mathcal{T}_{n_1}$ be a n_1-triangulation graph with vertex and edge sets $V_{\mathcal{T}_{n_1}}$ and $E_{\mathcal{T}_{n_1}}$ as defined in Equation 2. 1 (shown in Figure 1). Then the following order on the vertices and edges of the graph $\mathcal{T}_{n_1}$ is a lexicographic order on the set $V_{\mathcal{T}_{n_1}} \cup E_{\mathcal{T}_{n_1}}$:

$$\begin{array}{lc} x_{ij} > x_{kl} & \text{if } i < k \\ x_{ij} > x_{kl} & \text{if } i = k \text{ and } j < k \\ x_{ij} = x_{kl} & \text{if } i = k, j = l \end{array} \quad (2.\ 2)$$

where $i \leq j$ and $k \leq l$.

Theorem 2.2. Let $\mathcal{T}_{n_1}$ be a n_1-triangulation graph with vertex and edge sets $V_{\mathcal{T}_{n_1}}$ and $E_{\mathcal{T}_{n_1}}$ as defined in Equation 2. 1 (shown in Figure 1). Consider the binomial ideal on n_1-triangulation graph $\mathcal{T}_{n_1}$

$$P(\mathcal{T}_{n_1}) = \langle f_{ijk} = v_i v_j v_k - e_l e_m e_n \mid v_i, v_j, v_k \in V^{ijk}_{\mathcal{T}_{n_1}} \;\&\; e_l, e_m, e_n \in E^{ijk}_{\mathcal{T}_{n_1}} \rangle$$

in the polynomial ring $S = [x_{11}, x_{22}, \ldots, x_{\alpha\alpha}, x_{\beta\beta}, x_{\gamma\gamma}, x_{12}, x_{13}, x_{23}, \ldots, x_{\alpha\beta}, x_{\beta\gamma}, x_{\alpha\gamma}]$ (see Definition 1.2). Then the Grӧbner basis of the binomial ideal $P(\mathcal{T}_{n_1})$ is generated by binomials ideals with degree atmost 5, w.r.t. the lexicographic order defined in Equation 2. 2 .

Proof. The Buchberg's algorithm from Algorithm 1.12 will be applied to find Grӧbner-basis of $P(\mathcal{T}_{n_1})$.

Initially, the Grӧbner-basis contains all the ideals $f_{123}, f_{245}, f_{467}, \ldots, f_{\alpha\beta\gamma}$ such that the $\mathrm{ldm}(f_{ijk}) = x_{ii}x_{jj}x_{kk}$. The $S-$polynomials of the above said polynomials is of the type:

$$S(f_{ijk}, f_{jlm}) = x_{ii}x_{kk}x_{jl}x_{jm}x_{lm} - x_{ll}x_{mm}x_{ij}x_{ik}x_{jk}$$

Since none of leading monomials of f_{ijk} divides the $\mathrm{ldm}\Big(S(f_{ijk}, f_{jlm})\Big)$, therefore $S(f_{ijk}, f_{jlm})$ is added to Grӧbner-basis and denoted as follows

$$S(f_{ijk}, f_{jlm}) = f_{ijkjlm}, \quad \text{where } \mathrm{ldm}\,(f_{ijkjlm}) = x_{ii}x_{kk}x_{jl}x_{jm}x_{lm}.$$

Now following nontrivial cases, where the $\gcd$ of the polynomials in $S-$polynomials is non-zero, will be discused.

Case I:

First consider,

$$S(f_{ijk}, f_{ijkjlm}) = x_{jj}x_{ll}x_{mm}x_{ij}x_{ik}x_{jk} - x_{ij}x_{ik}x_{jk}x_{jl}x_{jm}x_{lm}$$

Since $\mathrm{ldm}\,(f_{ijk})$ divides the term of above $S-$polynomial, therefore

$$r_1 = S(f_{ijk}, f_{ijkjlm}) - x_{ij}x_{ik}x_{jk}f_{ijk} \implies r_1 = 0$$

and hence $S(f_{ijk}, f_{ijkjlm})$ reduces to zero.

Case II:

Next, consider the $S-$polynomial of the form

$$S(f_{ijk}, f_{jlmlst}) = x_{ii}x_{kk}x_{ss}x_{tt}x_{jl}x_{jm}x_{lm} - x_{mm}x_{ij}x_{ik}x_{jk}x_{ls}x_{lt}x_{st}$$

Since the $\mathrm{ldm}\,(f_{ijkjlm})$ divides the term of above, therefore

$$\begin{aligned} r_2 &= S(f_{ijk}, f_{jlmlst}) - x_{ss}x_{tt}f_{ijkjlm} \\ &= x_{ll}x_{mm}x_{ss}x_{tt}x_{ij}x_{ik}x_{jk} - x_{mm}x_{ij}xikx_{jk}x_{ls}x_{lt}x_{st}. \end{aligned}$$

Once again, it can be seen that $\mathrm{ldm}\,(f_{lst})$ divides a term of r_2, therefore

$$r_3 = r_2 - x_{mm}x_{ij}x_{ik}x_{jk}f_{lst} \implies r_2 = 0$$

i.e., $S(f_{ijk}, f_{jlmlst})$ reduces to zero.

Therefore, the Grӧbner-basis of $P(\mathcal{T}_{n_1})$ contains monomials with degree atmost 5. □

Example 2.3. *The Gröbner-basis of the binomial ideal $P(\mathcal{T}_3)$ on $3-$triangulation graph $\mathcal{T}_3$, using the notations and conventions from above theorem is given by;*

$$\{f_{123}, f_{245}, f_{467}, f_{123245}, f_{245467}\},$$

where, $f_{123} = x_{11}x_{22}x_{33} - x_{12}x_{13}x_{23}, f_{245} = x_{22}x_{44}x_{55} - x_{24}x_{25}x_{45}, f_{467} = x_{44}x_{66}x_{77} - x_{46}x_{47}x_{67}, f_{123245} = x_{11}x_{33}x_{24}x_{45}x_{25} - x_{44}x_{55}x_{12}x_{13}x_{23}, f_{245467} = x_{22}x_{55}x_{46}x_{47}x_{67} - x_{66}x_{77}x_{24}x_{25}x_{45}$

Corollary 2.4. The binomial ideal $P(\mathcal{T}_{n_1})$ on n_1-triangulation graph $\mathcal{T}_{n_1}$ is a radical ideal.

Proof. From the construction it is clear that the initial ideal in $\Big(P(\mathcal{T}_{n_1})\Big)$ of $P(\mathcal{T}_{n_1})$ is a square free monomial ideal generated by monomials $x_{ii}x_{jj}x_{kk}$ such that ijk is a triangle in $\mathcal{T}_{n_1}$. Therefore, from Proposition 3.3.7 of [9], the binomial ideal $P(\mathcal{T}_{n_1})$ is a radical ideal. □

3. Binomial Ideal on n_2-Triangulation Graph $\mathcal{T}_{\overline{n_2}}$

In this section, a binomial ideal on a subclass $\mathcal{T}_{\overline{n_2}}$ of the $n-$triangulation graph $\mathcal{T}_n$ is discussed. Here, $\mathcal{T}_{\overline{n_2}}$ is a finite simple graph consisting of a connected chain of n_2 triangles from the graph $\mathcal{T}_n$, such that all the triangles have one fixed vertex common between them. The vertex and edge sets of the graph $\mathcal{T}_{\overline{n_2}}$ are fixed as follows:

$$\begin{array}{rc} V_{\mathcal{T}_{\overline{n_2}}} = & \{x_{11}, x_{22}, \ldots, x_{\alpha\alpha}, x_{\beta\beta}\} \text{ and} \\ E_{\mathcal{T}_{\overline{n_2}}} = & \{x_{12}, x_{13}, x_{23}, x_{24}, x_{25}, x_{45}, \ldots, x_{2\alpha}, x_{2\beta}, x_{\alpha\beta}\}, \end{array} \tag{3. 3}$$

where, $i \leq j$ for all x_{ij} either in $V_{\mathcal{T}_{\overline{n_2}}}$ or in $E_{\mathcal{T}_{\overline{n_2}}}$. The graph of n_2-triangulation graph $\mathcal{T}_{\overline{n_2}}$ is shown in Figure 2.

Definition 3.1. ***(A Lexicographic Order on the Graph $\mathcal{T}_{\overline{n_2}}$)***
Let $\mathcal{T}_{\overline{n_2}}$ be a n_2-triangulation graph with vertex and edge sets $V_{\mathcal{T}_{\overline{n_2}}}$ and $E_{\mathcal{T}_{\overline{n_2}}}$ as defined in Equation 3. 3 (shown in Figure 2). Then the following order on the vertices and edges of the graph $\mathcal{T}_{\overline{n_2}}$ is a lexicographic order on the set $V_{\mathcal{T}_{\overline{n_2}}} \cup E_{\mathcal{T}_{\overline{n_2}}}$:

$$\begin{array}{lc} x_{ij} > x_{kl} & \text{if } i < k \\ x_{ij} > x_{kl} & \text{if } i = k \text{ and } j < k \\ x_{ij} = x_{kl} & \text{if } i = k, j = l \end{array} \tag{3. 4}$$

where $i \leq j$ and $k \leq l$.

Theorem 3.2. Let $\mathcal{T}_{\overline{n_2}}$ be a n_2-triangulation graph with vertex and edge sets $V_{\mathcal{T}_{\overline{n_2}}}$ and $E_{\mathcal{T}_{\overline{n_2}}}$ as defined in Equation 3. 3 (shown in Figure 2). Consider the binomial ideal on n_2-triangulation graph $\mathcal{T}_{\overline{n_2}}$

$$P(\mathcal{T}_{\overline{n_2}}) = \langle f_{ijk} = v_iv_jv_k - e_le_me_n \mid v_i, v_j, v_k \in V^{ijk}_{\mathcal{T}_{\overline{n_2}}} \;\&\; e_l, e_m, e_n \in E^{ijk}_{\mathcal{T}_{\overline{n_2}}}\rangle$$

in the polynomial ring $S = [x_{11}, x_{22}, \ldots, x_{\alpha\alpha}, x_{\beta\beta}, x_{12}, x_{13}, x_{23}, x_{24}, x_{25}, x_{45}, \ldots, x_{2\alpha}, x_{2\beta}, x_{\alpha\beta}]$ (see Definition 1.2). Then the Gröbner-basis of the binomial ideal $P(\mathcal{T}_{\overline{n_2}})$ is generated by binomials ideals with degree atmost 5, w.r.t. the lexicographic order defined in Equation 3. 4 .

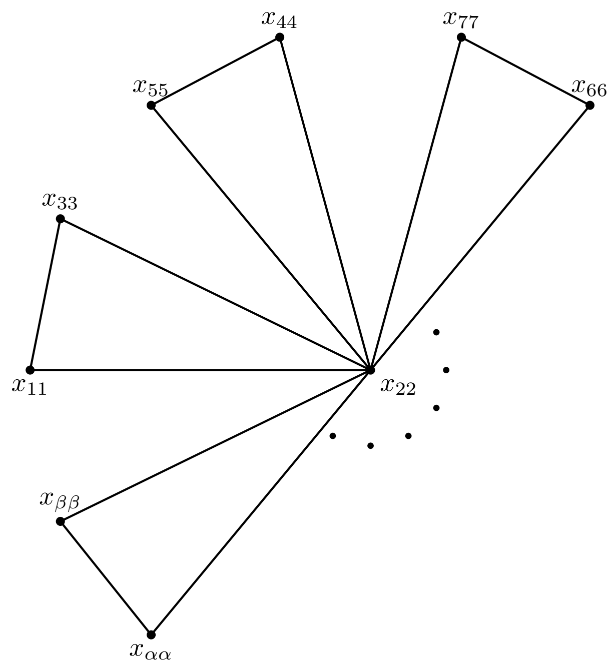

FIGURE 2. A n_2-triangulation graph $\mathcal{T}_{n_2}$

Proof. The Buchberg's algorithm from Algorithm 1.12 will be applied to find the Grӧbner-basis of $P(\mathcal{T}_{\overline{n_2}})$.

Initially, the Grӧbner-basis consist of only $f_{123}, f_{245}, f_{267}, \ldots, f_{2\alpha\beta}$. Since every two triangles have a common vertex x_{22}, therefore $S-$polynomials will be of the form

$$S(f_{123}, f_{2kl}) = x_{11}x_{33}x_{2k}x_{2l}x_{kl} - x_{kk}x_{ll}x_{12}x_{13}x_{23}$$

or

$$S(f_{2ij}, f_{2kl}) = x_{ii}x_{jj}x_{2k}x_{2l}x_{kl} - x_{kk}x_{ll}x_{2i}x_{2j}x_{ij}$$

Here, the $\operatorname{ldm}(f_{ijk}) = x_{ii}x_{jj}x_{kk}$ does not divide any term of the polynomials $S(f_{123}, f_{2kl})$ or $S(f_{2ij}, f_{2kl})$. Therefore, the polynomials $S(f_{123}, f_{2kl}) = f_{1232kl}$ and $S(f_{2ij}, f_{2kl}) = f_{2ij2kl}$ are included in Grӧbner-basis of $P(\mathcal{T}_{\overline{n_2}})$. Also it is visible that $\operatorname{ldm}(f_{1232kl}) = x_{11}x_{33}x_{2k}x_{2l}x_{kl}$ and $\operatorname{ldm}(f_{2ij2kl}) = x_{kk}x_{ll}x_{2i}x_{2j}x_{ij}$. Now following nontrivial cases, where the $\gcd$ of the polynomials in $S-$polynomials is non-zero, will be discused.

Case I:

Consider, $S(f_{123}, f_{1232kl}) = x_{2k}x_{2l}x_{kl}f_{123} - x_{22}f_{1232kl}$

$$\implies S(f_{123}, f_{1232kl}) = x_{kk}x_{ll}x_{22}x_{12}x_{13}x_{23} - x_{12}x_{13}x_{23}x_{2k}x_{2l}x_{kl}.$$

Since $\operatorname{ldm}(f_{2kl})$ divides term of $S(f_{123}, f_{1232kl})$, therefore the remainder

$$r = S(f_{123}, f_{1232kl}) - x_{12}x_{13}x_{23}f_{2kl}.$$

$S-$polynomial	ldm dividing term of $S-$polynomial	remainder
$S(f_{1232kl}, f_{1232mn})$	f_{2kl2mn}	0
$S(f_{1232kl}, f_{2kl2mn})$	f_{1232mn}	0
$S(f_{2ij2kl}, f_{2ij2mn})$	f_{2kl2mn}	0
$S(f_{2ij2kl}, f_{2mn2kl})$	f_{2ij2mn}	0

TABLE 1. Computation table for Gröbner Basis of $P(\mathcal{T}_{\overline{n_2}})$

A routine calculation shows that $r = 0$ and hence $S(f_{123}, f_{1232kl})$ reduces to zero.

Case II:
Next, the $S-$polynomial $S(f_{2kl}, f_{2ij2kl})$ is computed as follows

$$S(f_{2kl}, f_{2ij2kl}) = x_{2i}x_{2j}x_{ij}f_{2kl} + x_{22}f_{2ij2kl}$$

$$\implies S(f_{2kl}, f_{2ij2kl}) = x_{22}x_{ii}x_{jj}x_{2k}x_{2l}x_{kl} - x_{2i}x_{2j}x_{ij}x_{2k}x_{2l}x_{kl}$$

Since the $\text{ldm}\,(f_{2ij})$ divides term of $S(f_{2kl}, f_{2ij2kl})$, it's easy to see that its remainder

$$r = S(f_{2kl}, f_{2ij2kl}) - x_{2k}x_{2l}x_{kl}f_{2ij} = 0$$

and hence $S(f_{2kl}, f_{2ij2kl})$ reduces to zero. Similar computations reveal that the remaining $S-$polynomials also reduce to zero. The following table presents a brief summary of the remaining computations of $S-$polynomials, the leading monomial dividing its term and the remainder after division. Therefore, the Gröbner-basis of $P(\mathcal{T}_{\overline{n_2}})$ is

$$\{f_{123}, f_{245}, f_{267}, \ldots, f_{2\alpha\beta}, f_{1232kl}, f_{2ij2kl}\}$$

This shows that the degree of binomials in the Gröbner-basis is atmost five. □

Example 3.3. *The Gröbner-basis of the binomial ideal $P(\mathcal{T}_{\overline{3}})$ on $3-$triangulation graph $\mathcal{T}_{\overline{3}}$, using the notations and conventions from above theorem is given by;*

$$\{f_{123}, f_{245}, f_{267}, f_{123245}, f_{123267}, f_{245267}\}$$

where, $f_{123} = x_{11}x_{22}x_{33} - x_{12}x_{13}x_{23}$, $f_{245} = x_{22}x_{44}x_{55} - x_{24}x_{25}x_{45}$,
$f_{267} = x_{22}x_{66}x_{77} - x_{26}x_{27}x_{67}$, $f_{123245} = x_{11}x_{33}x_{24}x_{45}x_{25} - x_{44}x_{55}x_{12}x_{13}x_{23}$,
$f_{123267} = x_{11}x_{33}x_{26}x_{27}x_{67} - x_{66}x_{77}x_{12}x_{13}x_{23}$,
$f_{245267} = x_{44}x_{55}x_{26}x_{27}x_{67} - x_{66}x_{77}x_{24}x_{25}x_{45}$.

Corollary 3.4. The binomial ideal $P(\mathcal{T}_{\overline{n_2}})$ on n_2-triangulation graph $\mathcal{T}_{\overline{n_2}}$ is a radical ideal.

Proof. From the construction, it is clear that the initial ideal $\text{in}\left(P(\mathcal{T}_{\overline{n_2}})\right)$ of $P(\mathcal{T}_{\overline{n_2}})$ is a square free monomial ideal generated by monomials $x_{ii}x_{jj}x_{kk}$ such that ijk is a triangle in $\mathcal{T}_{\overline{n_2}}$. Therefore, from Proposition 3.3.7 of [9], the binomial ideal $P(\mathcal{T}_{\overline{n_2}})$ is a radical ideal. □

4. Some future directions

In this paper, the notion of binomial ideal on $n-$triangulation graph $P(\mathcal{T}_n)$ is introduced and discussed for some algebraic attributes on two particular classes of $n-$triangulations graphs: the class $\mathcal{T}_{n_1}$ and the class $\mathcal{T}_{\overline{n_2}}$. It is shown that degree of polynomials in the Gröbner-basis of $P(\mathcal{T}_{n_1})$ and $P(\mathcal{T}_{\overline{n_2}})$ is at most 5 and that these ideals are radical ideals. Some future research directions to work on the binomial ideal $P(\mathcal{T}_n)$ are given here:

The notion of binomial ideal on $n-$triangulation graph can be studied on some other classes of $n-$triangulation graphs of $2-$manifolds.

There are several binomial ideals linked to a graph in literature. However, none of these binomial ideals associated with a graph are generated by binomials having terms of degree 3. This is an entirely new idea and is in its infancy. The idea solicits further research and discussion for several algebraic attributes like its Cohan-Macaulayness, betti numbers, projective dimensions etc.

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