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A Paradigmatic approach to Quasi Topological Loops: Classification and Characterization

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Abstract.: In this paper, we discuss some more properties of quasi topological loops when multiplication mapping is separately irresolute, separately semi-continuous and separately $\mathcal{G}$-semi-continuous with their inverse mappings are irresolute, semi-continuous and $\mathcal{G}$-semi-continuous respectively. Moreover, we provide a comparative overview of these topological loops respecting three different forms of continuity based on Levine's semi-open set.

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1. Introduction

It is always captivating to go through the properties of topological spaces over various algebraic structures. Mostly, it requires continuity of algebraic operations. By and large, to generalize these structures, enfeeble form of continuity is used. Many mathematicians are interested to use the forms of continuity based on Levine's semi-open set [1, 9, 11]. Here, we utilize three forms of continuity to study the properties of topological spaces over a more stimulating algebraic structure; loop.

A subset M of X is semi-open, if there is an open set U in X such that

$$U \subseteq M \subseteq Cl(U)$$

or

$$M \subseteq Cl(Int(M)).$$

The class consisting of all the semi-open sets contained in Y is declared as $SO(Y)$. Arbitrary union of semi-open sets is semi-open, but their finite intersection may not be semi-open [14]. Intersection of a semi-open set and an open set is semi-open. Semi-closed set is the compliment of semi-open set. Each closed (open) set is semi-closed (semi-open). $M_1 \times M_2$ is semi-open in $X \times Y$, if $M_1 \subseteq X$ and $M_2 \subseteq Y$ are semi-open sets. $M_2 \subseteq X$ is a semi-open nbhd of $t \in X$, if there exists $M_1 \in SO(X)$ satiating

$$t \in M_1 \subseteq M_2.$$

A point $t \in X$ is a semi-interior point of M, if there exists a semi-open set M' satisfying

$$t \in M' \subseteq M.$$

$sInt(M)$ consists of all semi-interior points of M. For any subset M of X, $t \in sCl(M)$ if and only if $M_t \cap M \neq \phi$ for any semi-open nbhd M_t of t.

A subset M of a topological space X is semi-compact (semi-Lindelof), if there exists a finite (countable) subcover for any semi-open cover of M in X [6, 10]. $M \subseteq X$ is a semi-open nbhd of a subset $Q \subseteq X$, if M is a semi-open nbhd of t in X for each $t \in Q$; i.e., $Q \subseteq sInt(M)$. A space (L, τ) is s-regular, if for all $t \in L$ and each closed set $Q \subseteq L$ with $t \notin Q$, there exist semi-open nbhds M and N of t and Q, respectively, such that $M \cap N = \phi$.

Let X be a set and $\mathcal{G} \subseteq P(X)$. $\mathcal{G}$ is said to be generalized topology (denoted by $\mathcal{G}$-topology), if $\phi \in \mathcal{G}$ and $\mathcal{G}$ is closed under arbitrary union [2]. Members of $\mathcal{G}$ are said to be generalized open sets denoted by $\mathcal{G}$-open sets and their compliments are generalized closed sets ($\mathcal{G}$-closed sets). A subset M of X is generalized semi-open ($\mathcal{G}$-semi-open), if there is a $\mathcal{G}$-open set U in X satiating

$$U \subseteq M \subseteq Cl(U).$$

The class consisting of all $\mathcal{G}$-semi-open sets of a space X is declared as $\mathcal{G}SO(X)$. The class of all generalized semi-open sets comprising t is denoted by $\mathcal{G}SO(X, t)$. Finite intersection of $\mathcal{G}$-semi-open sets need not be $\mathcal{G}$-semi-open but any union of $\mathcal{G}$-semi-open sets remain $\mathcal{G}$-semi-open. The compliment of a $\mathcal{G}$-semi-open set is a $\mathcal{G}$-semi-closed set. Every $\mathcal{G}$-closed ($\mathcal{G}$-open) set is a $\mathcal{G}$-semi-closed ($\mathcal{G}$-semi-open). $M_2 \subseteq X$ is a $\mathcal{G}$-semi-open nbhd of $t \in X$, if there exists $M_1 \in \mathcal{G}SO(X)$ satiating

$$t \in M_1 \subseteq M_2.$$

If there is a $\mathcal{G}$-semi-open set M' satisfying $t \in M' \subseteq M$, then $t \in X$ is a $\mathcal{G}$-semi-interior point of M. Class of all $\mathcal{G}$-semi-interior points of M is declared as $\mathcal{G}sInt(M)$. For any subset M of X, $t \in \mathcal{G}sCl(M)$ if and only if $M_t \cap M \neq \phi$ for any $\mathcal{G}$-semi-open nbhd M_t of t [3]. A $\mathcal{G}$-topological space $(L, \mathcal{G})$ is $\mathcal{G}$-s-regular, if for each $t \in L$ and every $\mathcal{G}$-closed set $Q \subseteq L$ with $t \notin Q$, there exist $\mathcal{G}$-semi-open nbhds M and N of t and Q, respectively, such that $M \cap N = \phi$.

$\mathcal{G}$-topological space $(L, \mathcal{G})$ is $\mathcal{G}$-s regular, if for each $t \in L$ and every $\mathcal{G}$-closed set $Q \subseteq L$, there exists a $\mathcal{G}$-semi-open nbhds M, N containing t and Q respectively with $t \notin Q$ satiating $M \cap N = \phi$.

A map $f : A \to B$ is said to be:

- **Irresolute**, if $f^{-1}(M_2)$ is semi-open in A, for each semi-open set $M_2 \subseteq B$ [7];
- **Semi-continuous**, if the set $f^{-1}(M_2)$ is semi-open in A, for every open set $M_2 \subseteq B$ [14];
- **$\mathcal{G}$-semi-continuous**, if the set $f^{-1}(M_2)$ is a $\mathcal{G}$-semi-open in A for all $\mathcal{G}$-open set $M_2 \subseteq B$ [2];
- **Pre semi-open**, if $f(M)$ is semi-open in B, for every semi-open set M in A;
- **Semi-open**, if $f(M)$ is semi-open in B, for every open set M in A [4];
- **$\mathcal{G}$-semi-open**, if $f(M)$ is a $\mathcal{G}$-semi-open in B, for every $\mathcal{G}$-open set M in A [2];
- **Semi-homeomorphism**, if f is pre semi-open, bijective, and irresolute;
- **s-homeomorphism**, if f is pre semi-open, semi-continuous and bijective [8];
- **Quasi s-homeomorphism**, if it is bijective, semi-open and semi-continuous [8];
- **Quasi $\mathcal{G}$-s-homeomorphism**, if f is a $\mathcal{G}$-semi-open, $\mathcal{G}$-semi-continuous and bijective.

A groupoid $(L, *)$ is a loop if the following conditions are fulfilled:

- L contain an identity element.
- for every $t_1 \in L$, the maps $l_{t_1} : L \to L$ and $r_{t_1} : L \to L$ are bijective, where $l_{t_1}(t_2) = t_1 * t_2$ and $r_{t_1}(t_2) = t_2 * t_1$ for all $t_2 \in L$ [5].

An inverse property loop (IP-loop) L is a loop having two sided inverse t^{-1} such that $(r * t) * t^{-1} = r = t^{-1} * (t * r)$ for all $r, t \in L$. Left (right) translations and left (right) inverse maps are defined on a loop $(L, *)$ as follows:

- **Left translation** $l_{t_1} : L \to L$ is given by $l_{t_1}(t_2) = t_1 * t_2$;
- **Right translation** $r_{t_1} : L \to L$ is given as $r_{t_1}(t_2) = t_2 * t_1$;
- **Left inverse map** $i_L : L \to L$ is defined by $i_L(t) = (t)^{-1}_L$;
- **Right inverse map** $i_R : L \to L$ is defined as $i_R(t) = (t)^{-1}_R$;

where $t, t_1, t_2 \in L$.
We used the standard notions and terminologies as in [15].

This paper is aimed to characterize the properties of quasi topological loops with different forms of continuity. In *section II*, we discuss quasi topological loops with respect to irresoluteness. The *section III* of this paper is based on the properties of quasi topological loop with respect to semi-continuity. Quasi topological loop with respect to generalized semi-continuity is conferred in *section IV*. It is to be noted that, a quasi topological loop with respect to irresoluteness is a quasi topological loop with respect to semi-continuity which is a quasi topological loop with respect to generalized semi-continuity. Moreover, some examples are given to elaborate the concept.

2. Quasi Topological Loops with respect to Irresoluteness

Definition 2.1. *A triplet $(L, *, \tau)$ is called a quasi irresolute topological loop, if the following conditions are fulfilled:*

(1) *$(L, *)$ is a loop.*

(2) (L, τ) *is a topological space.*
(3) *Multiplication map is separately irresolute in* $(L, *, \tau)$.
(4) *Left and right inverse maps are irresolute in* $(L, *, \tau)$.

Example 2.2. *Consider a loop* $L_9(5)$ *of order* 10 *with the topology* $\tau = \{\phi, \{2,4\}, \{3,6,9\}, \{e,5\}, \{e,2,4,5\}, \{2,3,4,6,9\}, \{e,3,5,6,9\}, L_9(5)\}$. *The inverse mappings are irresolute in* $(L_9(5), *, \tau)$ *but multiplication mapping is not separately irresolute. Therefore,* $(L_9(5), *, \tau)$ *is not a quasi irresolute topological loop.* $(L_9(5), *, \tau')$ *is a quasi irresolute topological loop with the topology* $\tau' = \{\phi, L_9(5)\}$ *which is also a quasi s-topological loop and quasi* $\mathcal{G}$*-s-topological loop.*

Lemma 2.3. *In a quasi irresolute topological loop, left and right inverses of a semi-open set are also semi-open.*

Proof. Suppose that $(L, *, \tau)$ is a quasi irresolute topological loop. For $M \in SO(L)$, since $i_L : L \to L$ is semi-homeomorphism in $(L, *, \tau)$, so for every $M \in SO(L)$ we have $i_L(M) = (M)_L^{-1} \in SO(L)$. Consequently, $(M)_R^{-1} \in SO(L)$. □

Corollary 2.4. *Let* $(L, *, \tau)$ *be a quasi irresolute topological loop and* $M \subseteq L$. *Then* $sInt(M_L^{-1}) = (sInt(M))_L^{-1}$, *and* $sInt(M_R^{-1}) = (sInt(M))_R^{-1}$. *Also* $sCl(M_L^{-1}) = (sCl(M))_L^{-1}$, *and* $sCl(M_R^{-1}) = (sCl(M))_R^{-1}$.

Proof. Since, i_L is semi-homeomorphism and for a subset M of L, we have $sInt(M_L^{-1}) = (sInt(M))_L^{-1}$, and $sCl(M_L^{-1}) = (sCl(M))_L^{-1}$ [4]. Accordingly, $sInt(M_R^{-1}) = (sInt(M))_R^{-1}$, and $sCl(M_R^{-1}) = (sCl(M))_R^{-1}$. □

Theorem 2.5. *Every semi-open sub-loop in a quasi irresolute topological loop with pre semi-open left translation is semi-closed.*

Proof. Suppose that L is a quasi irresolute topological loop and L' is its semi-open sub-loop. By the definition of semi-closure, each semi-open nbhd of s meets L' if and only if $s \in sCl(L')$. As, $s * L'$ is a semi-open nbhd of s, it meets L'. Therefore, $\exists m, t \in L'$ such that $t = s * m$. But, then $s = t/m \in L'$. So, $sCl(L') = L'$. Hence, L' is semi-closed. □

Lemma 2.6. *Let* X *be a topological space and* $(L, *, \tau)$ *be a quasi irresolute topological loop. If* $f : X \to L$ *is an irresolute map, then the maps* $f_L^{-1} : X \to L$ *given by* $f_L^{-1}(x) = (f(x))_L^{-1}$, *and* $f_R^{-1} : X \to L$ *given by* $f_R^{-1}(x) = (f(x))_R^{-1}$ *are irresolute.*

Proof. As, $f_L^{-1} = i_L \circ f$, where f and i_L are irresolute, therefore f_L^{-1} is irresolute. Consequently, f_R^{-1} is irresolute. □

The theorems given below are about semi-compactness and semi-Lindelof.

Theorem 2.7. $t * N^{-1}$ *is semi-compact in a quasi irresolute topological loop* $(L, *, \tau)$ *with inverse property, if* N *is semi-compact.*

Proof. Consider a semi-open cover $\{M_i : i \in I\}$ of $t * N^{-1}$. So $t * N^{-1} \subseteq \cup_{i \in I} M_i$, $N^{-1} \subseteq t^{-1} * \cup_{i \in I} M_i = \cup_{i \in I} t^{-1} * M_i$. Hence, $N \subseteq \cup_{i \in I} M_i^{-1} * t$. By semi-compactness of N, there is a finite $I_0 \subseteq I$ such that $N \subseteq \cup_{i \in I_0} M_i^{-1} * t$, $N * t^{-1} \subseteq \cup_{i \in I_0} M_i^{-1}$. Thus $t * N^{-1} \subseteq \cup_{i \in I_0} M_i$. Therefore, $t * N^{-1}$ has semi-open finite subcover in L. □

Theorem 2.8. *Let β_e be a class of semi-open nbhds of e_L in a quasi irresolute topological loop $(L, *, \tau)$. Then,*

(1) *For any element M_1 of β_e, there exists an element M_2 in β_e such that M_1 contains both left and right inverses of M_2.*
(2) *For each $M_1 \in \beta_e$, there exists $M_2 \in \beta_e$ such that $t * M_2, M_2 * t \subseteq M_1$, where t is an arbitrary element of M_1.*

Proof.

(1) As given is that $(L, *, \tau)$ is a quasi irresolute topological loop, so for all $M_1 \in \beta_e$, there is $M_2 \in \beta_e$ such that $i_L(M_2) = (M_2)_L^{-1} \subseteq M_1$, and $i_R(M_2) = (M_2)_R^{-1} \subseteq M_1$.
(2) For all $M_1 \in \beta_e$ containing t, there exists $M_2 \in \beta_e$ such that $l_t(M_2) = t * M_2 \subseteq M_1$. Accordingly, $r_t(M_2) = M_2 * t \subseteq M_1 \; \forall t \in M_1$.

□

Theorem 2.9. *Let β_e be a class of semi-open nbhds of e_L in a quasi irresolute topological loop $(L, *, \tau)$. Then,*

(1) *For every element M_1 of β_e and for $t \in L$, if l_x is pre semi-open, then there exists $M_2 \in \beta_e$ such that $(t * M_2) * (t)_R^{-1} \subseteq M_1$.*
(2) *For all $M_1 \in \beta_e$ and $t \in L$, if the right translation is pre semi-open then there exists $M_2 \in \beta_e$ such that $(t)_L^{-1} * (M_2 * t) \subseteq M_1$.*

Proof.

(1) It follows that l_t is pre semi-open, r_t is irresolute in L, and $l_t(e) = t$. Then, there exists $M_2 \in SO(L, e)$ such that $l_t(M_2) = t * M_2$, $t * M_2$ is a semi-open nbhd of t, and $r_{(t)_R^{-1}}(t * M_2) = (t * M_2) * (t)_R^{-1} \subseteq M_1$.
(2) If r_t is pre semi-open, l_t is irresolute in L, and $r_t(e) = t$. Then there exists $M_2 \in SO(L, e)$ such that $r_t(M_2) = M_2 * t$, $M_2 * t$ is a semi-open nbhd of t and $l_{(t)_L^{-1}}(M_2 * t) = (t)_L^{-1} * (M_2 * t) \subseteq M_1$.

□

Theorem 2.10. *Let $(L, *)$ be an IP loop and (L, τ) be a topological space. If for each $M \in SO(L, e)$, there exist $N \in SO(L, e)$ satisfying $N^{-1} \subseteq M$, also left and right translations are irresolute in L then the inverse map is also irresolute in L.*

Proof. Suppose $t \in L$ and S is a semi-open nbhd of t^{-1}. Thus, there is a semi-open nbhd M of e satiating $l_{t^{-1}}(M) = t^{-1} * M \subseteq S$. Then $(l_t)^{-1}(N^{-1}) = l_{t^{-1}}(N^{-1}) = t^{-1} * N^{-1} = (N * t)^{-1}$ is a semi-open nbhd of t^{-1}, where $N * t$ is a semi-open nbhd of t and $i(N * t) = (N * t)^{-1} = t^{-1} * N^{-1} \subseteq t^{-1} * M \subseteq S$. Thus, the inverse map is irresolute in L. □

Theorem 2.11. *A quasi irresolute topological loop $(L, *, \tau)$ with inverse property is s-regular at e, if μ_e is a base at e and every $M \in \mu_e$ there exists a symmetric semi-open nbhd N of e satiating $N * N \subseteq M^{-1}$.*

Proof. Let $t \in sCl(N)$ and $t * N$ is a semi-open nbhd of t. Clearly, $t * N \cap N \neq \phi$. So, there exists $m, n \in N$ such that $n = t * m$, $t = n * m^{-1} \in N * N^{-1} = N * N \subseteq M^{-1}$. Thus $sCl(N) \subseteq M^{-1}$. It gives L is s-regular at e. □

3. **Quasi Topological Loops with respect to Semi-continuity**

Definition 3.1. [13] *A triplet $(L, *, \tau)$ is called a quasi s-topological loop, if the following conditions are fulfilled:*

(1) *$(L, *)$ is a loop.*
(2) *(L, τ) is a topological space.*
(3) *Multiplication map is separately semi-continuous in $(L, *, \tau)$.*
(4) *Left and right inverse maps i_L, i_R defined on $(L, *, \tau)$ are semi-continuous.*

Example 3.2. *Loop $L_7(4)$ of order 8 with the topologies $\tau_1 = P(L_7(4))$ and $\tau_2 = \{\phi, L_7(4)\}$ form a quasi s-topological loop.*

Theorem 3.3. *A quasi irresolute topological loop is a quasi s-topological loop.*

Proof. Suppose that $(L, *, \tau)$ is a quasi irresolute topological loop, therefore i_L, i_R, l_x and r_x, are irresolute in L. As every irresolute map is semi-continuous. Hence, i_L, i_R, r_x and l_x are semi-continuous in L. So, $(L, *, \tau)$ is a quasi s-topological loop. □

The subsequent corollary is the consequence of above result.

Corollary 3.4. *In a loop with topology, if left (right) inverse map is semi-open, then right (left) inverse map is semi-continuous.*

Proof. Suppose, left inverse map is semi-open. Then for each $M \in O(L)$, $(M)_L^{-1} \in SO(L)$. Therefore, $i_R((M)_L^{-1}) = M$. This implies right inverse map is semi-continuous. □

Next we will show that left and right inverses of an open set are semi-open in a quasi s-topological loop.

Lemma 3.5. *In a quasi s-topological loop, left and right inverses of an open set are semi-open.*

Proof. Suppose that $(L, *, \tau)$ is a quasi s-topological loop. For $M \in O(L)$, since $i_L : L \to L$ is quasi s-homeomorphism in $(L, *, \tau)$, so for every $M \in O(L)$ we have $i_L(M) = (M)_L^{-1} \in SO(L)$. Consequently, $(M)_R^{-1} \in SO(L)$. □

The next theorem is about the property of a semi-open sub-loop of a quasi s-topological loop.

Theorem 3.6. *Each open sub-loop in a quasi s-topological loop with semi-open left translation is semi-closed.*

Proof. Suppose, L' is an open sub-loop in a quasi s-topological loop L. By the definition of semi-closure, each semi-open nbhd of s meets L' if and only if $s \in sCl(L')$. As, $s * L'$ is a semi-open nbhd of s, it meets L'. Therefore, $\exists m, t \in L'$ such that $t = s * m$. But, then $s = t/m \in L'$. So, $sCl(L') = L'$. Hence, L' is semi-closed. □

Lemma 3.7. *Let $(L, *, \tau)$ be a quasi s-topological loop and X be a topological space. If $f : X \to L$ is an open and semi-continuous map, then the maps $f_L^{-1} : X \to L$ given by $f_L^{-1}(x) = (f(x))_L^{-1}$, and $f_R^{-1} : X \to L$ given by $f_R^{-1}(x) = (f(x))_R^{-1}$ are semi-continuous.*

Proof. As, $f_L^{-1} = i_L \circ f$, where i_L is semi-continuous and f is semi-continuous and open, therefore f_L^{-1} is semi-continuous. Consequently, f_R^{-1} is semi-continuous. □

Theorem 3.8. *Let β_e be a class of semi-open nbhds of e_L in a quasi s-topological loop $(L, *, \tau)$. Then,*

(1) *For any $M_1 \in O(L, e)$, there exists an element M_2 in β_e such that M_1 contains both left and right inverses of M_2.*
(2) *For each $M_1 \in O(L, e)$, there exists $M_2 \in \beta_e$ such that $t * M_2, M_2 * t \subseteq M_1$, where t is an arbitrary element of M_1.*

Proof.

(1) As given is that $(L, *, \tau)$ is a quasi s-topological loop, so for all $M_1 \in O(L, e)$, there exists $M_2 \in \beta_e$ such that $i_L(M_2) = (M_2)_L^{-1} \subseteq M_1$, and $i_R(M_2) = (M_2)_R^{-1} \subseteq M_1$.
(2) For all $M_1 \in O(L, e)$ containing t, there exists $M_2 \in \beta_e$ such that $l_t(M_2) = t * M_2 \subseteq M_1$. Accordingly, $r_t(M_2) = M_2 * t \subseteq M_1 \; \forall t \in M_1$.

□

Theorem 3.9. *Let β_e be a class of open nbhds of e_L in a quasi s-topological loop $(L, *, \tau)$. Then,*

(1) *For every element M_1 of β_e and for $t \in L$, if left translation is semi-open, then there exists $M_2 \in \beta_e$ such that $(t * M_2) * (t)_R^{-1} \subseteq M_1$.*
(2) *For all $M_1 \in \beta_e$ and $t \in L$, if the right translation is semi-open then there exists $M_2 \in \beta_e$ such that $(t)_L^{-1} * (M_2 * t) \subseteq M_1$.*

Proof.

(1) As r_t is semi-continuous and l_t is semi-open in L, and $l_t(e) = t$. Then there exists $M_2 \in O(L, e)$ such that $l_t(M_2) = t * M_2$, $t * M_2$ is a semi-open nbhd of t, and $r_{(t)_R^{-1}}(t * M_2) = (t * M_2) * (t)_R^{-1} \subseteq M_1$.
(2) If $r_t(e) = t$, l_t is semi-continuous and r_t is semi-open in L. Then there exists $M_2 \in SO(L, e)$ such that $r_t(M_2) = M_2 * t$, $M_2 * t$ is a semi-open nbhd of t, and $l_{(t)_L^{-1}}(M_2 * t) = (t)_L^{-1} * (M_2 * t) \subseteq M_1$.

□

Theorem 3.10. *Suppose that $f : (L, *, \tau_L) \to (M, *', \tau_M)$ is a homomorphism between a quasi s-topological loop M and a quasi irresolute topological loop L with pre semi-open left translation in L. f is semi-continuous in L, if f is irresolute at e_L.*

Proof. Suppose that for $s \in L$, W is an open nbhd of $f(s) = t$ in M. Therefore, there is a semi-open nbhd V of e_M satiating $l_t(V) = t * V \subseteq W$. Thus, $f(U) \subseteq V$ for $U \in SO(L, e_L)$. As $s * U \in SO(L, s)$. Therefore, $f(s * U) = f(s) * f(U) = t * f(U) \subseteq t * V \subseteq W$. So, f is irresolute for any $s \in L$. □

Corollary 3.11. *Suppose that $f : (L, *, \tau_L) \to (M, *', \tau_M)$ is a homomorphism between a quasi s-topological loop M and a quasi irresolute topological loop L with pre semi-open left translation in L. If f is semi-continuous at e_L then f is continuous in L.*

Theorem 3.12. *Let $(L,*)$ be an IP loop and (L,τ) be a space. If for all $M \in SO(L,e)$, there exists $N \in O(L,e)$ satiating $N^{-1} \subseteq M$, also right and left translations are semi-continuous in L then inverse map is also semi-continuous in L. Therefore L is a quasi s-topological loop.*

Proof. Suppose $t \in L$ and S is an open nbhd of t^{-1}. So, there exists a semi-open nbhd M of e satiating $l_{t^{-1}}(M) = t^{-1} * M \subseteq S$. Then $(l_t)^{-1}(N^{-1}) = l_{t^{-1}}(N^{-1}) = t^{-1} * N^{-1} = (N * t)^{-1}$ is a semi-open nbhd of t^{-1}, where $N * t$ is a semi-open nbhd of t and $i(N*t) = (N*t)^{-1} = t^{-1} * N^{-1} \subseteq t^{-1} * M \subseteq S$. Thus inverse map is semi-continuous in L. □

Theorem 3.13. *A quasi s-topological loop $(L,*,\tau)$ with inverse property is s-regular at e, if μ_e is a base at e and every $M \in \mu_e$ there exists a symmetric open nbhd N of e satisfying $N * N \subseteq M^{-1}$.*

Proof. Let $t \in sCl(N)$ and $t * N$ is a semi-open nbhd of t. Clearly, $t * N \cap N \neq \phi$. So, there exist $m, n \in N$ such that $n = t * m$, $t = n * m^{-1} \in N * N^{-1} = N * N \subseteq M^{-1}$. Thus $sCl(N) \subseteq M^{-1}$. It gives L is s-regular at e. □

4. Quasi Topological Loops with respect to Generalized Semi-continuity

Definition 4.1. [12] *A triplet $(L,*,\mathcal{G})$ is said to be a quasi $\mathcal{G}$-s-topological loop, if the following conditions are fulfilled:*

(1) *$(L,*)$ is a loop.*
(2) *$(L,\mathcal{G})$ is a $\mathcal{G}$-topological space.*
(3) *Multiplication map is separately $\mathcal{G}$-semi-continuous in $(L,*,\mathcal{G})$.*
(4) *Left and right inverse maps are $\mathcal{G}$-semi-continuous in $(L,*,\mathcal{G})$.*

Example 4.2. *Loop $L_7(3)$ of order 8 with the generalized topology $\mathcal{G} = \{\phi, \{1,5,7\}, \{e,6\}, \{e,1,5,6,7\}\}$ is not a quasi $\mathcal{G}$-s-topological loop. Moreover, with the topology $\mathcal{G}' = \{\phi\}$, $(L_7(3),*,\mathcal{G}')$ is a quasi $\mathcal{G}$-s-topological loop which is neither a quasi irresolute topological loop nor a quasi s-topological loop.*

Corollary 4.3. *In a loop with $\mathcal{G}$-topology, if left (right) inverse map is $\mathcal{G}$-semi-open, then right (left) inverse map is $\mathcal{G}$-semi-continuous.*

Proof. Suppose, left inverse map is $\mathcal{G}$-semi-open. Then for each $M \in \mathcal{G}O(L)$, $(M)_L^{-1} \in \mathcal{G}SO(L)$. Therefore, $i_R((M)_L^{-1}) = M$. This implies right inverse map is $\mathcal{G}$-semi-continuous. □

Remark 4.4. *It is to be noted that, unlike topology, in $\mathcal{G}$-topology the intersection of $\mathcal{G}$-open and $\mathcal{G}$-semi-open sets is not necessarily $\mathcal{G}$-semi-open.*

Next we show that left and right inverses of each $\mathcal{G}$-open set are $\mathcal{G}$-semi-open in a quasi $\mathcal{G}$-s-topological loop.

Lemma 4.5. *Left and right inverses of a $\mathcal{G}$-open set in a quasi $\mathcal{G}$-s-topological loop are $\mathcal{G}$-semi-open.*

Proof. Let $(L,*,\mathcal{G})$ be a quasi $\mathcal{G}$-s-topological loop. For $M \in \mathcal{G}O(L)$, since $i_L : L \to L$ is a quasi $\mathcal{G}$-s-homeomorphism in $(L,*,\mathcal{G})$, so $i_L(M) = (M)_L^{-1} \in \mathcal{G}SO(L)$. Consequently, $(M)_R^{-1} \in \mathcal{G}SO(L)$. □

The next theorem is about the property of a $\mathcal{G}$-semi-open sub-loop of a quasi $\mathcal{G}$-s-topological loop.

Theorem 4.6. *Every $\mathcal{G}$-open sub-loop in a quasi $\mathcal{G}$-s-topological loop with a $\mathcal{G}$-semi-open left translation is a $\mathcal{G}$-semi-closed.*

Proof. Suppose, L' is a $\mathcal{G}$-open sub-loop in a quasi $\mathcal{G}$-s-topological loop L. By the definition of a $\mathcal{G}$-semi-closure, each $\mathcal{G}$-semi-open nbhd of s meets L' if and only if $s \in \mathcal{G}sCl(L')$. As, $s * L'$ is a $\mathcal{G}$-semi-open nbhd of s, it meets L'. Thus, $\exists m, t \in L'$ such that $t = s * m$. But, then $s = t/m \in L'$. So, $\mathcal{G}sCl(L') = L'$. Hence, L' is a $\mathcal{G}$-semi-closed. □

Lemma 4.7. *Let $(L,*,\mathcal{G})$ be a quasi $\mathcal{G}$-s-topological loop and X be a $\mathcal{G}$-topological space. If $f : X \to L$ is a $\mathcal{G}$-open and $\mathcal{G}$-semi-continuous map, then the maps $f_L^{-1} : X \to L$ given by $f_L^{-1}(x) = (f(x))_L^{-1}$, and $f_R^{-1} : X \to L$ given by $f_R^{-1}(x) = (f(x))_R^{-1}$ are $\mathcal{G}$-semi-continuous.*

Proof. As, $f_L^{-1} = i_L \circ f$, where f is a $\mathcal{G}$-open and $\mathcal{G}$-semi-continuous and i_L is $\mathcal{G}$-semi-open, therefore f_L^{-1} is $\mathcal{G}$-semi-continuous. Thereupon, f_R^{-1} is $\mathcal{G}$-semi-continuous. □

Theorem 4.8. *Let β_e be a class of $\mathcal{G}$-semi-open nbhds of e_L in a quasi $\mathcal{G}$-s-topological loop $(L,*,\mathcal{G})$. Then,*

(1) *For any $M_1 \in \mathcal{G}O(L,e)$, there exists an element M_2 in β_e such that M_1 contains both left and right inverses of M_2.*
(2) *For each $M_1 \in \mathcal{G}O(L,e)$, there exists $M_2 \in \beta_e$ such that $t * M_2, M_2 * t \subseteq M_1$, where t is an arbitrary element of M_1.*

Proof.

(1) As given is that $(L,*,\mathcal{G})$ is a quasi $\mathcal{G}$-s-topological loop, so for all $M_1 \in \mathcal{G}O(L,e)$, there exists $M_2 \in \beta_e$ such that $i_L(M_2) = (M_2)_L^{-1} \subseteq M_1$, and $i_R(M_2) = (M_2)_R^{-1} \subseteq M_1$.
(2) For all $M_1 \in \mathcal{G}O(L,e)$ containing t, there exists $M_2 \in \beta_e$ such that $l_t(M_2) = t * M_2 \subseteq M_1$. Accordingly, $r_t(M_2) = M_2 * t \subseteq M_1 \ \forall t \in M_1$.

□

Theorem 4.9. *Let β_e be a class of $\mathcal{G}$-open nbhds of e_L in a quasi $\mathcal{G}$-s-topological loop $(L,*,\mathcal{G})$. Then,*

(1) *For every element M_1 of β_e and for $t \in L$, if left translation is $\mathcal{G}$-semi-open, then there exists $M_2 \in \beta_e$ such that $(t * M_2) * (t)_R^{-1} \subseteq M_1$.*
(2) *For all $M_1 \in \beta_e$ and $t \in L$, if the right translation is $\mathcal{G}$-semi-open then there exists $M_2 \in \beta_e$ such that $(t)_L^{-1} * (M_2 * t) \subseteq M_1$.*

Proof.

(1) As l_t is $\mathcal{G}$-semi-open and r_t is $\mathcal{G}$-semi-continuous in L, and $l_t(e) = t$. Then there exists $M_2 \in \beta_e$ such that $l_t(M_2) = t * M_2$, $t * M_2$ is a $\mathcal{G}$-semi-open nbhd of t, and $r_{(t)_R^{-1}}(t * M_2) = (t * M_2) * (t)_R^{-1} \subseteq M_1$.

(2) If r_t is $\mathcal{G}$-semi-open, l_t is $\mathcal{G}$-semi-continuous in L, and $r_t(e) = t$. Then there exists $M_2 \in \beta_e$ such that $r_t(M_2) = M_2 * t$, $M_2 * t$ is a $\mathcal{G}$-semi-open nbhd of t, and $l_{(t)_L^{-1}}(M_2 * t) = (t)_L^{-1} * (M_2 * t) \subseteq M_1$.

□

Theorem 4.10. *Let $(L, *)$ be an IP loop and $(L, \mathcal{G})$ is a $\mathcal{G}$-topological space. If for each $M \in \mathcal{G}SO(L, e)$, there exists $N \in \mathcal{G}O(L, e)$ satisfying $N^{-1} \subseteq M$, also r_x and l_x are $\mathcal{G}$-semi-continuous in L then inverse map is also $\mathcal{G}$-semi-continuous in L. Therefore L is a quasi $\mathcal{G}$-s-topological loop.*

Proof. Suppose $t \in L$ and S is a $\mathcal{G}$-open nbhd of t^{-1}. So, there is a $\mathcal{G}$-semi-open nbhd M of e satiating $l_{t^{-1}}(M) = t^{-1} * M \subseteq S$. Then $(l_t)^{-1}(N^{-1}) = l_{t^{-1}}(N^{-1}) = t^{-1} * N^{-1} = (N*t)^{-1}$, where $N*t$ is a $\mathcal{G}$-semi-open nbhd of t and $i(N*t) = (N*t)^{-1} = t^{-1}*N^{-1} \subseteq t^{-1} * M \subseteq S$. Thus inverse map is $\mathcal{G}$-semi-continuous in L. □

Theorem 4.11. *A quasi $\mathcal{G}$-s-topological loop $(L, *, \mathcal{G})$ with inverse property is $\mathcal{G}$-s regular at e, if μ_e is a $\mathcal{G}$-base at e and every $M \in \mu_e$ there exists a symmetric $\mathcal{G}$-open nbhd N of e satisfying $N * N \subseteq M$.*

Proof. Let $t \in \mathcal{G}sCl(N)$ and $t * N$ is a $\mathcal{G}$-semi-open nbhd of t. Clearly, $t * N \cap N \neq \phi$. So, there exist $m, n \in N$ such that $n = t * m$, $t = n * m^{-1} \in N * N^{-1} = N * N \subseteq M$. Thus $\mathcal{G}sCl(N) \subseteq M$. It gives L is $\mathcal{G}$-s regular at e. □

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