

Regularization Methods for Ill-Posed Problems with Monotone Nonlinear Part

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Abstract. We present regularization methods for solving ill-posed Hammerstein type operator equations under weaker conditions than in earlier studies such as [13]. Our semilocal convergence analysis is based on majorizing sequences. Numerical examples where the new convergence criteria are satisfied but the old criteria are not satisfied are also presented at the end of the study.

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1. INTRODUCTION

It is well known that many inverse problems typically leads to mathematical models that are ill-posed (according to Hadamard's definition [16]) in the sense that it is not possible to provide a unique solution.

In this paper we are interested in finding approximation for a solution of the nonlinear ill-posed operator equation of the Hammerstein type(see [10, 11, 12, 15]) equation

$$KF(x) = y \tag{1.1}$$

where $F : D(F) \subset X \rightarrow X$ is monotone and $K : X \rightarrow Y$ a bounded linear operator and X, Y are taken to be Hilbert spaces. Let $U(x, R)$ and $\overline{U}(x, R)$, stand respectively, for the open and closed ball in X with center x and radius $R > 0$. Let $L(X)$ denote the space of all bounded linear operators from X into itself. It is assumed that (1.1) has a solution $\hat{x} \in D(F)$. We are interested in the case when F is a monotone operator (cf. [24]). i.e., $F : D(F) \subset X \rightarrow X$ satisfies

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0, \quad \forall x_1, x_2 \in D(F).$$

A typical example of (1.1) is the Hammerstein equation of the form

$$\int_0^1 k(s, t)x^3(s)ds = y(t), \quad t \in [0, 1], \quad (1.2)$$

where $k(., .)$ is a nondegenerate kernel which is square integrable, that is,

$$\int_0^1 \int_0^1 |k(s, t)|^2 ds dt < \infty,$$

and $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is a suitable function. Note that (1.2) takes the form (1.1) with $K : L^2[0, 1] \rightarrow L^2[0, 1]$ defined by

$$Kx(t) = \int_0^1 k(s, t)x(s)ds, \quad x \in L^2[0, 1], \quad t \in [0, 1]$$

and $F : L^2[0, 1] \rightarrow L^2[0, 1]$ is the nonlinear operator defined by

$$F(x)(s) = x^3(s), \quad s \in [0, 1].$$

We assume throughout that $y^\delta \in Y$ are the available noisy data with

$$\|y - y^\delta\| \leq \delta. \quad (1.3)$$

Observe that (cf. [15]) the solution \hat{x} of (1.1) can be obtained by first solving the linear equation

$$Kz = y \quad (1.4)$$

for z and then solving the nonlinear equation

$$F(x) = z. \quad (1.5)$$

For the treatment of nonlinear ill-posed problems the standard regularization method is the method of Tikhonov regularization. But if the nonlinear operator is monotone then a simpler regularization strategy available is the Lavrentiev regularization. Note that KF need not be monotone even if F is monotone. So in the straight forward approach one has to consider Tikhonov regularization method for approximately solving (1.1).

What we show in this paper is that for the special case when K is linear and F is monotone, by splitting the equation (1.1) into (1.4) and (1.5), one can simplify the procedure by specifying a regularization strategy for linear part (1.4) and an iterative method for nonlinear part (1.5). More precisely, for fixed $\alpha > 0$, $\delta > 0$ we consider the regularized solution of (1.4) with y^δ in place of y as

$$z_\alpha^\delta = (K + \alpha I)^{-1}y^\delta \quad (1.6)$$

if the operator K in (1.4) is positive self adjoint and $X = Y$, otherwise we consider

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*y^\delta. \quad (1.7)$$

Note that (1.6) is the simplified or Lavrentiev regularization (see [8]) of the equation (1.4) and (1.7) is the Tikhonov regularization (see [6, 7, 9, 14, 22, 23]) of (1.4). The regularization parameter is chosen according to an adaptive method proposed by Pereverzev and Schock in [20].

In [15], it is assumed that the bounded inverse of $F'(x_0)$ exist and considered the sequence

$$x_{n+1, \alpha}^\delta = x_{n, \alpha}^\delta - F'(x_0)^{-1}(F(x_{n, \alpha}^\delta) - z_\alpha^\delta), \quad (1.8)$$

with $x_{0, \alpha}^\delta = x_0$ and proved that $(x_{n, \alpha}^\delta)$ converges linearly to the solution x_α^δ of

$$\text{inequation } F(x) = z_\alpha^\delta. \quad (1.9)$$

Later in [12], George and Kunhanadan considered the sequence $(x_{n,\alpha}^\delta)$ defined iteratively as

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_{n,\alpha}^\delta)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta), \quad (1.10)$$

with $x_{0,\alpha}^\delta = x_0$ and proved that $(x_{n,\alpha}^\delta)$ converges quadratically to the solution x_α^δ of (1.9) under the assumption that the bounded inverse of $F'(x)$ exist in a neighborhood of x_0 .

Recall that a sequence (x_n) is X with $\lim x_n = x^*$ is said to converge quadratically, if there exists positive number M , not necessarily less than 1, such that for all n sufficiently large

$$\|x_{n+1} - x^*\| \leq M\|x_n - x^*\|^2. \quad (1.11)$$

If the sequence (x_n) has the property that

$$\|x_{n+1} - x^*\| \leq q\|x_n - x^*\|, \quad 0 < q < 1$$

then (x_n) is said to be linearly convergent. For an extensive discussion of convergence rate, see Ortega and Rheinboldt [19].

Note that the ill-posedness of equation (1.1) in [15] and in [12] is due to the ill-posedness of the linear equation (1.4). In the present paper we assume that (1.1) is ill-posed in both the linear part (1.4) and the nonlinear part (1.5). Using the monotonicity of F , we carry out the convergence analysis by means of suitably constructed majorizing sequences, deviating from the methods used in [15] and [12]. An advantage of this approach is that the majorizing sequence gives an a priori error estimate which can be used to determine the number of iterations needed to achieve a prescribed solution accuracy before actual computation takes place.

In the present paper we expand the applicability of results in [13] using weaker conditions.

2. BACKGROUND

We consider that the operator F satisfies the following conditions:

(C0) There exists $R > 0$ such that $U(\hat{x}, R) \subseteq D(F)$ and F is Fréchet differentiable at all $x \in U(\hat{x}, R)$.

(C1) There exists a constant $L > 0$ such that for every $x, u \in U(\hat{x}, R)$ and $v \in X$, there exists an element $\Phi(x, u, v) \in X$ satisfying

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \quad \|\Phi(x, u, v)\| \leq L\|v\|\|x - u\|$$

for all $x, u \in U(\hat{x}, R)$ and $v \in X$.

(C1)' Let $x_0 \in D(F)$ be fixed. Let $R > 0$ be such that $U(\hat{x}, R) \subseteq D(F)$. There exists a constant $L_0 > 0$ such that for each $u \in U(\hat{x}, R)$ and $v \in X$, there exists an element $P(x, u, v) \in X$ such that

$$[F'(x) - F'(u)]v = F'(u)P(x, u, v), \quad \|P(x, u, v)\| \leq L_0\|v\|(\|x - x_0\| + \|x - u\|).$$

Here is the motivation for the introduction of (C1)'. We note that that since $\|u - x\| \leq \|u - x_0\| + \|x - x_0\|$ condition (C1) always implies (C1)' with $L_0 = L$ and $\Phi = P$ but not necessarily vice versa.

Note that

$$L_0 \leq L$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [1]-[5]. At the end of this study in Section 4 we provide numerical examples where (C1)' but not (C1).

(C2) There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \rightarrow (0, \infty)$ with $a \geq \|K\|^2$ satisfying;

- $\lim_{\lambda \rightarrow 0} \varphi(\lambda) = 0$
- $$\sup_{\lambda \geq 0} \frac{\alpha \varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha), \quad \forall \alpha \in (0, a].$$
- there exists $v \in X$ with $\|v\| \leq 1$ such that

$$F(\hat{x}) = \varphi(K^*K)v \quad (2.1)$$

Let

$$z_\alpha := (K^*K + \alpha I)^{-1} K^* y.$$

Hereafter we consider z_α^δ as in (1.7). We observe that

$$\begin{aligned} \|F(\hat{x}) - z_\alpha^\delta\| &\leq \|F(\hat{x}) - z_\alpha\| + \|z_\alpha - z_\alpha^\delta\| \\ &\leq \|F(\hat{x}) - z_\alpha\| + \frac{\delta}{\sqrt{\alpha}}, \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} F(\hat{x}) - z_\alpha &= F(\hat{x}) - (K^*K + \alpha I)^{-1} K^* K F(\hat{x}) \\ &= [I - (K^*K + \alpha I)^{-1} K^* K] F(\hat{x}) \\ &= \alpha (K^*K + \alpha I)^{-1} F(\hat{x}). \end{aligned} \quad (2.3)$$

So by (C2),

$$\|F(\hat{x}) - z_\alpha\| \leq \varphi(\alpha). \quad (2.4)$$

Thus we have the following theorem.

Theorem 1. *Let z_α^δ be as in (1.7) and the (C2) holds. Then*

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq (\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}). \quad (2.5)$$

2.1. A priori choice of the parameter. Note that the estimate $\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}$ in (2.5) attains minimum for the choice $\alpha := \alpha_\delta$ which satisfies $\varphi(\alpha_\delta) = \frac{\delta}{\sqrt{\alpha_\delta}}$. Let $\psi(\lambda) := \lambda \sqrt{\varphi^{-1}(\lambda)}$, $0 < \lambda \leq \|K\|^2$. Then we have $\delta = \sqrt{\alpha_\delta} \varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, and

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta)). \quad (2.6)$$

So Theorem 1 and the above observation lead to the following.

Theorem 2. *Let $\psi(\lambda) := \lambda \sqrt{\varphi^{-1}(\lambda)}$, $0 < \lambda \leq \|K\|^2$ and the assumptions of Theorem 1 are satisfied. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Then*

$$\|F(\hat{x}) - z_{\alpha_\delta}^\delta\| \leq O(\psi^{-1}(\delta)).$$

2.2. An adaptive choice of the parameter. The error estimate in the above Theorem has optimal order with respect to δ . Unfortunately, an a priori parameter choice (2.6) cannot be used in practice since the smoothness properties of the unknown solution \hat{x} reflected in the function φ are generally unknown. There exist many parameter choice strategies in the literature (cf. [8], [12], [20], [24], etc.).

In [20], Pereverzev and Schock considered an adaptive selection of the parameter which does not involve even the regularization method in an explicit manner. In this method the regularization parameter α_i are selected from some finite set $\{\alpha_i : 0 < \alpha_0 < \alpha_1 < \dots < \alpha_N\}$ and the corresponding regularized solution, say $z_{\alpha_i}^\delta$ are studied on-line. In this paper also, we consider the adaptive method for selecting the parameter α in z_α^δ .

Let $i \in \{0, 1, 2, \dots, N\}$ and $\alpha_i = \mu^{2i}\alpha_0$ where $\mu > 1$ and $\alpha_0 = \delta^2$. Let

$$l := \max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}\} \quad (2.7)$$

and

$$k := \max\{i : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \leq \frac{4\delta}{\sqrt{\alpha_j}}, j = 0, 1, 2, \dots, i\}. \quad (2.8)$$

We will be using the following theorem from [12]

Theorem 3. (cf. [12], Theorem 4.2) Let l be as in (2.7), k be as in (2.8) and $z_{\alpha_k}^\delta$ be as in (1.7) with $\alpha = \alpha_k$. Then $l \leq k$ and

$$\|F(\hat{x}) - z_{\alpha_k}^\delta\| \leq (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta).$$

3. SEMILOCAL CONVERGENCE

Now consider the nonlinear equation (1.5) with $z_{\alpha_k}^\delta$ in place of z . It can be seen as in [24], Theorem 1.1, that for monotone operator F , the equation

$$F(x) + \frac{\alpha_k}{c}(x - x_0) = z_{\alpha_k}^\delta \quad (3.1)$$

where $0 < c < \alpha_k$ has a unique solution $x_{c\alpha_k}^\delta$. It is interesting to note that the presence of regularization parameter α_k , in (3.1) relieves us of the labour of choosing another regularization parameter for Lavrentiev regularization in the nonlinear part. We propose the following iterative method for computing the solution $x_{c\alpha_k}^\delta$. For $n \geq 0$, let

$$x_{n+1, \alpha_k}^\delta = x_{n, \alpha_k}^\delta - (F'(x_{n, \alpha_k}^\delta) + \frac{\alpha_k}{c})^{-1}(F(x_{n, \alpha_k}^\delta) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(x_{n, \alpha_k}^\delta - x_0)), \quad (3.2)$$

where $x_{0, \alpha_k}^\delta := x_0$ is a starting point of the iteration. The main goal of this section is to provide sufficient conditions for the convergence of method (3.2) to $x_{c\alpha_k}^\delta$ and obtain an error estimate for $\|x_{c\alpha_k}^\delta - x_{n, \alpha_k}^\delta\|$. We use a majorizing sequence for proving our results [1]-[5]. We need the following result on majorizing sequences for method (3.2)

Lemma 4. Let $L_0 > 0$ and $\eta > 0$. Suppose that

$$h_0 = 16L_0\eta \leq 1. \quad (3.3)$$

Set

$$q = \frac{1 - \sqrt{1 - h_0}}{2}. \quad (3.4)$$

Then, scalar sequence $\{t_n\}$ given by

$$t_0 = 0, t_1 = \eta, t_{n+1} = t_n + 2L_0(t_n + t_{n-1})(t_n - t_{n-1}), \quad \forall n = 1, 2, \dots \quad (3.5)$$

is increasing, bounded from above by $t^{**} = \frac{\eta}{1-q}$ and converges to its unique least upper bound t^* which satisfies

$$\eta \leq t^* \leq t^{**}. \quad (3.6)$$

Moreover, the following assertion hold for each $n = 1, 2, \dots$

$$0 < t_{n+1} - t_n \leq q(t_n - t_{n-1}) \leq q^n\eta \quad (3.7)$$

and

$$t^* - t_n \leq \frac{q^n}{1-q}\eta. \quad (3.8)$$

Proof. We shall prove (3.7) using mathematical induction. Estimate (3.7) holds for $n = 0$ by the initial conditions. Then, we have by (3.4) and (6.3) that

$$t_2 \leq t_1 + q(t_1 - t_0) = \eta + q\eta = \frac{1 - q^2}{1 - q}\eta \leq t^{**}.$$

Let us assume (3.7) holds for all $k \leq n$. Then, we have by (3.4) that

$$\begin{aligned} t_{k+1} - t_k &= 2L_0(t_k + t_{k-1})(t_k - t_{k-1}) \\ &\leq 2L_0\left(\frac{1 - q^k}{1 - q}\eta + \frac{1 - q^{k-1}}{1 - q}\eta\right)(t_k - t_{k-1}) \\ &\leq \frac{4L_0\eta}{1 - q}(t_k - t_{k-1}) \leq q(t_k - t_{k-1}). \end{aligned}$$

Moreover, we obtain that

$$\begin{aligned} t_{k+1} &\leq t_k + q(t_k - t_{k-1}) \leq \dots \leq \eta + q\eta + \dots + q^k\eta \\ &\leq \frac{1 - q^{k+1}}{1 - q}\eta < t^{**}. \end{aligned}$$

Hence, sequence $\{t_k\}$ is increasing and bounded above by t^{**} and as such it converges to t^* satisfying (3.6). Estimate (3.8) follows from (3.7) by using standard majorization techniques [1]-[5]. The proof of the Lemma is complete, Let

$$R_\alpha(x) = F'(x) + \frac{\alpha_k}{c}. \quad (3.9)$$

□

Next we show the main semilocal convergence result for method (3.2).

Theorem 5. *Under $(C1)'$ and the hypotheses in Lemma 4 with*

$$\|R_\alpha(x_0)^{-1}(F(x_0) - z_{\alpha_k}^\delta)\| \leq \eta, \quad (3.10)$$

further suppose that $t^ \leq q$. Then, sequence $\{x_{n,\alpha_k}^\delta\}$ generated by (3.2) is well defined, remains in $U(x_0, t^*)$ for each $n = 0, 1, 2, \dots$ and converges to a solution $x_{c\alpha_k}^\delta \in \overline{U}(x_0, t^*)$ of equation (3.1). Moreover, the following estimates hold for each $n = 0, 1, 2, \dots$*

$$\|x_{n+1,\alpha_k}^\delta - x_{n,\alpha_k}^\delta\| \leq a_n \|x_{n,\alpha_k}^\delta - x_{n-1,\alpha_k}^\delta\| \quad (3.11)$$

$$\|x_{n+1,\alpha_k}^\delta - x_{n,\alpha_k}^\delta\| \leq t_{n+1} - t_n \quad (3.12)$$

and

$$\|x_{n,\alpha_k}^\delta - x_{c\alpha_k}^\delta\| \leq t^* - t_n, \quad (3.13)$$

where

$$a_n = L_0 \left\{ \int_0^1 \|x_{n,\alpha_k}^\delta - x_0 + \theta(x_{n-1,\alpha_k}^\delta - x_{n,\alpha_k}^\delta)\| d\theta + 2\|x_{n,\alpha_k}^\delta - x_0\| + \|x_{n-1,\alpha_k}^\delta - x_0\| \right\}. \quad (3.14)$$

Proof. Let $G(x) = x - R_\alpha(x)^{-1}[F(x) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(x - x_0)]$. Using (3.2), we have, we have for $u, v \in U(x_0, t^*)$,

$$\begin{aligned}
G(u) - G(v) &= u - v - R_\alpha(u)^{-1}[F(u) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(u - x_0)] \\
&\quad + R_\alpha(v)^{-1}[F(v) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(v - x_0)] \\
&= u - v - [R_\alpha(u)^{-1} - R_\alpha(v)^{-1}](F(v) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(v - x_0)) \\
&\quad - R_\alpha(u)^{-1}(F(u) - F(v) + \frac{\alpha_k}{c}(u - v)) \\
&= R_\alpha(u)^{-1}[F'(u)(u - v) - (F(u) - F(v))] \\
&\quad - R_\alpha(u)^{-1}[F'(v) - F'(u)]R_\alpha(v)^{-1}(F(v) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(v - x_0)) \\
&= R_\alpha(u)^{-1}[F'(u)(u - v) - (F(u) - F(v))] \\
&\quad + R_\alpha(u)^{-1}[F'(v) - F'(u)](v - G(v)) \\
&= R_\alpha(u)^{-1}[F'(u)(u - v) + \int_0^1 (F'(u + t(v - u))(v - u)dt] \\
&\quad + R_\alpha(u)^{-1}[F'(v) - F'(u)](v - G(v)) \\
&= \int_0^1 R_\alpha(u)^{-1}[(F'(u + t(v - u)) - F'(u))(v - u)dt] \\
&\quad + R_\alpha(u)^{-1}[F'(v) - F'(u)](v - G(v)). \tag{3.15}
\end{aligned}$$

Let in particular $u = x_{n, \alpha_k}^\delta$ and $v = x_{n-1, \alpha_k}^\delta$. Then by (C1)' and $\|R(u)^{-1}F'(u)\| \leq 1$, we obtain (3.11).

Next, we shall prove that the sequence $\{t_n\}$ defined in Lemma 4 is a majorizing sequence of the sequence $\{x_{n, \alpha_k}^\delta\}$.

Note that $\|x_{1, \alpha_k}^\delta - x_0\| = \|R_\alpha(x_0)^{-1}(F(x_0) - z_{\alpha_k}^\delta)\| \leq \eta = t_1 - t_0$. Assume that $\|x_{i+1, \alpha_k}^\delta - x_{i, \alpha_k}^\delta\| \leq t_{i+1} - t_i$ for all $i \leq k$ for some k . Then by (3.11)

$$\|x_{k+2, \alpha_k}^\delta - x_{k+1, \alpha_k}^\delta\| \leq a_k \|x_{k+1, \alpha_k}^\delta - x_{k, \alpha_k}^\delta\| \leq t_{k+2} - t_{k+1}. \tag{3.16}$$

Thus by induction $\|x_{n+1, \alpha_k}^\delta - x_{n, \alpha_k}^\delta\| \leq t_{n+1} - t_n$ for all $n \geq 0$. Hence, it follows from Lemma 4 $\{t_n\}, n \geq 0$ is a majorizing sequence of $\{x_{n, \alpha_k}^\delta\}$. $\{x_{n, \alpha_k}^\delta\}, n \geq 0$ is a complete sequence in a Hilbert space and as such it converges to some $x_{c\alpha_k}^\delta \in \overline{U(x_0, t^*)}$ (since and $\overline{U(x_0, t^*)}$ is a closed set). Now by letting $n \rightarrow \infty$ in (3.2) we obtain $F(x_{c\alpha_k}^\delta) = z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(x_0 - x_{c\alpha_k}^\delta)$. Estimate (3.13) follows from (3.12) by using standard majorization techniques [1]- [5]. This completes the proof of the Theorem. \square

Remark 6. The convergence order of method (1.10) is two [13]. In Theorem 5 the error bounds are too pessimistic. That is why in practice we shall use the computational order of convergence (COC) (see eg. [5]) defined by

$$\varrho \approx \ln \left(\frac{\|x_{n+1} - x_\alpha^\delta\|}{\|x_n - x_\alpha^\delta\|} \right) / \ln \left(\frac{\|x_n - x_\alpha^\delta\|}{\|x_{n-1} - x_\alpha^\delta\|} \right).$$

The (COC) ϱ will then be close to 2 which is the order of convergence for(1.10).

3.1. Linear Convergence. In this subsection, we consider the sequence (\tilde{x}_n^δ) defined iteratively by

$$\tilde{x}_{n+1}^\delta := \tilde{x}_n^\delta - (F'(x_0) + \frac{\alpha_k}{c}I)^{-1}(F(\tilde{x}_n^\delta) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(\tilde{x}_n^\delta - x_0)), \quad (3.17)$$

where $\tilde{x}_0^\delta = x_0$ is a starting point of the iteration. We prove that the sequence (\tilde{x}_n^δ) converge to the unique solution $x_{c\alpha_k}^\delta$ of (3.1) and obtain an error estimate for $\|x_{c\alpha_k}^\delta - \tilde{x}_n^\delta\|$. The proof of the following lemma is analogous to the proof of Lemma 4.

Lemma 7. *Assume there exist $\tilde{r} \in [0, 1)$ and nonnegative numbers L_0, η, α such that*

$$\frac{L_0}{(1 - \tilde{r})}\eta \leq \tilde{r}. \quad (3.18)$$

Then the sequence (\tilde{t}_n) defined by

$$\tilde{t}_{n+1} = \tilde{t}_n + \frac{L_0}{(1 - \tilde{r})}\eta(\tilde{t}_n - \tilde{t}_{n-1}) \quad (3.19)$$

*is increasing, bounded above by $\tilde{t}^{**} := \frac{\eta}{1 - \tilde{r}}$, and converges to some \tilde{t}^* such that $0 < \tilde{t}^* \leq \frac{\eta}{1 - \tilde{r}}$. Moreover, for $n \geq 0$;*

$$0 \leq \tilde{t}_{n+1} - \tilde{t}_n \leq \tilde{r}(\tilde{t}_n - \tilde{t}_{n-1}) \leq \tilde{r}^n \eta, \quad (3.20)$$

and

$$\tilde{t}^* - \tilde{t}_n \leq \frac{\tilde{r}^n}{1 - \tilde{r}}\eta. \quad (3.21)$$

We shall assume that

$$\begin{aligned} \frac{L_0}{2}\rho^2 + \rho + (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta) &\leq \eta \\ &\leq \min\{r(1 - \tilde{r}), \frac{\tilde{r}(1 - \tilde{r})}{L_0}\}. \end{aligned} \quad (3.22)$$

Let

$$\tilde{G}(x) := x - R_\alpha(x_0)^{-1}[F(x) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(x - x_0)]. \quad (3.23)$$

Note that with the above notation, $\tilde{G}(\tilde{x}_n^\delta) = \tilde{x}_{n+1}^\delta$ and $\|\tilde{R}(x_0)^{-1}\| \leq 1$.

Theorem 8. *Suppose (C0) and (C1)' hold. Let the assumptions in Lemma 7 are satisfied with η as in (3.22). Then the sequence (\tilde{x}_n^δ) defined in (3.17) is well defined and $\tilde{x}_n^\delta \in U(x_0, \tilde{t}^*)$ for all $n \geq 0$. Further (\tilde{x}_n^δ) is Cauchy sequence in $U(x_0, \tilde{t}^*)$ and hence converges to $x_{c\alpha_k}^\delta \in \overline{U(x_0, \tilde{t}^*)} \subset U(x_0, \tilde{t}^{**})$ and $F(x_{c\alpha_k}^\delta) + \frac{\alpha_k}{c}(x_{c\alpha_k}^\delta - x_0) = z_{\alpha_k}^\delta$.*

Moreover, the following estimates hold for all $n \geq 0$,

$$\|\tilde{x}_{n+1}^\delta - \tilde{x}_n^\delta\| \leq \tilde{t}_{n+1} - \tilde{t}_n, \quad (3.24)$$

and

$$\|\tilde{x}_n^\delta - x_{c\alpha_k}^\delta\| \leq \tilde{t}^* - \tilde{t}_n \leq \frac{\tilde{r}^n \eta}{1 - \tilde{r}}. \quad (3.25)$$

Proof. Let \tilde{G} be as in (3.23). Then for $u, v \in U(x_0, \tilde{t}^*)$,

$$\begin{aligned}
\tilde{G}(u) - \tilde{G}(v) &= u - v - R_\alpha(x_0)^{-1}[F(u) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(u - x_0)] \\
&\quad + R_\alpha(x_0)^{-1}[F(v) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(v - x_0)] \\
&= R_\alpha(x_0)^{-1}[R_\alpha(x_0)(u - v) - (F(u) - F(v))] + R_\alpha(x_0)^{-1}\frac{\alpha_k}{c}(v - u) \\
&= R_\alpha(x_0)^{-1}[F'(x_0)(u - v) - (F(u) - F(v)) + \frac{\alpha_k}{c}(u - v)] \\
&\quad + R_\alpha(x_0)^{-1}\frac{\alpha_k}{c}(v - u) \\
&= R_\alpha(x_0)^{-1}[F'(x_0)(u - v) - (F(u) - F(v))]
\end{aligned}$$

Thus by (C1)' we have

$$\|\tilde{G}(u) - \tilde{G}(v)\| \leq L_0 \tilde{t}^* \|u - v\|. \quad (3.26)$$

The rest of the proof is analogous to the proof of Theorem 5. \square

Remark 9. Now by taking $u = x_{c\alpha_k}^\delta$ and $v = \tilde{x}_{n-1}$ in (3.2), we obtain linear convergence of \tilde{x}_n to $x_{c\alpha_k}^\delta$.

Remark 10. For the remainder of the paper we shall consider only the quadratically convergent sequence (x_{n,α_k}^δ) defined in (3.2) for detailed analysis. The results verbatim hold good in the case of linearly convergent sequence (\tilde{x}_n^δ) defined in (3.17).

Remark 11. (a) The semilocal convergence condition given in [13] under the (C1) conditions are given by

$$\frac{3L\eta}{2} \leq q_1 \quad \text{and} \quad h = \frac{3L\eta}{2} \leq 1. \quad (3.27)$$

By comparing (3.3) and (3.27) we see that

$$\frac{h_0}{h} \rightarrow 0 \quad \text{as} \quad \frac{L_0}{L} \rightarrow 0.$$

Hence, under our convergence criteria the applicability of method (3.2) is expanded infinitely many times.

- (b) If $L_0 = L$ the results for method (3.2) coincide with the ones in [13] (with $c = \alpha_k$). Otherwise, i.e., if $L_0 < L$ our results constitute an improvement. Note that in this case the differences $t_{n+1} - t_n$ are tighter than in [13] since we use L_0 instead of L and $L_0 < L$.

4. ERROR BOUNDS UNDER SOURCE CONDITIONS

(C3) There exists a continuous, strictly monotonically increasing function $\varphi_1 : (0, b] \rightarrow (0, \infty)$ with $b \geq \|F'(x_0)\|$ satisfying;

- $\lim_{\lambda \rightarrow 0} \varphi_1(\lambda) = 0$,
-

$$\sup_{\lambda \geq 0} \frac{\alpha \varphi_1(\lambda)}{\lambda + \alpha} \leq \varphi_1(\alpha) \quad \forall \lambda \in (0, b]$$

and

- there exists $v \in X$ with $\|v\| \leq 1$ (cf. [18]) such that

$$x_0 - \hat{x} = \varphi_1(F'(x_0))v.$$

- for each $x \in U(x_0, R)$ there exists a bounded linear operator $G(x, x_0)$ (cf.[21]) such that

$$F'(x) = F'(x_0)G(x, x_0)$$

with $\|G(x, x_0)\| \leq k_1$.

Assume that $k_1 < \frac{1-L_0R}{1-c}$ and for the sake of simplicity assume that $\varphi_1(\alpha) \leq \varphi(\alpha)$ for $\alpha > 0$.

Theorem 12. Suppose $x_{c\alpha_k}^\delta$ is the solution of

$$F(x) + \frac{\alpha_k}{c}(x - x_0) = z_{\alpha_k}^\delta$$

and (C1)' and (C3) holds. Then

$$\|\hat{x} - x_{c\alpha_k}^\delta\| \leq \frac{\varphi_1(\alpha_k) + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1 - (1-c)k_1 - L_0R}.$$

Proof. Note that $c(F(x_{c\alpha_k}^\delta) - z_{\alpha_k}^\delta) + \alpha_k(x_{c\alpha_k}^\delta - x_0) = 0$, so

$$\begin{aligned} \|x_{c\alpha_k}^\delta - \hat{x}\| &\leq \|\alpha_k(F'(x_0) + \alpha_k I)^{-1}(x_0 - \hat{x})\| \\ &\quad + \|(F'(x_0) + \alpha_k I)^{-1}c(F(\hat{x}) - z_{\alpha_k}^\delta)\| \\ &\quad + \|(F'(x_0) + \alpha_k I)^{-1}[F'(x_0)(x_{c\alpha_k}^\delta - \hat{x}) \\ &\quad - c(F(x_{c\alpha_k}^\delta) - F(\hat{x}))]\| \\ &\leq \|\alpha_k(F'(x_0) + \alpha_k I)^{-1}(x_0 - \hat{x})\| \\ &\quad + \|F(\hat{x}) - z_{\alpha_k}^\delta\| + \Gamma \end{aligned} \quad (4.1)$$

where $\Gamma := \|(F'(x_0) + \alpha_k I)^{-1} \int_0^1 [F'(x_0) - cF'(\hat{x} + t(x_{c\alpha_k}^\delta - \hat{x}))](x_{c\alpha_k}^\delta - \hat{x}) dt\|$. So by (C3), we obtain

$$\begin{aligned} \Gamma &\leq \|(F'(x_0) + \alpha_k I)^{-1}s_1\| \\ &\quad + (1-c)\|(F'(x_0) + \alpha_k I)^{-1}s_2\| \\ &\leq L_0R\|x_{c\alpha_k}^\delta - \hat{x}\| + (1-c)k_1\|x_{c\alpha_k}^\delta - \hat{x}\| \end{aligned} \quad (4.2)$$

where

$$\begin{aligned} s_1 &:= \int_0^1 [F'(x_0) - F'(\hat{x} + t(x_{c\alpha_k}^\delta - \hat{x}))](x_{c\alpha_k}^\delta - \hat{x}) dt, \\ s_2 &:= F'(x_0) \int_0^1 G(\hat{x} + t(x_{c\alpha_k}^\delta - \hat{x}), x_0)(x_{c\alpha_k}^\delta - \hat{x}) dt. \end{aligned}$$

Hence by (4.1) and (4.2) we have

$$\begin{aligned} \|x_{c\alpha_k}^\delta - \hat{x}\| &\leq \frac{\tau_{x_0}}{1 - (1-c)k_1 - L_0R} \\ &\leq \frac{\varphi_1(\alpha_k) + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1 - (1-c)k_1 - L_0R}, \end{aligned}$$

where

$$\tau_{x_0} := \|\alpha_k(F'(x_0) + \alpha_k I)^{-1}(x_0 - \hat{x})\| + \|F(\hat{x}) - z_{\alpha_k}^\delta\|.$$

This completes the proof of the theorem.

The following Theorem is a consequence of Theorem 5 and Theorem 12. \square

Theorem 13. Let x_n be defined as in (3.2). If all assumptions of the Theorems 5 and 12 are fulfilled, then

$$\|\hat{x} - x_n\| \leq \frac{q^n \eta}{1 - q} + O(\psi^{-1}(\delta)).$$

4.1. **Stopping index.** Let

$$n_k = \min\{n : q^n \leq \frac{1}{\mu^k}\}. \quad (4.3)$$

Then we have the following

Theorem 14. Let $x_{c\alpha_k}^\delta$ be the unique solution of (3.1) and x_{n,α_k}^δ be as in (3.2). Let the assumptions in Theorem 1, (C1), (C1)', (C2) be satisfied. Let n_k be as in (4.3). Then we have

$$\|x_{n_k,\alpha_k}^\delta - \hat{x}\| = O(\psi^{-1}(\delta)). \quad (4.4)$$

5. IMPLEMENTATION OF ADAPTIVE CHOICE RULE

The main goal of this section is to provide a starting point for the iteration approximating the unique solution $x_{c\alpha}^\delta$ of (3.1) and then to provide an algorithm for the determination of a parameter fulfilling the balancing principle (2.8).

For $i, j \in \{0, 1, 2, \dots, N\}$, we have

$$z_{\alpha_i}^\delta - z_{\alpha_j}^\delta = (\alpha_j - \alpha_i)(K^*K + \alpha_i I)^{-1}(K^*K + \alpha_j I)^{-1}K^*y^\delta.$$

The implementation of our method involves the following steps:

Step I

- $i=1$
- Solve for $w_i : (K^*K + \alpha_i I)w_i = K^*y^\delta$
- Solve for $z_{i,j} : (K^*K + \alpha_i I)^{-1}z_{i,j} = (\alpha_j - \alpha_i)w_i, j \leq i$
- If $\|z_{i,j}\| > \frac{4}{\mu^j}$, then take $k = i - 1$.
- Otherwise, repeat with $i + 1$ in place of i .

Step II

- Choose $x_0 \in D(F)$ such that $\|R_\alpha(x_0)^{-1}(F(x_0) - z_{\alpha_k}^\delta)\| \leq \eta$ for some $\eta > 0$.
- Choose $q \leq \frac{1 - \sqrt{1 - 16L_0\eta}}{2}$

Step III

- $n = 1$
- If $q^n \leq \frac{1}{\mu^k}$, then take $n_k := n$
- Otherwise, repeat with $n + 1$ in place of n

Step IV

- Solve $x_{j,\alpha_k}^\delta : (F'(x_{j-1,\alpha_k}^\delta) + \frac{\alpha_k}{c}I)(x_{j,\alpha_k}^\delta - x_{j-1,\alpha_k}^\delta) = F(x_{j-1,\alpha_k}^\delta) - w_k + \frac{\alpha_k}{c}(x_{j-1,\alpha_k}^\delta - x_0)$ for $j = 1, 2, \dots, n_k$.

6. EXAMPLES

In this section we present two examples where (C1) is not satisfied but (C1)' is satisfied.

Example 15. Let $X = Y = \mathbb{R}$, $D = [0, \infty)$, $x_0 = 1$ and define function F on D by

$$F(x) = \frac{x^{1+\frac{1}{i}}}{1+\frac{1}{i}} + c_1x + c_2, \quad (6.1)$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D . However central Lipschitz condition $(C2)'$ holds for $L_0 = 1$.

Indeed, we have

$$\begin{aligned} \|F'(x) - F'(x_0)\| &= |x^{1/i} - x_0^{1/i}| \\ &= \frac{|x - x_0|}{x_0^{\frac{i-1}{i}} + \dots + x^{\frac{i-1}{i}}} \end{aligned}$$

so

$$\|F'(x) - F'(x_0)\| \leq L_0|x - x_0|.$$

Example 16. We consider the integral equations

$$u(s) = f(s) + \lambda \int_a^b G(s, t)u(t)^{1+1/n} dt, \quad n \in \mathbb{N}. \quad (6.2)$$

Here, f is a given continuous function satisfying $f(s) > 0, s \in [a, b], \lambda$ is a real number, and the kernel G is continuous and positive in $[a, b] \times [a, b]$.

For example, when $G(s, t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$\begin{aligned} u'' &= \lambda u^{1+1/n} \\ u(a) &= f(a), u(b) = f(b). \end{aligned}$$

These type of problems have been considered in [1], [3], [17].

Equation of the form (6.2) generalize equations of the form

$$u(s) = \int_a^b G(s, t)u(t)^n dt \quad (6.3)$$

studied in [1], [3], [17]. Instead of (6.2) we can try to solve the equation $F(u) = 0$ where

$$F : \Omega \subseteq C[a, b] \rightarrow C[a, b], \Omega = \{u \in C[a, b] : u(s) \geq 0, s \in [a, b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \lambda \int_a^b G(s, t)u(t)^{1+1/n} dt.$$

The norm we consider is the max-norm.

The derivative F' is given by

$$F'(u)v(s) = v(s) - \lambda \left(1 + \frac{1}{n}\right) \int_a^b G(s, t)u(t)^{1/n}v(t) dt, \quad v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω . Let us consider, for instance, $[a, b] = [0, 1], G(s, t) = 1$ and $y(t) = 0$. Then $F'(y)v(s) = v(s)$ and

$$\|F'(x) - F'(y)\| = |\lambda| \left(1 + \frac{1}{n}\right) \int_a^b x(t)^{1/n} dt.$$

If F' were a Lipschitz function, then

$$\|F'(x) - F'(y)\| \leq L_1 \|x - y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n} dt \leq L_2 \max_{x \in [0,1]} x(s), \quad (6.4)$$

would hold for all $x \in \Omega$ and for a constant L_2 . But this is not true. Consider, for example, the functions

$$x_j(t) = \frac{t}{j}, \quad j \geq 1, \quad t \in [0, 1].$$

If these are substituted into (6.4)

$$\frac{1}{j^{1/n}(1+1/n)} \leq \frac{L_2}{j} \Leftrightarrow j^{1-1/n} \leq L_2(1+1/n), \quad \forall j \geq 1.$$

This inequality is not true when $j \rightarrow \infty$.

Therefore, condition (6.4) is not satisfied in this case. However, condition (C1)' holds. To show this, let $x_0(t) = f(t)$ and $\gamma = \min_{s \in [a,b]} f(s)$, $\alpha > 0$. Then for $v \in \Omega$,

$$\begin{aligned} \|[F'(x) - F'(x_0)]v\| &= |\lambda| \left(1 + \frac{1}{n}\right) \max_{s \in [a,b]} \left| \int_a^b G(s,t) (x(t)^{1/n} - f(t)^{1/n}) v(t) dt \right| \\ &\leq |\lambda| \left(1 + \frac{1}{n}\right) \max_{s \in [a,b]} G_n(s,t) \end{aligned}$$

where $G_n(s,t) = \frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n} + x(t)^{(n-2)/n} f(t)^{1/n} + \dots + f(t)^{(n-1)/n}} \|v\|$.

Hence,

$$\begin{aligned} \|[F'(x) - F'(x_0)]v\| &= \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}} \max_{s \in [a,b]} \int_a^b G(s,t) dt \|x - x_0\| \\ &\leq L_0 \|x - x_0\|, \end{aligned}$$

where $L_0 = \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}} N$ and $N = \max_{s \in [a,b]} \int_a^b G(s,t) dt$. Then condition (C1)' holds for sufficiently small λ .

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