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New Integral Inequalities of the Type of Hermite-Hadamard Through Quasi Convexity

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Abstract. In this paper, we establish some new integral inequalities of Hermite Hadamard type for twice differentiable functions through quasi convexity by using Riemann-Liouville fractional integrals. Applications to special means of real numbers are also given.

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1. Introduction

Let $f : \Phi \neq I \subseteq \mathrm{R} \to \mathrm{R}$ be a function defined on the interval I of real numbers. Then f is called convex, if

$$f\left(\lambda x + (1-\lambda)\, y\right) \leq \lambda f\left(x\right) + (1-\lambda)\, f\left(y\right),$$

for all $x, y \in I$ and $\lambda \in [0,1]$. Geometrically, this means that if P, Q and R are three distinct points on graph of f with Q between P and R, then Q is on or below chord PR. There are many results associated with convex functions in the area of inequalities.

The notion of quasi-convex functions generalized the notion of convex functions. More precisely, a function $f : [a,b] \to \mathrm{R}$ is said to be quasi-convex on $[a,b]$, if

$$f\left(\lambda x + (1-\lambda)\, y\right) \leq \max\left\{f\left(x\right), f\left(y\right)\right\}, \quad \forall x, y \in [a,b].$$

Any convex function is a quasi-convex function but the converse is not true. Because there exist quasi-convex functions which is not convex, (see [2]). For example, the function $f : \mathrm{R}^{+} \to \mathrm{R}$, defined by $f\left(x\right) = \ln x, \quad x \in \mathrm{R}^{+}$ is quasi-convex. However f is not a convex function.

There are many results associated with convex functions in the area of inequalities, but one of those is the classical Hermite Hadamard inequality. This inequality is defined as: Let $f : I \subset \mathrm{R} \to \mathrm{R}$ be a convex function defined on the interval $I$of real numbers with $a, b \in I$ and $a < b$. Then f satisfies the following well-known Hermite Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1. 1}$$

for $a, b \in I$, with $a < b$.

For several recent results concerning the above inequality (1.1) we refer the interested reader to [1, 4, 8, 9, 10, 11, 12].

Recently, D. A. Ion [14] obtained the following two inequalities of the right hand side of Hermite-Hadamard's type functions whose derivatives in absolute values are quasi-convex.

Theorem 1. *Let $f : I^0 \subseteq \mathrm{R} \to \mathrm{R}$ be a differentiable function on I^0 with $a, b \in I^0$ and $a < b$. If $|f'|$ is a quasi-convex function on $[a, b]$, then we have:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\max\left[\left|f'(a)\right|, \left|f'(b)\right|\right]. \tag{1. 2}$$

Theorem 2. *Let $f : I^0 \subseteq \mathrm{R} \to \mathrm{R}$ be a differentiable function on I^0 with $a, b \in I^0$ and $a < b$. If $|f'|^p$ is a quasi-convex function on $[a, b]$ for some fixed $p > 1$, then we have:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \frac{1}{b-a}\int_a^b f(x)g(x)\,dx\right| \\ \leq\ & \frac{b-a}{2(p+1)^{1/p}}\left[\max\left\{\left|f'(a)\right|^{p/(p-1)}, \left|f'(b)\right|^{p/(p-1)}\right\}\right]^{(p-1)/p}.\end{aligned} \tag{1. 3}$$

In [2], Alomari, Draus and Kirmaci established the following Hermite-Hadamard inequalities for quasi-convex functions which give refinements of above Theorems 1 and 2.

Theorem 3. *Let $f : I^0 \subseteq \mathrm{R} \to \mathrm{R}$ be a differentiable function on I^0 with $a, b \in I^0$ and $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then we have*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{8}\left[\max\left\{\left|f'(a)\right|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\}\right. \\ &+ \left.\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, \left|f'(b)\right|\right\}\right]\end{aligned} \tag{1. 4}$$

Theorem 4. *Let $f : I^0 \subseteq \mathrm{R} \to \mathrm{R}$ be a differentiable function on I^0 with $a, b \in I^0$ and $a < b$. If $|f'|^q$ is quasi-convex on $[a, b]$ and $p > 1$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)}{4}\left(\frac{1}{(1+p)}\right)^{1/p}\left[\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}},\left|f'(a)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right.$$
$$\left.+\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right] \quad (1.\ 5)$$

Theorem 5. *Let $f : I^0 \subseteq \mathrm{R} \to \mathrm{R}$ be a differentiable function on I^0 with $a, b \in I^0$ and $a < b$. If $|f'(x)|$ is quasi-convex on $[a, b]$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left[\left\{max(|f'(a)|^q,|f'(\frac{a+b}{2})|^q)\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{max(|f'(\frac{a+b}{2})|^q,|f'(b)|^q)\right\}^{\frac{1}{q}}\right] \quad (1.\ 6)$$

Alomari, Darus and Dragomir in [3] introduced the following theorems for twice differentiable quasi-convex functions.

Theorem 6. *Let $f : I^0 \subseteq \mathrm{R} \to \mathrm{R}$ be a differentiable function on I^0 with $a, b \in I^0$ and $a < b$. If $|f''|$ is quasi-convex on $[a, b]$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\max\left\{\left|f''(a)\right|,\left|f''(b)\right|\right\}. \quad (1.\ 7)$$

Theorem 7. *Let $f : I^0 \subseteq \mathrm{R} \to \mathrm{R}$ be a differentiable function on I^0 with $a, b \in I^0$ and $a < b$. If $|f''|^q$ is quasi-convex on $[a, b]$ and $q \geq 1$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{1/q} \quad (1.\ 8)$$

In [13], R. Gorenflo, F. Mainardi defined the Riemann-Liouville fractional integrals as: Let $f \in L_1[a, b]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $\alpha \geq 0$ are defined by

$$J^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt, \quad (a<x),$$

and

$$J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt, \quad (x<b),$$

respectively. Here $\Gamma(\alpha)=\int\limits_0^{\infty} e^{-u}u^{\alpha-1}du$ and $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$.
Note that if $\alpha = 1$, the fractional integral reduces to the classical integral.

In this paper, we establish some new integral inequalities of Hermite Hadamard type for twice differentiable functions through quasi convexity by using Riemann-Liouville fractional integrals. Applications to special means of real numbers are also given.

2. Main results

In order to prove our main results, we use the following Lemma of [17].

Lemma 8. *Suppose $f : [a,b] \to \mathrm{R}$ be a twice differentiable mapping on (a,b) with $a < b$. If $f'' \in L[a,b]$, then we have the following fractional integral inequality:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \\ &\leq \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-\lambda)^{\alpha+1}-\lambda^{\alpha+1}}{\alpha+1} f''(\lambda a + (1-\lambda)b)\, d\lambda. \end{aligned} \tag{2.1}$$

Theorem 9. *Suppose $f : [a,b] \to \mathrm{R}$ be a differentiable mapping on (a,b) with $a < b$ and $f'' \in L_1[a,b]$. If $|f''|$ is quasi-convex on $[a,b]$, for $\alpha > 0$, then we have the following fractional integral inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \leq \frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)} \max\{|f''(a)|, |f''(b)|\}. \tag{2.2}$$

Proof. Using Lemma 8 and quasi convexity of $|f''|$ on $[a,b]$, we get

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ &\leq \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-\lambda)^{\alpha+1}-\lambda^{\alpha+1}}{\alpha+1} |f''(\lambda a + (1-\lambda)b)|\, d\lambda. \\ &\leq \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-\lambda)^{\alpha+1}-\lambda^{\alpha+1}}{\alpha+1} \max\{|f''(a)|, |f''(b)|\} d\lambda \\ &= \frac{(b-a)^2}{2} \max\{|f''(a)|, |f''(b)|\} \int_0^1 \frac{1-(1-\lambda)^{\alpha+1}-\lambda^{\alpha+1}}{\alpha+1} d\lambda \\ &= \frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)} \max\{|f''(a)|, |f''(b)|\}. \end{aligned}$$

The proof is completed. □

Note that, If we take $\alpha = 1$, in above Theorem 9 with the properties of gamma functions, we get inequality (1.7).

Theorem 10. *Suppose $f : [a,b] \to \mathrm{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $f'' \in L_1[a,b]$. If $|f''|^q$ is quasi-convex on $[a,b]$, and $p > 1$, then we have the following fractional integral inequality:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ &\leq \frac{(b-a)^2}{2(\alpha+1)}\left(1 - \frac{2}{p(\alpha+1)+1}\right)^{1/p} \left(\max\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}, \end{aligned} \tag{2.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 8, we get

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ &\leq \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-\lambda)^{\alpha+1}-\lambda^{\alpha+1}}{\alpha+1} |f''(\lambda a + (1-\lambda)b)|\, d\lambda. \end{aligned}$$

By Holders inequality and quasi convexity of $|f''|$ on $[a,b]$ with $\lambda \in [0,1]$, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\\ &\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1\left(1-(1-\lambda)^{\alpha+1}-\lambda^{\alpha+1}\right)^p d\lambda\right)^{1/p}\left(\int_0^1 f''\left|\lambda a+(1-\lambda)b\right|^q d\lambda\right)^{1/q}\\ &\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1\left(1-(1-\lambda)^{\alpha+1}-\lambda^{\alpha+1}\right)^p d\lambda\right)^{1/p}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\\ &=\frac{(b-a)^2}{2(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{1/p}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\end{aligned}$$

The proof is completed. □

Theorem 11. *Suppose $f:[a,b]\to \mathrm{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f''\in L_1[a,b]$. If $|f''|^q$ is quasi-convex on $[a,b]$ and $q\geq 1$, then we have the following fractional integral inequality:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\\ &\leq \frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}.\end{aligned}\tag{2.4}$$

Proof. Using Lemma 8 and Holder's inequality, we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\\ &\leq \frac{(b-a)^2}{2}\int_0^1\frac{1-(1-\lambda)^{\alpha+1}-\lambda^{\alpha+1}}{\alpha+1}\left|f''(\lambda a+(1-\lambda)b)\right|d\lambda\\ &\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1\left(1-(1-\lambda)^{\alpha+1}-\lambda^{\alpha+1}\right)^p d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f''(\lambda a+(1-\lambda)b)\right|^q d\lambda\right)^{1/q}\end{aligned}$$

Using quasi-convexity of $|f''|$ on $[a,b]$ and $\lambda\in[0,1]$, we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\\ &\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1\left(1-(1-\lambda)^{\alpha+1}-\lambda^{\alpha+1}\right)^p d\lambda\right)^{1-\frac{1}{q}}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\\ &=\frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}.\end{aligned}$$

The proof is completed . □

Note that, If we take $\alpha=1$, in above Theorem 11 with the properties of gamma functions, we get inequality (1.8).

3. Applications to Some Special Means

First we recall the arithmetic mean A(a, b), logarithmic mean $L(a,b)$and p-logarithmic mean $L_p(a,b)$ for arbitrary real numbers a and b as follows:

A = A (a, b): = $\frac{a+b}{2}$, a, b >0,

$$L=L(a,b)=\begin{cases} a\ , & ifa=b\\ \frac{b-a}{\ln b-\ln a}\ , & ifa\neq b\end{cases}\quad ,\quad a,b\quad >0$$

$$L_p\equiv L_p(a,b)=\begin{cases} a\ , & ifa=b\\ \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\ , & ifa\neq b\end{cases}\quad ,$$

$p\quad\in\quad \mathrm{Re}\quad\setminus\{-1,0\}$: $a,b\quad >0$

Now we present some new inequalities for the above means by using the results of sections 2.

The following proposition follows from Theorem 9 applied to quasi-convex mapping $f(x) = x^n, \quad x \in \mathrm{N}$ and $\alpha = 1$.

Proposition 12. *Let* $a, b \in \mathrm{R}^+$, $0 < a < b$, *and* $n \in \mathrm{N}$. *Then, we have*
$|A(a^n, b^n) - L_n^n(a,b)| \leq \frac{n(n-1)}{12}(b-a)^2 \max\left\{|a|^{n-2}, |b|^{n-2}\right\}.$

The following proposition follows from Theorem 11 applied to the mapping $f(x) = x^n, \quad x \in \mathrm{N}$ and $\alpha = 1$.

Proposition 13. *Let* $a, b \in \mathrm{R}^+$, $0 < a < b$, *and* $n \in \mathrm{N}$. *Then for all* $q \geq 1$, *we have*
$|A(a^n, b^n) - L_n^n(a,b)| \leq \frac{n(n-1)}{12}(b-a)^2 \left(\max\left\{|a|^{(n-2)q}, |b|^{(n-2)q}\right\}\right)^{\frac{1}{q}}.$

References

[1] M. Alomari and M. Darus, *On the Hadamard's inequality for log-convex functions on the coordinates*, J. Ineq. Appl., Volume 2009, Article ID 283147, 13 pages. Doi:10.1155/2009/283147.

[2] M. Alomari, M. Darus and U.S. Kirmaci, *Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means*, Comp. Math. Appl., **59** (2010), 225-232.

[3] M. Alomari, M. Darus and S.S. Dragomir, *New inequalities of Hermite-Hadamard's type for functions whose second derivatives absolute values are quasiconvex*, Tamk. J. Math., **41** (2010) 353-359.

[4] M. K. Bakula, M Emin Ozdemir and J. Pecaric', *Hadamard type inequalities for m-convex and*(α, m)*-convex*, J. Inequal. Pure and Appl. Math., **9**(2008), Article 96. [ONLINE: http://jipam.vu.edu.au].

[5] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities*, J. Ineq. Pure and Appl. Math., **10(3)** (2009), Article 86.

[6] L. Chun and F. Qi, *Integral inequalities for Hermite-Hadamard type for functions whose 3rd derivatives are s-convex*, Applied Mathematics, **3** (2012), 1680-1885.

[7] S. S. Dragomir, R. P. Agarwal and P. Cerone, *On Simpson's inequality and applications*, J. Ineq. Appl., **5**(2000), 533-579.

[8] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. Online: http://www.staff.vu.edu.au/RGMIA/monographs/hermite hadamad.

[9] Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. **1(1)** (2010), 51-58.

[10] Z. Dahmani, L. Tabharit, S. Taf, *New generalizations of Gruss inequality using Riemann-Liouville fractional integrals*, Bull. Math. Anal. Appl., **2(3)** (2010), 93-99.

[11] Z. Dahmani, L. Tabharit, S. Taf, *Some fractional integral inequalities*, Non l. Sci. Lett. A.,**1(2)** (2010), 155-160.

[12] Z. Dahmani, *New inequalities in fractional integrals*, International Journal of Nonlinear Science, **9(4)** (2010), 493-497.

[13] R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien (1997), 223-276.

[14] D. A. Ion, *Some estimates on the Hermite-Hadamard inequality through quasi-convex functions*, Annals of University of Craiova Math. Comp. Sci. Ser., **34**(2007), 82-87.

[15] J. Pecaric, F. Proschan and Y.L. Tong, *Convex functions, partial ordering and statistical applications*, Academic Press, New York, 1991.

[16] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model. (2012), Online, doi:10.1016/j.mcm.2011.12.048.

[17] JinRong Wang, Xuezhu Li, Michal Fe, *Hermite–Hadamard-type inequalities for Riemann–Liouville fractional integrals via two kinds of convexity*, Applicable Analysis, DOI:10.1080/00036811.2012.727986.