On The Total Differential of Almost Quasiconformal Mappings

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Abstract. We give some sufficient conditions of existence of total differential at a point (which may be a boundary point) for almost quasiconformal mappings.

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1. AUXILIARY CONCEPTS

1.1. Let $D$ be a domain in $\mathbb{R}^n$ and $f : D \to \mathbb{R}^m$ be a vector function. The vector function $f : D \to \mathbb{R}^m$ has at a point $a \in D$ a total differential, if there exists a constant matrix

$$C = \{C_{ij}\}_{1 \leq i,j \leq m}$$

such that

$$f(x) - f(a) = C \cdot (x - a) + o(|x - a|) \quad (x \to a, \quad x \in D). \quad (1.1)$$

It is known, that a function $f$ has a total differential at a point $a \in D$, if in a neighborhood of $a$ there exist partial derivatives $\partial f_i / \partial x_j \ (i = 1, \ldots, n, j = 1, \ldots, m)$, which are continuous at $a$. There are examples which show that the continuity of partial derivatives at $a$ is not a necessary condition for the existence a total differential at $a$.

1.2. Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a mapping of the class $W^{1,n}_{\text{loc}}(D)$. We put

$$f'(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \quad \text{and, next,} \quad \|f'(x)\| = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \frac{\partial f_i}{\partial x_j}(x) \right)^2 \right)^{1/2}.$$

We shall say, that a mapping $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ belongs to a class $W^{1,n}_{\text{loc}}(D)$, if for an arbitrary subdomain $D' \subset \subset D$ there exists a constant $p > n$ which, in general, depends on $D'$, such that $f \in W^{1,p}(D')$. A continuous mapping $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is almost quasiconformal in $D$ with a measurable function $K(x) \geq 0$ and locally integrable function
\[ \delta(x) : D \to \mathbb{R}, \text{if } f \in W^{1,n}_{\text{loc}}(D) \text{ and almost everywhere on } D \text{ the following property holds} \]
\[ \|f'(x)\|_n \leq K(x) |J(x,f)| + \delta(x), \tag{1.2} \]

where
\[ J(x,f) = \det(f'(x)). \]

The concept of almost quasiconformal maps belongs to Callender [6], however we note that the condition (1.2) has in [6] a different form. Namely, in [6] it is assumed that \( K(x) \equiv \text{const} \), and instead of \(|J(x,f)|\) it is written \( J(x,f) \). Thus the class of maps considered here is essentially wider than the class considered by Callender in [6]. In particular, our definition permits to consider degenerate quasiconformal maps.

Under condition of preservation of the Jacobian sign and the assumptions
\[ K \equiv \text{const} > 0, \quad \delta \equiv 0, \]
the supposition (1.2) means that the mapping \( f \) is quasiregular [23, §3 Ch. I], [25, Sect. 14.1]. It should be noted that in the case of quasiregular maps it is assumed only that the vector-function \( f \) is continuous and belongs to \( W^{1,n}_{\text{loc}}(D) \), and the supposition \( W^{1,n}_{\text{loc}}(D) \) holds automatically.

The assumption (1.2) does not require that the sign of \( \det(f'(x)) \) is constant. Thus, almost quasiconformal maps can change their orientation.

The following simple statement [12, Sect. 8.1] shows that the class of considered maps is very wide.

**Proposition 1.** Let \( f : D \to \mathbb{R}^n \) be a mapping and moreover
\[ f \in \text{ACL}(D) \quad \text{and} \quad \text{ess sup}_{x \in D} \|f'(x)\| \leq q < \infty. \]
Then \( f \) is almost quasiconformal with \( K = \epsilon n^{n/2} \) and \( \delta = (1 + \epsilon) q^n \), where \( \epsilon = \text{const} > 0 \) is arbitrary.

### 1.3. In the case, if \( D \subset \mathbb{R}^2 \) and \( a \in \partial D \) is a multiple point of a boundary, the relation (1.1) can depend on a direction of the approach to the point \( a \) from \( D \) and, consequently, the definition of the total differential must be more precise.

We define ends of a domain \( D \) using analogy with the Carathéodory theory of prime ends (see, for example, [13, §3]).

For an arbitrary set \( U \subset D \) we put \([U] = \overline{U} \setminus \partial D\), where \( \overline{U} \) is the closure with respect to \( \mathbb{R}^n \). Let \( \{U_k\}, k = 1, 2, \ldots \) be a family of subdomains \( U_k \subset D \) with properties:

(i) for every \( k = 1, 2, \ldots \) \([U_{k+1}] \subset U_k\),

(ii) \( \bigcap_{k=1}^{\infty} [U_k] = \emptyset \).

An arbitrary sequence \( \{U_k\} \) with these properties is called a *chain* in \( D \).

Let \( \{U'_k\}, \{U''_k\} \) be two chains of subdomains of \( D \). We say, that \( U'_k \) is *contained* in \( U''_k \), if for every \( m \geq 1 \) there is a number \( k(m) \) such that for all \( k > k(m) \) the following property holds \( U'_k \subset U''_m \). Two chains are called *equivalent*, if each of them is contained in the other one. The classes of equivalence \( \xi \) of chains are called *ends* of \( D \).

To define an end \( \xi \) it is sufficient to set even one representative of the class of equivalence. If an end \( \xi \) is defined with a chain \( \{U_k\} \), then we write \( \xi \asymp \{U_k\} \).

A *body* of an end \( \xi \asymp \{U_k\} \) is the set
\[ |\xi| = \cap_{i=1}^{\infty} \overline{U_i}. \]
It is clear, that this set does not depend on the choice of a chain \( U_k \).

Let \( \{x_m\}_{m=1}^\infty \) be a sequence of points \( x_m \in D \) which does not have condensation point in \( D \). Such sequences are called nonconvergent in \( D \).

Let \( a_\xi \in |\xi| \) be an arbitrary point. A nonconvergent in \( D \) sequence of points \( x_k \in D \) converges to a point \( a_\xi \) with respect to the topology of \( \xi \), if \( x_k \to a_\xi \) (with respect to the topology \( \mathbb{R}^n \)) and for some chain \( \{U_k\} \in \xi \) the following property holds: for every \( k = 1, 2, \ldots \) there is a number \( m(k) \) such that \( x_m \in U_k \) for arbitrary \( m > m(k) \).

Let \( D \) be a domain in \( \mathbb{R}^n \), \( \xi \) be an end of \( D \), \( a_\xi \in |\xi| \) be a point. We shall say, that a subdomain \( D' \) of \( D \) adjoins to \( a_\xi \), if \( a_\xi \in \partial D' \) and any sequence of points \( x_k \in D' \), converging to \( a_\xi \) with respect to the topology \( \mathbb{R}^n \), converges to this point with respect to the topology \( \xi \). We shall say, that a vector-function \( f : D \to \mathbb{R}^m \) satisfies to the property

\[
\lim_{x \to a_\xi} f(x) = A, \quad A = (A_1, \ldots, A_m)
\]

if

\[
f(x_k) \to A \quad \text{as} \quad x_k \to a_\xi
\]

along every sequence of point \( x_k \in D \), which converges to \( a_\xi \) with respect to the topology of \( \xi \). The vector \( A \) is denoted by \( f(a_\xi) \).

Suppose that a vector-function \( f : D \to \mathbb{R}^m \) and a point \( a_\xi \) are such that \( f(a_\xi) \) exists.

We say, that \( f \) has a total differential at a boundary point \( a_\xi \), if there exists a constant matrix \( C = \{C_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq m} \), for which

\[
f(x) - f(a_\xi) = C \cdot (x - a_\xi) + o(|x - a_\xi|) \quad (x \to a_\xi, \quad x \in D). \tag{1.3}
\]

As in the case of an inner point, we shall say that

\[
df(a_\xi) = C \cdot (x - a_\xi)
\]

is a differential of \( f \) at \( a_\xi \).

The differential of the vector-function at the boundary point need not be unique (see corresponding examples in [9]).

2. The weighted modulus

2.1. Recall the definition of the class \( ACL_p \). Let \( D \subset \mathbb{R}^n \) be an open set. Fix \( i, 1 \leq i \leq n \), and denote by \( D_i^* \) an orthogonal projection of \( D \) onto the hyperplane \( x_i = 0 \). For an arbitrary locally summable in \( D \) function \( f \) we put

\[
f^*_i(x_i', t, x''_i) \equiv f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n),
\]

\[
x_i' = (x_1, \ldots, x_{i-1}), \quad x''_i = (x_{i+1}, \ldots, x_n).
\]

Next, let

\[
D_i(x_i', x''_i) \equiv \{(x_i', t, x''_i) \in \mathbb{R}^n : (x_i', 0, x''_i) \in D_i^*\}.
\]

A continuous function \( f : D \to \mathbb{R} \) is called absolutely continuous on lines (or shortly, ACL), if for every \( i = 1, \ldots, n \), the coordinate functions \( f^*_i(x_i', t, x''_i) \) are absolutely continuous (with respect to the variable \( t \)) inside the union of linear intervals \( D \cap D_i(x_i', x''_i) \) for \( \mathcal{H}^{n-1} \)-almost all points \( (x_i', 0, x''_i) \in D_i^* \). (Here and below the symbol \( d\mathcal{H}^p \) means an element of \( p \)-dimensional Hausdorff measure.)

Every ACL-function \( f : D \to \mathbb{R} \) has partial derivatives \( \partial f/\partial x_i \) (\( i = 1, \ldots, n \)) almost everywhere in \( D \). By the symbol \( f' \equiv (\partial f_i/\partial x_j) \) we denote a formal derivative of \( f \) at
points, where all partial derivatives exist. At points, in which the matrix $f'$ is not defined, let us agree to take all $\partial f_i/\partial x_j = +\infty$ ($i = 1, \ldots, n; j = 1, \ldots, m$).

Let $\sigma : D \to \mathbb{R}^1$ be a nonmeasurable negative function, which is defined almost everywhere in $D$, and let $p \geq 1$ be a constant. The class $\text{ACL}^p(D)$ is the set of ACL-functions in $D$, for which

$$\int_D \|f'(x)\|^p \sigma(x) \, d\mathcal{H}^n < \infty.$$ 

In the case, if the weight function $\sigma \equiv 1$, we have the well known class $\text{ACL}^p(D)$, which coincides with the set of continuous $W_{p,1}(D)$-functions [17, Theorems 5.6.2-3].

2.2. Let $D$ be a domain in $\mathbb{R}^m$, $m > 1$, let $U \subset D$ be a countably $(\mathcal{H}^k, k)$-rectifiable set, $1 \leq k \leq m$, and let $\sigma : U \to \mathbb{R}^1$ be a nonnegative $\mathcal{H}^k$-measurable function. Fix a constant $p > 1$ and for an arbitrary family $\Gamma$ of locally rectifiable arcs $\gamma \subset U$, we define a $(p, \sigma)$-modulus

$$\text{mod}_{p,\sigma}(\Gamma; U) = \inf_{\rho} \left( \frac{\inf_{\gamma \in \Gamma} \int_{\gamma} \rho \, d\mathcal{H}^1}{\int_{U} \rho^p \sigma \, d\mathcal{H}^k} \right)^{1/p},$$

(2.4)

where the infimum is taken over all nonnegative, Borel measurable functions $\rho$ in $U$. If $\Gamma = \emptyset$, then we put $\text{mod}_{p,\sigma}(\Gamma; U) = \infty$.

In the case $U = D$ we have a standard definition of the weighted $(p, \sigma)$-modulus of the family $\Gamma$ in $\mathbb{R}^n$ (see, for example, [18, Sect. 3.2]).

2.3. Let $y$ and $a$ be a pair of points such that $y \in D$ and either $a$ is an interior point of $D$, or $a = a_{\xi} \in \partial D$, where $\xi$ is an end of the domain $D \subset \mathbb{R}^n$. We say that a simple Jordan arc $\gamma$, defined by a parametrization $x(\tau) : [0, 1) \to D$, leads from $y$ to $a$, if $x(0) = y$ and

$$\lim_{\tau \to 1} x(\tau) = a \quad \text{as} \quad a \in D$$

and there is a sequence $\tau_k \to 1$, along which

$$\lim_{\tau_k \to 1} x(\tau_k) = a_{\xi} \quad \text{as} \quad a \in \partial D.$$

We consider a family $\Gamma$ of all locally rectifiable, simple Jordan arcs $\gamma \subset D$, leading from $y$ to $a$. We put

$$\text{mod}_{p,\sigma}(\Gamma; y, a; D) = \text{mod}_{p,\sigma}(\Gamma; D).$$

(2.5)

2.4. Let $D \subset \mathbb{R}^n$ be a domain and $a = a_{\xi}$ be its interior or boundary point. Fix a continuous vector-function $\nu : \overline{D} \to \mathbb{R}^k$, $1 \leq k < \infty$ and put $B^\nu(a, r) = \{x \in D : |\nu(x) - \nu(a)| < r\}$. By $B^\nu_{D}(a, r)$ we denote a connected component of $B^\nu(a, r)$, containing $a$ if $a$ is an interior point of $D$, and adjoining at $a$ if $a \in \partial D$.

By $S^\nu_{D}(a, r)$ we denote the relative boundary

$$S^\nu_{D}(a, r) = \partial B^\nu_{D}(a, r) \setminus \partial D.$$

In the case $\nu(x) \equiv x$ we shall use notations $B^0(a, r)$, $B^0_{D}(a, r)$ and $S^0_{D}(a, r)$, respectively.

Suppose that $\nu(x)$ is locally Lipschitz. Let $h(x) = |\nu(x) - \nu(a)|$ and let

$$0 < \text{ess inf}_{x \in D} |\nabla h(x)| \leq \text{ess sup}_{x \in D} |\nabla h(x)| < \infty$$

(2.6)
on every subset $D' \subset D$.

By Theorem 3.2.15 [16] (see also [18, Theorem 1.6.1]) for a.e. $t \in \mathbb{R}^3$ the sets $S_D^\nu(a, t)$ are countably $(\mathcal{H}^{n-1}, n - 1)$-rectifiable.

Fix a countably $(\mathcal{H}^{n-1}, n - 1)$-rectifiable set $S_D^\nu(a, t)$ and a measurable function $\sigma : S_D^\nu(a, t) \rightarrow \mathbb{R}^3$. Let $U$ be a connected component of $S_D^\nu(a, t)$. For a pair of points $a_1, a_2 \in U$ let $\Gamma = \Gamma(a_1, a_2)$ stands for the family of all locally rectifiable arcs $\gamma \subset U$, joining points $a_1$ and $a_2$. We define a weighted modulus

$$\text{mod}(a_1, a_2; \sigma) = \text{mod}_{n, \sigma}(a_1, a_2).$$

(2.7)

Next, let

$$\kappa(S_D^\nu(a, t), \sigma) = \inf_U \inf_{a_1, a_2 \in U} \text{mod}(a_1, a_2; \sigma),$$

(2.8)

where the first of infimums is taken over the collection $\{U\}$ of all connected components $U$ of $S_D^\nu(a, t)$. We put

$$\kappa^\nu(a, t) = \kappa(S_D^\nu(a, t), \sigma^*), \quad \sigma^* = \frac{\sigma}{|\nabla h|},$$

where $\sigma : D \rightarrow \mathbb{R}^3$ is a nonnegative measurable function.

2.5. We shall need the following multidimensional version of known "Length and Area Principle" (see, for example, [19], [13]).

**Lemma 2.** [7] Let $D$ be a domain in $\mathbb{R}^n$, let $a = a_\xi \in \overline{D}$, let a vector-function $\nu : D \rightarrow \mathbb{R}^k$ satisfy (2.6) and $\sigma(x) : D \rightarrow \mathbb{R}^3$ is a nonnegative, measurable function. Let $f : D \rightarrow \mathbb{R}^m$ be a vector-function of the class $\text{ACL}_n(D)$. Then for arbitrary $t', t'' \in h(D), t' < t''$, the following inequality holds

$$\int_{t'}^{t''} \Omega^\nu(f, S_D^\nu(a, t)) \kappa^\nu(a, t) \, dt \leq \int_{D(t', t'')} \|f'(x)||^n \sigma(x) \, d\mathcal{H}^n(x).$$

(2.9)

Here

$$D(t', t'') = \{x \in D : t' < |\nu(x) - \nu(a)| < t''\},$$

$$\Omega(f, S_D^\nu(a, t)) = \sup_U \text{osc}(f, U)$$

and the infimum is taken over all connected components $U$ of $S_D^\nu(a, t)$.

3. Main results

3.1. Let $D \subset \mathbb{R}^n$ be an open set. We say that a function $f : D \rightarrow \mathbb{R}^m, m \geq 1$, is monotone, if for every subdomain $U \subset D$ the following property holds

$$\text{osc}(f, U) \leq \text{osc}(f, \partial'U), \quad \partial'U = \partial U \setminus D.$$

Here and below by the symbol

$$\text{osc}(\phi, E) = \sup_{x, y \in E} |\phi(x) - \phi(y)|$$

we denote the oscillation of a function $\phi$ on $E$. 
Let \( h(t) : [0, \infty) \to [0, \infty) \) be an upper semicontinuity function. We shall say, that \( f : D \to \mathbb{R}^m, m \geq 1, \) is \( h \)-monotone, if for every subdomain \( U \subset D \) we have
\[
    h(\text{osc}(f, U)) \leq \text{osc}(f, \partial U),
\]
and is \( \alpha \)-monotone, \( 0 < \alpha \equiv \text{const} < \infty, \) if \( f \) is \( h \)-monotone with \( h(t) = t^\alpha. \)

Some examples of \( \alpha \)-monotone functions were given in [8].

Fix a continuous vector-function \( \nu : D \to \mathbb{R}^k. \) We say, that a vector-function \( f : D \to \mathbb{R}^m, m \geq 1, \) is weakly \((h, \nu)\)-monotone close to a point \( a = a_\xi \) (interior or boundary), if
\[
    \limsup_{r \to 0} \frac{h(\text{osc}(f, B_{\nu}(a, r)))}{\text{osc}(f, S_{\nu}(a, r))} < \infty, \quad (3.10)
\]
and weakly \((\alpha, \nu)\)-monotone close to a point \( a, \) if \( f \) is weakly \((h, \nu)\)-monotone close to \( a \) for \( h(t) = t^\alpha. \)

It is clear, that every monotone in the Lebesgue sense function is weakly \((\alpha, \nu)\)-monotone, \( \alpha = 1, \) close to every point.

3.2. For an arbitrary continuous mapping \( y = \varphi(x) : D \subset \mathbb{R}^n \to \mathbb{R}^n \) and for a set \( A \subset D \) by
\[
    N(y; \varphi, A),
\]
we shall denote the number of preimages of a point \( y \in \mathbb{R}^n \) in \( A. \) Next we put
\[
    n(x; \varphi, A) = N(y; \varphi, A), \quad \text{where} \quad y = \varphi(x).
\]

3.3. The following statement is the main result of this paper

**Theorem 3.** Suppose that a vector-function \( f : D \to \mathbb{R}^n \) is an almost quasiconformal mapping of a domain \( D \subset \mathbb{R}^n \) in the sense (1.2), for which
\[
    \int_D \frac{\delta(x) \, dx}{K(x)} < \infty. \quad (3.11)
\]
Then for every subdomain \( A \subset D \) the following inequality holds
\[
    \int_A \frac{\|f'(x)\|^n \, dx}{K(x) \, n(x; f, A)} \leq \text{mes}_n(f(A)) + \int_A \frac{\delta(x) \, dx}{K(x) \, n(x; f, A)}. \quad (3.12)
\]

On the other hand, let \( a = a_\xi \in \overline{D} \) be an interior or boundary point of the domain and let \( \nu : \overline{D} \to \mathbb{R}^k \) be a continuous vector-function satisfying (2.6). If

i) for some \( p > n \) and some constant matrix \( C = (C_{ij})_{i,j=1}^n \) the following assumption holds
\[
    \limsup_{y \to a} \int_{B_{\nu}(a, r(a, y))} \frac{\|f'(x) - C\|^p \, dx}{K(x) \, n(x; f, B_{\nu}(a, r(a, y)))} = 0, \quad (3.13)
\]
where
\[
    r(a, y) = \inf\{t > 0 : y \in B_{D}(a, t)\}, \quad \sigma_r(x) = \frac{1}{K(x) \, n(x; f, B_{D}(a, r))}, \quad (3.14)
\] or
ii) the vector-function \( f(x) - C \cdot x \) is weakly \((\alpha, \nu)\)-monotone close to \( a \) and there is a constant \( \lambda > 1 \), for which

\[
\limsup_{y \to a} \int_{B'_D(a, \lambda r(a,y))} \frac{||f'(x) - C||^n \, dx}{K(x) n(x; f, B'_D(a, r(a,y)))} \left/ r^{n\alpha}(a,y) \right/ \int_{r(a,y)}^\infty \kappa''(a,t) \frac{dt}{t} = 0,
\]

(3.15)

then \( f \) has at the point \( a = a_\xi \) the total differential \( C \cdot dx \).

3.4. Consider some particular cases of Theorem. Let \( w = f(z) : D \subset \mathbb{C}^1 \to \mathbb{C}^1 \) be a generalized solution of a Beltrami equation

\[
f_z = \mu(z) f_z,
\]

(3.16)

where \( \mu(z) \) is a measurable complex-valued function, and by symbols

\[
f_z = \frac{1}{2} (f_x - i f_y), \quad f_\bar{z} = \frac{1}{2} (f_x + i f_y)
\]

we denote formal derivatives.

Note that in contrast to the traditional case (see, for example, [10, Ch. V], [11, Ch. 1]) we do not assume here, that \(|\mu(z)| < 1\).

We have

\[
J(z, f) = |f_z|^2 - |f_\bar{z}|^2 = (1 - |\mu(z)|^2)|f_z|^2
\]

and

\[
||f'||^2 = |f_z|^2 + |f_\bar{z}|^2 = (1 + |\mu(z)|^2)|f_z|^2.
\]

Thus,

\[
||f'||^2 \leq \frac{1 + |\mu|^2}{|1 - |\mu|^2|} |J(z, f)|
\]

and (1.2) holds with

\[
K(z) = \frac{1 + |\mu(z)|^2}{|1 - |\mu(z)|^2|}, \quad \delta(z) \equiv 0.
\]

The assumption (3.11) holds always.

Here we have also

\[
\sigma_r(z) = \frac{1}{K(x) n(x; f, B'_D(a, r))} = \frac{|1 - |\mu(z)|^2|}{(1 + |\mu(z)|^2)n(x; f, B'_D(a, r))}.
\]

For schlicht maps \( n(x; f, B'_D(a, r)) \equiv 1 \) and

\[
\sigma_r(z) = \frac{|1 - |\mu(z)|^2|}{1 + |\mu(z)|^2}.
\]

Theorem connects the differentiability of \( f \) at a singular point \( a = a_\xi \) with the behavior of the characteristic \( \mu(z) \) close to its neighborhood. In the case, if \( a \) is an inner point, see [1], [2], [3, Ch. VI], [4, Ch. 11]. In the case, if \( a = a_\xi \) is a boundary point and \( \mu(z) \equiv 0 \), see related results in [5, Ch. 11].

In the case, if the matrix \( C \) is orthogonal, Theorem gives conditions, under which maps \( f : D \to \mathbb{R}^n \) are conformal at \( a = a_\xi \).

For space quasiregular maps, near questions were being considered in [24, Ch. VI].
4. PROOF OF THEOREM

Let \( \varphi : D \subset \mathbb{R}^n \to \mathbb{R}^n \) be a continuous mapping. This mapping \( \varphi \) is called absolutely continuous, if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for an arbitrary measurable set \( E \subset D \), \( \text{mes}_n E < \delta \), we have \( \text{mes}_n \varphi(E) < \varepsilon \). In particular, every absolutely continuous mapping possesses the Lusin \( N \)-property.

**Lemma 4.** [20] If a mapping \( \varphi \) is continuous and belongs to the class \( W_{1,n}^{1,n} \), then \( \varphi \) is absolutely continuous on every subdomain \( D' \subset D \).

Applying Lemma 2, we conclude that the following statement holds (see, for example, [22]).

**Lemma 5.** If a mapping \( \varphi \) is continuous and belongs to \( W_{1,n}^{1,n} \), then for an arbitrary integrable in \( \varphi(D) \) function \( u(y) \), the function \( (u \circ \varphi)(x) |J(x, \varphi)| \) is integrable in \( D \), and moreover

\[
\int_{\varphi(D)} u(y) N(y; \varphi, A) \, dy = \int_A (u \circ \varphi)(x) |J(x, \varphi)| \, dx.
\]

In particular, if we observe that \( J(x, \varphi) J(y, \varphi^{-1}) = 1 \), \( y = \varphi(x) \), and set

\[
u(y) = \frac{1}{N(y; \varphi, A)},
\]

then using (4.17) we have

\[
\text{mes}_n(\varphi(A)) = \int_{\varphi(A)} dy = \int_A \frac{|J(x, \varphi)|}{n(x; \varphi, A)} \, dx.
\]

From this, using (1.2) we conclude that

\[
\int_A \frac{||f'(x)||^n - \delta(x)}{K(x) n(x; f, A)} \, dx \leq \text{mes}_n(f(A)),
\]

and, thus, we obtain (3.12).

Consider the family of locally rectifiable arcs \( \Gamma(y, a; B_D'(a, |y-a|)) \), lying in \( B_D'(a, |y-a|) \) and joining the point \( y \in B_D'(a, |y-a|) \) with the point \( a \). Choose in (2.4) the function \( \rho(x) = ||f'(x) - C|| \). We find

\[
\text{mod}_{p,\sigma}(y, a; B_D'(a, |y-a|)) \leq \frac{\int_{B_D'(a, |y-a|)} ||f'(x) - C|| P(x) \, dH^n}{\gamma \in \Gamma(y, a; B_D'(a, |y-a|)) \left( \int_{\gamma} ||f'(x) - C|| \, dx \right)^\frac{1}{p}}.
\]

If \( \gamma \) is an arc of the family \( \Gamma(y, a; B_D'(a, |y-a|)) \), then

\[
|f(y) - f(a) - C \cdot (y-a)| \leq \int_\gamma ||f'(x) - C|| \, dH^1.
\]
Thus using (4.18), for every point $y \in D$ we have

$$|f(y) - f(a) - C \cdot (y - a)|^p \leq \int \|f'(x) - C\|^p \sigma(x) \, d\mathcal{H}^n$$

(4.19)

By virtue of (4.19) the assumption (3.13) implies realization (1.1) (respectively, (1.3)) and, thus, the existence in the case $i$ of the total differential at $a = a_\xi$.

We first prove the statement in the case $ii$. Fix a point $y \in D$ and consider the subdomain $B^\nu_D(a, r), a = a_\xi, r = \rho(a, y)$, adjoining at the end $\xi$ and containing $y$ in its closure. We put $f^*(x) = f(x) - f(a) - C \cdot (x - a)$. Applying Lemma 1 by virtue of (2.9), we have

$$\lambda r(a, y) \int_{r(a, y)} \Omega^n (f^*, S^\nu_D(a, t)) \kappa^\nu(a, t) \, dt \leq \int_{D(r(a, y), \lambda r(a, y))} \| (f^*)'(x) \|^n \sigma_{\lambda r(a, y)}(x) \, d\mathcal{H}^n(x),$$

where $\sigma_{\lambda r(a, y)}(x)$ is defined in (3.14).

The mapping $f^*(x)$ is weakly $(\alpha, \nu)$-monotone close to $a$ and by virtue of (3.10) for every $t, r(a, y) < t < \lambda r(a, y)$, and some constant $A < \infty$ the following estimates hold

$$|f^*(y) - f^*(a)|^\alpha \leq \text{osc}_a (f^*, B^\nu_D(a, t)) \leq A \Omega (f^*, S^\nu_D(a, t)).$$

From this we obtain

$$|f^*(y) - f^*(a)|^\alpha \int_{r(a, y)} \kappa^\nu(a, t) \, dt \leq$$

$$\leq A \int_{D(r(a, y), \lambda r(a, y))} \|f'(x) - C\|^n \sigma_{\lambda r(a, y)}(x) \, d\mathcal{H}^n(x).$$

The assumption (3.15) implies (1.1) (and, respectively, (1.3)). Theorem is proved. □

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**References**


