Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 44 (2012) pp. 39-50

Subordination Properties for Certain Subclasses of Prestarlike Functions

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Abstract. Making use of generalized Sălăgean derivative operator in this paper, we define a unified class of starlike functions with negative coefficients and obtain subordination results, partial sums, integral transforms for this class. Further integral means and square root transformation results are discussed.

AMS (MOS) Subject Classification Codes: 30C45
Key Words: Univalent, convex, starlike, Hadamard product, pre-starlike, Sălăgean derivative operator, subordination, integral means.

1. Introduction

Let $\mathcal{S}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1. 1}$$

which are analytic and univalent in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Also, T denote the subclass of $\mathcal{S}$ consisting functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \;\; a_n \geq 0 \tag{1. 2}$$

introduced and studied by Silverman [7]. We denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of $\mathcal{S}$ consisting of all functions which are, respectively starlike and convex functions of order α. Thus,

$$S^*(\alpha) = \left\{ f \in \mathcal{S} : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \;\; 0 \leq \alpha < 1, \;\; z \in \mathcal{U} \right\}$$

and

$$K(\alpha) = \left\{ f \in \mathcal{S} : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ \ 0 \le \alpha < 1,\ \ z \in \mathcal{U} \right\}.$$

For functions $f \in \mathcal{S}$ given by (1. 1) and $g \in \mathcal{S}$ of the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}. \tag{1. 3}$$

In 1977, Ruscheweyh [5] introduced and studied the class of prestarlike functions of order α, which are the function f such that $f * S_\alpha$ is a starlike function of order α, where

$$\mathcal{S}_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}, \quad 0 \le \alpha < 1, \quad z \in \mathcal{U}. \tag{1. 4}$$

We also note that $\mathcal{S}_\alpha(z)$ can be written in the form

$$\mathcal{S}_\alpha(z) = z + \sum_{n=2}^{\infty} C_n(\alpha) z^n, \tag{1. 5}$$

where

$$C_n(\alpha) = \frac{\prod_{i=2}^{n}(i - 2\alpha)}{(n-1)!}, \quad n \ge 2. \tag{1. 6}$$

Clearly, $C_n(\alpha)$ is decreasing in α and satisfies

$$\lim_{n\to\infty} C_n(\alpha) = \begin{cases} \infty & \text{if } \alpha < \frac{1}{2} \\ 1 & \text{if } \alpha = \frac{1}{2} \\ 0 & \text{if } \alpha > \frac{1}{2} \end{cases}. \tag{1. 7}$$

Denote by D_δ^m the Al-Oboudi operator [1] for $m \in \mathbb{N}_0$ and $\delta \ge 0$ defined by $D_\delta^m : A \to A$, $D_\delta^0 f(z) = f(z); D_\delta^1 f(z) \ = (1-\delta)f(z) + \delta z f'(z) = D_\delta f(z); D_\delta^m f(z) = D_\delta(D_\delta^{m-1} f(z))$. Note that for $f(z)$ given by (1. 1),

$$D_\delta^m f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\delta]^m a_n z^n, \quad m \in \mathbb{N}_0. \tag{1. 8}$$

For $\delta = 1$, D_δ^m is Sălăgean operator [6] defined as:

$$D^0 f(z) = f(z); \quad D^1 f(z) = Df(z) = zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$$

$$D^2 f(z) = D(Df(z)) = z + \sum_{n=2}^{\infty} n^2 a_n z^n$$

$$D^m f(z) = D(D^{m-1} f(z)) = z + \sum_{n=2}^{\infty} n^m a_n z^n, \quad m \in \mathbb{N}_0. \tag{1. 9}$$

Making use of Al-Oboudi operator (1. 8), Sălăgean differential operator (1. 9) and prestarlike function (1. 5), and motivated by Darus [2], Silverman and Silvia [10], we define the following unified class of starlike function.

Definition 1. Let $\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$ be the subclass of $\mathcal{S}$ consisting of functions $f(z)$ of the form (1. 2) and satisfying the analytic criterion

$$\left|\frac{\frac{z(D_\delta^m f(z)*S_\alpha)'}{D_\delta^m f(z)*S_\alpha}-1}{2\gamma(B-A)\left(\frac{z(D_\delta^m f(z)*S_\alpha)'}{D_\delta^m f(z)*S_\alpha}-\lambda\right)-B\left(\frac{z(D_\delta^m f(z)*S_\alpha)'}{D_\delta^m f(z)*S_\alpha}-1\right)}\right|<\beta,\quad z\in\mathcal{U}, \quad (1.\ 10)$$

where $0\le\lambda<1$, $0<\beta\le1$,

$$\frac{B}{2(B-A)}<\gamma\le\begin{cases}\frac{B}{2(B-A)\lambda} & \lambda\neq0,\\ 1 & \lambda=0.\end{cases}$$

for fixed $-1\le A\le B\le1$ and $0<B\le1$. We also let $T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)=\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)\cap T$.

Now we obtain the coefficient bounds for the class $T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$.

Theorem 2. *Let the function $f(z)$ be defined by (1. 2), then it is in the class $T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$ if and only if*

$$\sum_{n=2}^{\infty}\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)|a_n|\le2\beta\gamma(1-\lambda)(B-A), \quad (1.\ 11)$$

where

$$\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)=[2\beta\gamma(B-A)(n-\lambda)+(1-B\beta)(n-1)][1+(n-1)\delta]^mC_n(\alpha). \quad (1.\ 12)$$

The proof of Theorem 2 is much akin to the proof of theorem on coefficient bounds established in [4], so we omit the details.

Now we recall the following results which are very much needed for our study.

Definition 3. *(Subordination)* For analytic functions g and h with $g(0)=h(0)$, g is said to be subordinate to h, denoted by $g\prec h$, if there exists an analytic function w such that $w(0)=0$, $|w(z)|<1$ and $g(z)=h(w(z))$, for all $z\in\mathcal{U}$.

Definition 4. [12]*(Subordinating Factor Sequence)* A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z)=\sum\limits_{n=1}^{\infty}a_nz^n$, $a_1=1$ is regular, univalent and convex in $\mathcal{U}$, we have

$$\sum_{n=1}^{\infty}b_na_nz^n\prec f(z),\quad z\in\mathcal{U}. \quad (1.\ 13)$$

Lemma 5. [12] *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left\{1+2\sum_{n=1}^{\infty}b_nz^n\right\}>0,\quad z\in\mathcal{U}. \quad (1.\ 14)$$

2. Subordination Results

Theorem 6. *Let $f\in T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$ and $g(z)$ be any function in the usual class of convex functions K, then*

$$\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{2[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}(f*g)(z)\prec g(z) \quad (2.\ 1)$$

where $0 \leq \gamma < 1$; $k \geq 0$, $0 \leq \lambda < 1$,*and*

$$Re\ \{f(z)\} > -\frac{[2\beta\gamma(1-\lambda)(B-A) + \Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}, \quad z \in \mathcal{U}. \quad (2.\ 2)$$

The constant factor $\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{2[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}$ *in (2. 1) cannot be replaced by a larger number.*

Proof. Let $f \in T\mathcal{D}^{\alpha,\beta}_{\lambda,\gamma}(m,A,B)$ and suppose that $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n \in K$. Then

$$\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{2[2\beta\gamma(1-\lambda)(B-A) + \Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}(f*g)(z)$$
$$= \frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{2[2\beta\gamma(1-\lambda)(B-A) + \Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}\left(z + \sum_{n=2}^{\infty} b_n a_n z^n\right). \quad (2.\ 3)$$

Thus, by Definition 4, the subordination result holds true if

$$\left\{\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{2[2\beta\gamma(1-\lambda)(B-A) + \Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 5, this is equivalent to the following inequality

$$\text{Re}\left\{1 + \sum_{n=1}^{\infty}\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{[2\beta\gamma(1-\lambda)(B-A) + \Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}a_n z^n\right\} > 0, \quad z \in \mathcal{U}. \quad (2.\ 4)$$

By noting the fact that $\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)}{2\beta\gamma(1-\lambda)(B-A)}$ is increasing function for $n \geq 2$ and in particular

$$\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{2\beta\gamma(1-\lambda)(B-A)} \leq \frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)}{2\beta\gamma(1-\lambda)(B-A)}, \quad n \geq 2,$$

therefore, for $|z| = r < 1$, we have

$$\text{Re}\left\{1 + \frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}\sum_{n=1}^{\infty} a_n z^n\right\}$$

$$= \text{Re}\left\{1 + \frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}z \right.$$

$$\left. + \frac{\sum_{n=2}^{\infty}\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)a_n z^n}{[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}\right\}$$

$$\geq 1 - \frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}r$$

$$-\frac{1}{[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}\sum_{n=2}^{\infty}|\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)a_n|\, r^n$$

$$\geq 1 - \frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}r$$

$$-\frac{2\beta\gamma(1-\lambda)(B-A)}{[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}r$$

$$> 0, \quad |z| = r < 1,$$

where we have also made use of the assertion (1. 11) of Theorem 2. This evidently proves the inequality (2. 4) and hence also the subordination result (2. 1) asserted by Theorem 6. The inequality (2. 2) follows from (2. 1) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in K.$$

Next we consider the function

$$F(z) := z - \frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}z^2$$

where $0 \leq \gamma < 1,\ k \geq 0, 0 \leq \lambda < 1$. Clearly $F \in T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$. For this function (2. 1)becomes

$$\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{2[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}F(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\text{Re}\left(\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{2[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}F(z)\right)\right\} = -\frac{1}{2}, \quad z \in \mathcal{U}.$$

This shows that the constant $\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}{2[2\beta\gamma(1-\lambda)(B-A)+\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)]}$ cannot be replaced by any larger one. □

3. Partial Sums

Following the earlier works by Silverman [8] and Silvia [11] on partial sums of analytic functions, we consider in this section partial sums of functions in the class $\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_n(z)$ and $f'(z)$ to $f'_k(z)$.

Theorem 7. *Let $f(z) \in \mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m, A, B)$. Define the partial sums $f_1(z)$ and $f_k(z)$, by*

$$f_1(z) = z; \text{ and } f_k(z) = z + \sum_{n=2}^{k} a_n z^n, \ (k \in N/1). \tag{3. 1}$$

Suppose also that

$$\sum_{n=2}^{\infty} d_n |a_n| \leq 1,$$

where

$$d_n := \frac{[2\beta\gamma(B-A)(n-\lambda) + (1-B\beta)(n-1)][1+(n-1)\delta]^m C_n(\alpha)}{2\beta\gamma(1-\lambda)(B-A)}. \tag{3. 2}$$

Then $f \in T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m, A, B)$. Furthermore,

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} > 1 - \frac{1}{d_{k+1}} \ \ z \in \mathcal{U}, k \in N \tag{3. 3}$$

and

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} > \frac{d_{k+1}}{1 + d_{k+1}}. \tag{3. 4}$$

Proof. For the coefficients d_n given by (3. 2) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \tag{3. 5}$$

Therefore we have

$$\sum_{n=2}^{k} |a_n| + d_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} d_n |a_n| \leq 1 \tag{3. 6}$$

by using the hypothesis (3. 2). By setting

$$\begin{aligned} g_1(z) &= d_{k+1}\left\{\frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{d_{k+1}}\right)\right\} \\ &= 1 + \frac{d_{k+1} \sum\limits_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{k} a_n z^{n-1}} \end{aligned} \tag{3. 7}$$

and applying (3. 6), we find that

$$\begin{aligned} \left|\frac{g_1(z)-1}{g_1(z)+1}\right| &\leq \frac{d_{k+1} \sum\limits_{n=k+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=2}^{k} |a_n| - d_{k+1} \sum\limits_{n=k+1}^{\infty} |a_n|} \\ &\leq 1, \ \ z \in \mathcal{U}, \end{aligned} \tag{3. 8}$$

which readily yields the assertion (3. 3) of Theorem 7. In order to see that

$$f(z) = z + \frac{z^{k+1}}{d_{k+1}} \tag{3. 9}$$

gives sharp result, we observe that for $z = re^{i\pi/k}$ that $\frac{f(z)}{f_k(z)} = 1 + \frac{z^k}{d_{k+1}} \to 1 - \frac{1}{d_{k+1}}$ as $z \to 1^-$. Similarly, if we take

$$g_2(z) = (1 + d_{k+1}) \left\{ \frac{f_k(z)}{f(z)} - \frac{d_{k+1}}{1 + d_{k+1}} \right\} = 1 - \frac{(1 + d_{k+1}) \sum_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \tag{3. 10}$$

and making use of (3. 6), we can deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_{k+1}) \sum_{n=k+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^{k} |a_n| - (1 - d_{k+1}) \sum_{n=k+1}^{\infty} |a_n|} \tag{3. 11}$$

which leads us immediately to the assertion (3. 4) of Theorem 7.

The bound in (3. 4) is sharp for each $k \in N$ with the extremal function $f(z)$ given by (3. 9). The proof of the Theorem 7, is thus complete. □

Theorem 8. *If $f(z)$ of the form (1. 1) satisfies the condition (1. 11), then*

$$Re \left\{ \frac{f'(z)}{f'_k(z)} \right\} \geq 1 - \frac{k + 1}{d_{k+1}}. \tag{3. 12}$$

Proof. By setting

$$g(z) = d_{k+1} \left\{ \frac{f'(z)}{f'_k(z)} - \left(1 - \frac{k + 1}{d_{k+1}} \right) \right\} = \frac{1 + \frac{d_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n a_n z^{n-1} + \sum_{n=2}^{\infty} n a_n z^{n-1}}{1 + \sum_{n=2}^{k} n a_n z^{n-1}} = 1 + \frac{\frac{d_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n a_n z^{n-1}}{1 + \sum_{n=2}^{k} n a_n z^{n-1}}.$$

$$\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq \frac{\frac{d_{k+1}}{k+1} \sum_{n=k+1}^{\infty} k|a_n|}{2 - 2 \sum_{n=2}^{k} k|a_n| - \frac{d_{k+1}}{k+1} \sum_{n=k+1}^{\infty} k|a_n|}. \tag{3. 13}$$

Now

$$\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq 1$$

if

$$\sum_{n=2}^{k} n|a_n| + \frac{d_{k+1}}{k + 1} \sum_{n=k+1}^{\infty} n|a_n| \leq 1 \tag{3. 14}$$

since the left hand side of (3. 14) is bounded above by $\sum_{n=2}^{k} d_n|a_n|$ if

$$\sum_{n=2}^{k}(d_n - n)|a_n| + \sum_{n=k+1}^{\infty} d_n - \frac{d_{k+1}}{k+1} n|a_n| \geq 0, \tag{3. 15}$$

and the proof is complete. The result is sharp for the extremal function $f(z) = z + \frac{z^{k+1}}{c_{k+1}}$. □

Theorem 9. *If $f(z)$ of the form (1.1) satisfies the condition (1. 11) then*

$$Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{d_{k+1}}{k+1+d_{k+1}}. \tag{3. 16}$$

Proof. By setting

$$\begin{aligned} g(z) &= [(n+1)+d_{k+1}]\left\{\frac{f_k'(z)}{f'(z)} - \frac{d_{k+1}}{k+1+d_{k+1}}\right\} \\ &= 1 - \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} na_n z^{n-1}}{1+\sum\limits_{n=2}^{k} na_n z^{n-1}} \end{aligned}$$

and making use of (3. 15), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{k} n|a_n| - \left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|} \leq 1,$$

which leads us immediately to the assertion of the Theorem 9. □

4. Integral Transform of the class $T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m, A, B)$

For $f \in \mathbb{A}$ we define the integral transform

$$V_\mu(f)(z) = \int_0^1 \mu(t)\frac{f(tz)}{t}dt,$$

where $\mu(t)$ is a real valued, non-negative weight function normalized so that $\int_0^1 \mu(t)dt = 1$. Since special cases of $\mu(t)$ are particularly interesting such as $\mu(t) = (1+c)t^c$, $c > -1$, for which V_μ is known as the Bernardi operator, and

$$\mu(t) = \frac{(c+1)^\delta}{\Gamma(\delta)} t^c \left(\log\frac{1}{t}\right)^{\delta-1}, \quad c > -1, \quad \delta \geq 0$$

which gives the Komatu operator.

First we show that the class $T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m, A, B)$ is closed under $V_\mu(f)$.

Theorem 10. *Let $f \in T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m, A, B)$. Then $V_\mu(f) \in T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m, A, B)$.*

Proof. By definition, we have

$$V_\mu(f)(z) = \frac{(c+1)^\delta}{\Gamma(\delta)} \int_0^1 (-1)^{\delta-1} t^c (\log t)^{\delta-1} \left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1} \right) dt$$

$$= \frac{(-1)^{\delta-1}(c+1)^\delta}{\Gamma(\delta)} \lim_{r\to 0^+} \left[\int_r^1 t^c (\log t)^{\delta-1} \left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1} \right) dt \right].$$

A simple calculation gives

$$V_\mu(f)(z) = z - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right)^\delta a_n z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty} \frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)}{2\beta\gamma(1-\lambda)(B-A)} \left(\frac{c+1}{c+n} \right)^\delta a_n \le 1. \tag{4. 1}$$

On the other hand by Theorem 2, $f(z) \in T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)}{2\beta\gamma(1-\lambda)(B-A)} a_n \le 1.$$

Hence $\frac{c+1}{c+n} < 1$. Therefore (4. 1) holds and the proof is complete.
The above theorem yields the following two special cases.

Theorem 11. *If $f(z)$ is starlike of order γ then $V_\mu f(z)$ is also starlike of order α.*

Theorem 12. *If $f(z)$ is convex of order γ then $V_\mu f(z)$ is also convex of order α.*

Theorem 13. *Let $f \in T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$. Then $V_\mu f(z)$ is starlike of order $0 \le \xi < 1$ in $|z| < R_1$, where*

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\xi)\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)}{(n-\xi)(2\beta\gamma(1-\lambda)(B-A))} \right]^{\frac{1}{n-1}}.$$

Proof. It is sufficient to prove

$$\left| \frac{z(V_\mu(f)(z))'}{V_\mu(f)(z)} - 1 \right| < 1 - \xi. \tag{4. 2}$$

For the left hand side of (4. 2) we have

$$\left| \frac{z(V_\mu(f)(z))'}{V_\mu(f)(z)} - 1 \right| = \left| \frac{\sum\limits_{n=2}^{\infty} (1-n) \left(\frac{c+1}{c+n}\right)^\delta a_n z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^\delta a_n z^{n-1}} \right|$$

$$\le \frac{\sum\limits_{n=2}^{\infty} (1-n) \left(\frac{c+1}{c+n}\right)^\delta |a_n|\, |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^\delta |a_n|\, |z|^{n-1}}.$$

The last expression is less than $1-\xi$ since,

$$|z|^{n-1} < \left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)}{(n-\xi)(2\beta\gamma(1-\lambda)(B-A))}.$$

Therefore, the proof is complete.

Using the fact that $f(z)$ is convex if and only if $zf'(z)$ is starlike, we obtain the following.

Theorem 14. *Let $f \in T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$. Then $V_\mu f(z)$ is convex of order $0 \le \xi < 1$ in $|z| < R_2$, where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)}{n(n-\xi)(2\beta\gamma(1-\lambda)(B-A))}\right]^{\frac{1}{n-1}}.$$

Motivated by Silverman [9] in the following section we obtain integral means inequality for the class $T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$.

5. Integral Means

In 1925, Littlewood [3] proved the following subordination theorem.

Lemma 15. *If the functions f and g are analytic in $\mathcal{U}$ with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta \le \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta. \tag{5. 1}$$

Applying Lemma 15 and Lemma 2, we prove the following result.

Theorem 16. *Suppose $f \in T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$, $\eta > 0$, $0 \le \lambda < 1$, $0 \le \gamma < 1$, $k \ge 0$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{1-\gamma}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)} z^2,$$

where $\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)$ is defined in Theorem 2. Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^{\eta} d\theta \le \int_0^{2\pi} |f_2(z)|^{\eta} d\theta. \tag{5. 2}$$

Proof. For $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, (5. 2) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} |a_n| z^{n-1}\right|^{\eta} d\theta \le \int_0^{2\pi} \left|1 - \frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\lambda,\gamma,k,2)} z\right|^{\eta} d\theta.$$

By Lemma 15, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)} z.$$

Setting

$$1-\sum_{n=2}^{\infty}|a_n|z^{n-1}=1-\frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}w(z). \tag{5. 3}$$

From 5. 3 and (1. 11), we obtain

$$\begin{aligned}|w(z)| &= \left|\sum_{n=2}^{\infty}\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)}{2\beta\gamma(1-\lambda)(B-A)}a_n z^{n-1}\right| \\ &\leq |z|\sum_{n=2}^{\infty}\frac{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,n,A,B)}{2\beta\gamma(1-\lambda)(B-A)}|a_n| \\ &\leq |z|.\end{aligned}$$

This completes the proof of the Theorem 16 by the Theorem 2. □

6. Square Root Transformation

Definition 17. If $f\in S$ and $h(z)=\sqrt{f(z^2)}$, then $h\in S$ and $h(z)=z+\sum\limits_{n=2}^{\infty}c_{2n-1}z^{2n-1}$, $|z|<1$. The function h is called a square-root transformation of f.

Theorem 18. *If $f\in T\mathcal{D}_{\lambda,\gamma}^{\alpha,\beta}(m,A,B)$, $2\beta\gamma(1-\lambda)(B-A)\leq\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)$ and h be the square root transformation of f, then*

$$r\sqrt{1-\frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}r^2}\leq|h(z)|\leq r\sqrt{1+\frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}r^2} \tag{6. 1}$$

with equality for

$$f(z)=z-\frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}z^2;\quad (|z|=\pm r). \tag{6. 2}$$

Proof. In the view of [4, Theorem 3.1], we have

$$r^2-\frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}r^4\leq|f(z^2)|\leq r^2+\frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}r^4. \tag{6. 3}$$

Using this inequality in the definition we find

$$\begin{aligned}|h(z)| &= \sqrt{|f(z^2)|} \\ &\leq \sqrt{r^2+\frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}r^4} \\ &= r\sqrt{1+\frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}r^2}.\end{aligned} \tag{6. 4}$$

Since, $2\beta\gamma(1-\lambda)(B-A)\leq\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)$ and $r=|z|<1$, we have

$$1-\frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}r^2\geq 1+\frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)}\geq 0 \tag{6. 5}$$

and hence,

$$\begin{aligned} |h(z)| &= \sqrt{|f(z^2)|} \\ &\geq \sqrt{r^2 - \frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)} r^4} \\ &= r\sqrt{1 - \frac{2\beta\gamma(1-\lambda)(B-A)}{\Phi(\alpha,\beta,\lambda,\gamma,\delta,m,2,A,B)} r^2}. \qquad (6.\ 6) \end{aligned}$$

It can be seen that the result follows from (6. 4) and (6. 6). □

Acknowledgment: The authors would like to thank the referee(s) for their insightful suggestions.

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