Coupled Coincidence Point Theorems for \((\psi, \alpha, \beta)\)-Weak Contractions in Partially Ordered Partial Metric Spaces

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Abstract. In this paper, we obtain coupled coincidence and coupled fixed points of mappings satisfying a non linear contractive condition in partially ordered partial metric spaces. Our results generalize the theorems of H.Aydi [4] and W. Shatanawi et.al. [20]. We also provide an example to support our results.

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1. INTRODUCTION

In 1994 the partial metric space was introduced by Matthews [16] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation.

The notion of coupled fixed point of a mapping \(F : X \times X \to X\) was introduced by Bhaskar and Lakshmikantham in [17] and investigated some coupled fixed point theorems in partially ordered sets. Later on, many authors investigated many coupled fixed point results in different spaces such as usual metric spaces, fuzzy metric spaces, partial metric spaces and partially ordered metric spaces(see [2 - 21]).

In this paper, we prove results of coupled coincidence points of mappings satisfying a nonlinear contractive condition in partially ordered partial metric spaces. Our results generalize the results of H.Aydi [4] and W. Shatanawi et.al.[20]. Consistent with [12, 16, 17, 18], the following definitions and results will be needed in sequel.

A set with a partial order \(\preceq\) is called partially ordered set.
Definition 1. Any two elements \(x\) and \(y\) of a set \(X\), which is partially ordered by a binary relation \(\leq\), are either comparable or incomparable. Specifically, the elements \(x\) and \(y\) are comparable if and only if \(x \preceq y\) or \(y \preceq x\). Otherwise, \(x\) and \(y\) are incomparable.

Definition 2 ([16]). A partial metric on a nonempty set \(X\) is a function \(p : X \times X \to \mathbb{R}^+\) such that for all \(x, y, z \in X\):

\begin{align*}
(p_1) : & \quad x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y), \\
(p_2) : & \quad p(x, x) \leq p(x, y), p(y, y) \leq p(x, y), \\
(p_3) : & \quad p(x, y) = p(y, x), \\
(p_4) : & \quad p(x, y) \leq p(x, z) + p(z, y) - p(z, z).
\end{align*}

\((X, p)\) is called a partial metric space.

It is clear that \(p(x, y) = 0\) implies \(x = y\) from \((p_1)\) and \((p_2)\).

But if \(x = y\), \(p(x, y)\) may not be zero. A basic example of a partial metric space is the pair \((R^+, p)\), where \(p(x, y) = \max\{x, y\}\) for all \(x, y \in R^+\).

Each partial metric \(p\) on \(X\) generates \(\tau_p\) topology \(\tau_p\) on \(X\) which has a base the family of open \(p\)-balls \(\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}\) for all \(x \in X\) and \(\varepsilon > 0\), where \(B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}\) for all \(x \in X\) and \(\varepsilon > 0\).

If \(p\) is a partial metric on \(X\), then the function \(d_p : X \times X \to R^+\) given by \(d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)\) is a metric on \(X\).

Definition 3 ([16]). Let \((X, p)\) be a partial metric space.

(i) A sequence \(\{x_n\}\) in \((X, p)\) is said to converge to a point \(x \in X\) if and only if \(p(x, x) = \lim_{n \to \infty} p(x, x_n)\).

(ii) A sequence \(\{x_n\}\) in \((X, p)\) is said to be Cauchy sequence if \(\lim_{n,m \to \infty} p(x_n, x_m)\) exists and is finite.

(iii) \((X, p)\) is said to be complete if every Cauchy sequence \(\{x_n\}\) in \(X\) converges, w. r. to \(\tau_p\), to a point \(x \in X\) such that \(p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)\).

Lemma 4 ([16]). Let \((X, p)\) be a partial metric space.

(a) \(\{x_n\}\) is a Cauchy sequence in \((X, p)\) if and only if it is a Cauchy sequence in the metric space \((X, d_p)\).

(b) \((X, p)\) is complete if and only if the metric space \((X, d_p)\) is complete. Furthermore,

\[\lim_{n \to \infty} d_p(x_n, x) = 0\quad \text{if and only if}\quad \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m).\]

Lemma 5 ([3]). Let \((X, p)\) be a partial metric space and \(x_n \to z\) as \(n \to \infty\) in \((X, p)\) such that \(p(z, z) = 0\). Then \(\lim_{n \to \infty} p(x_n, y) = p(z, y)\) for every \(y \in X\).

Definition 6 ([17]). An element \((x, y) \in X \times X\) is called a coupled fixed point of mapping \(F : X \times X \to X\) if \(x = F(x, y)\) and \(y = F(y, x)\).

Definition 7 ([12]). An element \((x, y) \in X \times X\) is called

\(g_1\) a coupled coincident point of mappings \(F : X \times X \to X\) and \(f : X \to X\) if \(f x = F(x, y)\) and \(f y = F(y, x)\).

\(g_2\) a common coupled fixed point of mappings \(F : X \times X \to X\) and \(f : X \to X\) if \(x = f x = F(x, y)\) and \(y = f y = F(y, x)\).

Definition 8 ([12]). The mappings \(F : X \times X \to X\) and \(f : X \to X\) are called

\(w\) - compatible if \(f(F(x, y)) = F(fx, fy)\) and \(f(F(y, x)) = F(fy, fx)\) whenever \(fx = F(x, y)\) and \(fy = F(y, x)\).
**Definition 9** ([17]). Let \((X, \preceq)\) be a partially ordered set and \(F : X \times X \to X\). Then the map \(F\) is said to have mixed monotone property if \(F(x, y)\) is monotone non-decreasing in \(x\) and is monotone non-increasing in \(y\); that is, for any \(x, y \in X\),
\[x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X\]
and
\[y_1 \preceq y_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X.\]

**Definition 10** ([18]). Let \((X, \preceq)\) be a partially ordered set and \(F : X \times X \to X\). Then the map \(F\) is said to have mixed g-monotone property if \(F(x, y)\) is monotone g-non-decreasing in \(x\) and is monotone g-non-increasing in \(y\); that is, for any \(x, y \in X\),
\[g x_1 \preceq g x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X\]
and
\[g y_1 \preceq g y_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X.\]

In recent year 2011, Aydi [4], proved the following coupled fixed point theorems in partial metric spaces.

**Theorem 11.** Let \((X, p)\) be a complete partial metric space. Suppose that the mapping \(F : X \times X \to X\) satisfies the following contractive condition for all \(x, y, u, v \in X\)
\[p(F(x, y), F(u, v)) \leq k p(x, u) + l p(y, v),\]
where \(k, l\) are nonnegative constants with \(k + l < 1\). Then \(F\) has a unique coupled fixed point.

**Theorem 12.** Let \((X, p)\) be a complete partial metric space. Suppose that the mapping \(F : X \times X \to X\) satisfies the following contractive condition for all \(x, y, u, v \in X\)
\[p(F(x, y), F(u, v)) \leq k p(F(x, y), x) + l p(F(u, v), u),\]
where \(k, l\) are non-negative constants with \(k + l < 1\). Then \(F\) has a unique coupled fixed point.

The main theorems of Shatanawi et. al. [20], are

**Theorem 13** (Theorem 3, [20]). Let \((X, \preceq)\) be a partially ordered set and \(p\) be a partial metric on \(X\) such that \((X, p)\) is a complete partial metric space. Let \(F : X \times X \to X\) be a continuous mapping having the mixed monotone property on \(X\). Assume that for \(x, y, u, v \in X\) with \(x \preceq u\) and \(y \succeq v\), we have
\[\psi(p(F(x, y), F(u, v))) \leq \psi(\max\{p(x, u), p(y, v)\}) - \phi(\max\{p(x, u), p(y, v)\}),\]
where \(\psi\) and \(\phi\) are altering distance functions. If there exists \((x_0, y_0) \in X \times X\) where \(x_0 \preceq F(x_0, y_0)\) and \(y_0 \succeq F(y_0, x_0)\), then \(F\) has a coupled fixed point.

**Theorem 14** (Theorem 4, [20]). Let \((X, \preceq)\) be a partially ordered set and \((X, p)\) be a complete partial metric space. Let \(F : X \times X \to X\) be a mapping having the mixed monotone property. Assume there are two altering distance functions such that
\[(14.1)\psi(p(F(x, y), F(u, v))) \leq \psi(\max\{p(x, u), p(y, v)\}) - \phi(\max\{p(x, u), p(y, v)\}),\]
for all \(x, y, u, v \in X\) with \(x \preceq u\) and \(v \succeq y\). Also, suppose that \(X\) has the following properties:
(i) if \(\{x_n\}\) is a non-decreasing sequence and \(x \in X\) with
\[\lim_{n \to \infty} p(x_n, x) = p(x, x) = 0,\] then \(x_n \preceq x\) for all \(n\),
The function φ: \( R^+ \to R^+ \) is called an altering distance function if the following properties are satisfied:

(i) φ is continuous and non-decreasing.
(ii) \( φ(t) = 0 \) if and only if \( t = 0 \).

Theorem 16. Let \( (X, \preceq) \) be a partially ordered set and \( p \) be a partial metric such that \( (X, p) \) is complete metric. Let \( F: X \times X \to X \), \( g: X \to X \) be such that

\[
\psi(p(F(x, y), F(u, v))) \leq \alpha(M(x, y, u, v)) - \beta(M(x, y, u, v)),
\]

\( \forall x, y, u, v \in X, gx \preceq gu \text{ and } gy \preceq gv \), where \( \psi, \alpha, \beta : [0, \infty) \to [0, \infty) \) are such that \( \psi \) is an altering distance function, \( \alpha \) is continuous, \( \beta \) is lower semi continuous, \( \alpha(0) = \beta(0) = 0 \) and \( \psi(t) - \alpha(t) + \beta(t) > 0 \), for all \( t > 0 \) and

\[
M(x, y, u, v) = \max \left\{ p(gx, gu), p(gy, gv), p(gx, F(x, y)), p(gy, F(y, x)), p(gx, F(u, v)), p(gu, F(u, v)), p(gu, F(v, u)), p(guv, F(x, y)) \right\}.
\]

(16.2) \( F(X \times X) \subseteq g(X) \) and \( g(X) \) is a complete subspace of \( X \).

(16.3) \( F \) has a mixed \( g \)-monotone property.

(16.4) (a) If a non-decreasing sequence \( \{x_n\} \to x \), then \( x_n \preceq x \) for all \( n \).

(b) If a non-increasing sequence \( \{y_n\} \to y \), then \( y_n \preceq y \) for all \( n \).

If there exist \( x_0, y_0 \in X \) such that \( gx_0 \preceq F(x_0, y_0) \) and \( gy_0 \preceq F(y_0, x_0) \), then \( F \) and \( g \) have coupled coincidence point in \( X \times X \).

Proof. Let \( x_0, y_0 \in X \) such that \( gx_0 \preceq F(x_0, y_0) \) and \( gy_0 \preceq F(y_0, x_0) \).

Since \( F(X \times X) \subseteq g(X) \), we choose \( x_1, y_1 \in X \) such that

\[
gx_0 \preceq F(x_0, y_0) = gx_1 \quad \text{and} \quad gy_0 \preceq F(y_0, x_0) = gy_1
\]

and choose \( x_2, y_2 \in X \) such that \( gx_2 = F(x_1, y_1) \) and \( gy_2 = F(y_1, x_1) \).

Since \( F \) has the mixed \( g \)-monotone property, we obtain \( gx_0 \preceq gx_1 \preceq gx_2 \) and \( gy_0 \preceq gy_1 \preceq gy_2 \). Continuing this process, we can construct the sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
a_n = gx_n = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad b_n = gy_n = F(y_{n-1}, x_{n-1}), n = 0, 1, 2, \ldots
\]

with

\[
\begin{align*}
gx_0 \preceq gx_1 \preceq gx_2 \preceq \cdots \quad \text{and} \\
gy_0 \preceq gy_1 \preceq gy_2 \preceq \cdots
\end{align*}
\]

Choose \( R_n = \max \{p(a_n, a_{n+1}), p(b_n, b_{n+1})\} \).
Since $\psi$ is non-decreasing, we have $\alpha(R_{n-1}) - \beta(R_{n-1}) < \psi(R_{n-1})$. Thus from (2.1) we get, 

$$\psi(R_n) \leq \psi(R_{n-1}).$$ 

Since $\psi$ is non-decreasing, we obtain 

$$R_n \leq R_{n-1}.$$
Thus \( \{ R_n \} \) is non-increasing sequence of non-negative real numbers and must converge to a real number \( r \geq 0 \) say.

Suppose \( r > 0 \).

Letting \( n \to \infty \) in (2.1), we get

\[
\psi(r) \leq \alpha(r) - \beta(r),
\]

which implies that \( r = 0 \), a contradiction.

Hence \( r = 0 \). Thus

\[
\lim_{n \to \infty} \max \{ p(a_n, a_{n+1}), p(b_n, b_{n+1}) \} = 0.
\]

Hence

\[
\lim_{n \to \infty} p(a_n, a_{n+1}) = 0 = \lim_{n \to \infty} p(b_n, b_{n+1}).
\]  

From (2.2) and (2.3) and by definition of \( d_p \), we get

\[
\lim_{n \to \infty} d_p(a_n, a_{n+1}) = 0 = \lim_{n \to \infty} d_p(b_n, b_{n+1}).
\]

Now we prove that \( \{ a_n \} \) and \( \{ b_n \} \) are Cauchy sequences.

We shall show that for every \( \epsilon > 0 \), there exists \( k \in N \) such that if \( m, n \geq k \),

\[
\max \{ d_p(a_n, a_m), d_p(b_n, b_m) \} < \epsilon.
\]

Suppose the above statement is false.

Then there exist an \( \epsilon > 0 \) and monotone increasing sequences of natural numbers \( \{ m_k \} \) and \( \{ n_k \} \) such that \( n_k > m_k > k \),

\[
\max \{ d_p(a_{m_k}, a_{n_k}), d_p(b_{m_k}, b_{n_k}) \} \geq \epsilon.
\]  

and

\[
\max \{ d_p(a_{m_k}, a_{n_k-1}), d_p(b_{m_k}, b_{n_k-1}) \} < \epsilon.
\]

From (2.5) and (2.6), we have

\[
\epsilon \leq \max \{ d_p(a_{m_k}, a_{n_k}), d_p(b_{m_k}, b_{n_k}) \} \\
\leq \max \{ d_p(a_{m_k}, a_{n_k-1}), d_p(b_{m_k}, b_{n_k-1}) \} \\
+ \max \{ d_p(a_{n_k-1}, a_{n_k}), d_p(b_{n_k-1}, b_{n_k}) \} \\
< \epsilon + \max \{ d_p(a_{n_k-1}, a_{n_k}), d_p(b_{n_k-1}, b_{n_k}) \}.
\]

Letting \( k \to \infty \) and using (2.4), we get

\[
\lim_{k \to \infty} \max \{ d_p(a_{m_k}, a_{n_k}), d_p(b_{m_k}, b_{n_k}) \} = \epsilon.
\]

By definition of \( d_p \) and using (2.3) we get

\[
\lim_{k \to \infty} \max \{ p(a_{m_k}, a_{n_k}), p(b_{m_k}, b_{n_k}) \} = \frac{\epsilon}{2}.
\]

Since

\[
d_p(a_{n_k+1}, a_{m_k+1}) \leq d_p(a_{n_k+1}, a_{n_k}) + d_p(a_{n_k}, a_{m_k}) + d_p(a_{m_k}, a_{m_k+1})
\]

and

\[
d_p(b_{n_k+1}, b_{m_k+1}) \leq d_p(b_{n_k+1}, b_{n_k}) + d_p(b_{n_k}, b_{m_k}) + d_p(b_{m_k}, b_{m_k+1})
\]

we have
\[
\max \left\{ \frac{d_p(a_{n_k+1}, a_{m_{k+1}})}{d_p(b_{m_{k+1}}, b_{m_{k+1}})} \right\} \leq \max \left\{ \frac{d_p(a_{n_k+1}, a_{n_k}), d_p(b_{n_{k+1}}, b_{n_k})}{d_p(a_{n_k}, a_{n_k+1}), d_p(b_{n_k}, b_{n_{k+1}})} \right\} \\
+ \max \left\{ \frac{d_p(a_{n_k}, a_{n_k+1}), d_p(b_{n_k}, b_{n_{k+1}})}{d_p(a_{m_{k+1}}, a_{m_k}), d_p(b_{m_{k+1}}, b_{m_{k+1}})} \right\}.
\]

Letting \( k \to \infty \), using (2.4) and (2.7), we get
\[
\lim_{k \to \infty} \max \left\{ d_p(a_{n_{k+1}}, a_{m_{k+1}}), d_p(b_{n_{k+1}}, b_{m_{k+1}}) \right\} \leq \epsilon.
\] (2.11)

Again since
\[
d_p(a_{n_k}, a_{m_k}) \leq d_p(a_{n_k}, a_{n_{k+1}}) + d_p(a_{n_{k+1}}, a_{n_{k+1}}) + d_p(a_{n_{k+1}}, m_{k})
\] (2.12)

and
\[
d_p(b_{n_k}, b_{m_k}) \leq d_p(b_{n_k}, b_{n_{k+1}}) + d_p(b_{n_{k+1}}, b_{n_{k+1}}) + d_p(b_{n_{k+1}}, b_{m_k})
\] (2.13)

we have
\[
\max \left\{ \frac{d_p(a_{n_k}, a_{m_k})}{d_p(b_{n_k}, b_{m_k})} \right\} \leq \max \left\{ \frac{d_p(a_{n_k}, a_{n_{k+1}}), d_p(b_{n_k}, b_{n_{k+1}})}{d_p(a_{n_k}, a_{n_{k+1}}), d_p(b_{n_k}, b_{n_{k+1}})} \right\} \\
+ \max \left\{ \frac{d_p(a_{n_{k+1}}, a_{m_k}), d_p(b_{n_{k+1}}, b_{m_k})}{d_p(a_{n_{k+1}}, a_{m_k}), d_p(b_{n_{k+1}}, b_{m_k})} \right\}.
\]

Letting \( k \to \infty \) and using (2.4), we have
\[
\epsilon \leq \lim_{k \to \infty} \max \left\{ d_p(a_{n_{k+1}}, a_{m_{k+1}}), d_p(b_{n_{k+1}}, b_{m_{k+1}}) \right\}.
\] (2.14)

Now, from (2.11) and (2.14), we get
\[
\lim_{k \to \infty} \max \left\{ d_p(a_{n_{k+1}}, a_{m_{k+1}}), d_p(b_{n_{k+1}}, b_{m_{k+1}}) \right\} = \epsilon.
\] (2.15)

By definition of \( d_p \) and using (2.3) we get
\[
\lim_{k \to \infty} \max \left\{ p(a_{n_{k+1}}, a_{n_{k+1}}), p(b_{m_{k+1}}, b_{m_{k+1}}) \right\} = \frac{\epsilon}{2}.
\] (2.16)

As \( n_k > m_k \), we have \( g x_{m_k} \leq g x_{m_k} \) and \( g y_{m_k} \geq g y_{m_k} \). Putting \( x = x_{m_k}, y = y_{m_k} \),
\( u = x_{n_k}, v = y_{n_k} \) in (16.1), we have
\[
\psi(p(a_{n_{k+1}}, a_{n_{k+1}})) = \psi\left(p(F(x_{m_k}, y_{m_k}), F(x_{n_k}, y_{n_k}))\right) \leq \alpha \left(M(x_{m_k}, y_{m_k}, x_{n_k}, y_{n_k}) - \beta \left(M(x_{m_k}, y_{m_k}, x_{n_k}, y_{n_k})\right)\right).
\]

\[
= \alpha \left( \max \left\{ \frac{p(a_{m_k}, a_{n_k}), p(b_{n_k}, b_{n_k}), p(a_{m_{k+1}}, a_{n_{k+1}})}{\frac{1}{2}[p(a_{m_k}, a_{n_{k+1}}) + p(a_{n_k}, a_{m_{k+1}})]} \right\} \right.
\]
and
\[
= \beta \left( \max \left\{ \frac{p(a_{m_k}, a_{n_k}), p(b_{n_k}, b_{n_k}), p(a_{m_{k+1}}, a_{n_{k+1}})}{\frac{1}{2}[p(a_{m_k}, a_{n_{k+1}}) + p(a_{n_k}, a_{m_{k+1}})]} \right\} \right).
\]
By using \((p_4)\) we get

\[
\psi(p(a_{m_{k+1}}, a_{n_{k+1}}) \leq \alpha \left( \max \left\{ \begin{array}{l}
p(a_{m_k}, a_{n_k}), p(b_{m_k}, b_{n_k}), p(a_{m_k}, a_{m_{k+1}}), \\
p(b_{m_k}, b_{m_{k+1}}), p(a_{m_k}, a_{n_{k+1}}), p(b_{n_k}, b_{n_{k+1}}), \\
\frac{1}{2}[p(a_{m_k}, a_{n_k}) + p(a_{n_k}, a_{m_{k+1}})] \end{array} \right\} \right) - \beta \left( \max \left\{ \begin{array}{l}
p(a_{m_k}, a_{n_k}), p(b_{m_k}, b_{n_k}), p(a_{m_k}, a_{m_{k+1}}), \\
p(b_{m_k}, b_{m_{k+1}}), p(a_{m_k}, a_{n_{k+1}}), p(b_{n_k}, b_{n_{k+1}}), \\
\frac{1}{2}[p(a_{m_k}, a_{n_k}) + p(a_{n_k}, a_{m_{k+1}})] \end{array} \right\} \right)
\]

Similarly we have

\[
\psi(p(b_{m_{k+1}}, b_{n_{k+1}}) \leq \alpha \left( \max \left\{ \begin{array}{l}
p(b_{m_k}, b_{n_k}), p(a_{m_k}, a_{n_k}), p(b_{m_k}, b_{m_{k+1}}), \\
p(a_{m_k}, a_{m_{k+1}}), p(b_{m_k}, b_{n_{k+1}}), p(a_{n_k}, a_{n_{k+1}}), \\
\frac{1}{2}[p(b_{m_k}, b_{n_k}) + p(b_{n_k}, b_{n_{k+1}})] \end{array} \right\} \right) - \beta \left( \max \left\{ \begin{array}{l}
p(b_{m_k}, b_{n_k}), p(a_{m_k}, a_{n_k}), p(b_{m_k}, b_{m_{k+1}}), \\
p(a_{m_k}, a_{m_{k+1}}), p(b_{m_k}, b_{n_{k+1}}), p(a_{n_k}, a_{n_{k+1}}), \\
\frac{1}{2}[p(b_{m_k}, b_{n_k}) + p(b_{n_k}, b_{n_{k+1}})] \end{array} \right\} \right)
\]

Now we have

\[
\psi(\max \left\{ p(a_{m_{k+1}}, a_{n_{k+1}}), p(b_{m_{k+1}}, b_{n_{k+1}}) \right\}) \leq \alpha \left( \max \left\{ \begin{array}{l}
p(b_{m_k}, b_{n_k}), p(a_{m_k}, a_{n_k}), p(b_{m_k}, b_{m_{k+1}}), \\
p(a_{m_k}, a_{m_{k+1}}), p(b_{m_k}, b_{n_{k+1}}), p(a_{n_k}, a_{n_{k+1}}), \\
\frac{1}{2}[p(b_{m_k}, b_{n_k}) + p(b_{n_k}, b_{n_{k+1}})] \end{array} \right\} \right) - \beta \left( \max \left\{ \begin{array}{l}
p(b_{m_k}, b_{n_k}), p(a_{m_k}, a_{n_k}), p(b_{m_k}, b_{m_{k+1}}), \\
p(a_{m_k}, a_{m_{k+1}}), p(b_{m_k}, b_{n_{k+1}}), p(a_{n_k}, a_{n_{k+1}}), \\
\frac{1}{2}[p(b_{m_k}, b_{n_k}) + p(b_{n_k}, b_{n_{k+1}})] \end{array} \right\} \right).
\]

Letting \(k \to \infty\) and using (2.4),(2.8),(2.16), we have

\[
\psi(\frac{\epsilon}{2}) \leq \alpha(\frac{\epsilon}{2}) - \beta(\frac{\epsilon}{2}).
\]

Thus \(\psi(\frac{\epsilon}{2}) - \alpha(\frac{\epsilon}{2}) + \beta(\frac{\epsilon}{2}) \leq 0\) and hence \(\epsilon = 0\), which is a contradiction. Consequently, \(\{a_n\}\) and \(\{b_n\}\) are Cauchy.

Hence \(\{gx_n\}\) and \(\{gy_n\}\) are Cauchy sequences in the metric space \((X, d_p)\).

Hence we have \(\lim_{n \to \infty} d_p(gx_n, gx_m) = 0 = \lim_{n \to \infty} d_p(gy_n, gy_m)\).

Now from definition of \(d_p\) and from (2.3) we have

\[
\lim_{n \to \infty} p(gx_n, gx_m) = 0 = \lim_{n \to \infty} p(gy_n, gy_m). \tag{2.17}
\]

Suppose \(g(X)\) is a complete sub space of \(X\). Since \(\{gx_n\}\) and \(\{gy_n\}\) are Cauchy sequences in complete metric space \((g(X), d_p)\), it follows that \(\{gx_n\}\) and \(\{gy_n\}\) are converge to some \(r\) and \(s\) in \(g(X)\).

Thus

\[
\lim_{n \to \infty} d_p(gx_n, r) = 0
\]

and

\[
\lim_{n \to \infty} d_p(gy_n, s) = 0
\]
From Lemma 4(b) and (2.17), we obtain
\[ p(r, r) = \lim_{n \to \infty} p(gx_n, r) = p(s, s) = \lim_{n \to \infty} p(gy_n, s) = 0. \]  
(2.18)

Now from Lemma 5 and (2.18), we have
\[ \lim_{n \to \infty} p(F(x, y), gx_n) = p(F(x, y), r) \text{ and} \]
\[ \lim_{n \to \infty} p(F(y, x), gy_n) = p(F(y, x), s). \]

From (p4), we have that
\[ p(r, F(x, y)) \leq p(r, gx_{n+1}) + p(gx_{n+1}, F(x, y)) - p(gx_{n+1}, gx_{n+1}) \]
\[ \leq p(r, gx_{n+1}) + p(gx_{n+1}, F(x, y)). \]

Letting \( n \to \infty \), we have
\[ p(r, F(x, y)) \leq 0 + \lim_{n \to \infty} p(F(x, y), F(x, y)). \]

Also from (16.4), we get \( gx_n \preceq gx \) and \( gy_n \succeq gy \). Since \( \psi \) is continuous and non-decreasing function, we get
\[ \psi\left(p(r, F(x, y))\right) \leq \lim_{n \to \infty} \psi\left(p(F(x, y), F(x, y))\right) \]
\[ \leq \lim_{n \to \infty} \left[ \psi\left(M(x_n, y_n, x, y)\right) - \beta(M(x_n, y_n, x, y))\right]. \]

Therefore
\[ \psi\left(p(r, F(x, y))\right) \leq \alpha\left(\max\left\{ p(r, F(x, y)), \frac{p(gx_n, F(x, y))}{p(s, F(y, x))}\right\}\right) - \beta\left(\max\left\{ p(r, F(x, y)), \frac{p(gy_n, F(x, y))}{p(s, F(y, x))}\right\}\right). \]

Similarly,
\[ \psi\left(p(s, F(y, x))\right) \leq \alpha\left(\max\left\{ p(r, F(x, y)), \frac{p(s, F(y, x))}{p(s, F(y, x))}\right\}\right) - \beta\left(\max\left\{ p(r, F(x, y)), \frac{p(s, F(y, x))}{p(s, F(y, x))}\right\}\right). \]

Hence
\[ \psi\left(\max\{p(r, F(x, y)), p(s, F(y, x))\}\right) \]
\[ = \max\{\psi(p(r, F(x, y))), \psi(p(s, F(y, x)))\} \]
\[ \leq \alpha\left(\max\left\{ p(r, F(x, y)), \frac{p(r, F(x, y))}{p(s, F(y, x))}\right\}\right) - \beta\left(\max\left\{ p(r, F(x, y)), \frac{p(s, F(y, x))}{p(s, F(y, x))}\right\}\right). \]

Hence from our assumptions, we have
\[ \max\{ p(r, F(x, y)), p(s, F(y, x))\} = 0 \]
In addition to the hypothesis of Theorem 16, suppose that for every point in $X \times X$ there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^1, y^1), F(y^1, x^1))$. If $(x, y)$ and $(x^1, y^1)$ are coupled coincidence points of $F$ and $g$, then

$$F(x, y) = gx = gx^1 = F(x^1, y^1) \text{ and } T(y, x) = gy = gy^1 = F(y^1, x^1).$$

Moreover if $(F, g)$ is $w$-compatible, then $F$ and $g$ have a unique common coupled fixed point in $X \times X$.

**Proof.** From Theorem 16, there exists $(x, y) \in X \times X$ such that $F(x, y) = gx = r$ and $F(y, x) = gy = s$. Thus the existence of coupled coincidence point of $F$ and $g$ is confirmed. Now if $(x^1, y^1)$ be another coupled coincidence point of $F$ and $g$, we will prove that

$$gx = gx^1 \text{ and } gy = gy^1 \quad (2.19)$$

By additional assumption, there is $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^1, y^1), F(y^1, x^1))$. Let $u_0 = u, v_0 = v, x_0 = x, y_0 = y, x_0^1 = x^1$ and $y_0^1 = y^1$. Since $F(X \times X) \subseteq g(X)$, we can construct the sequences $\{gu_n\}, \{gv_n\}, \{gx_n\}, \{gy_n\}, \{gx_n^1\}$ and $\{gy_n^1\}$ such that

$$gu_{n+1} = F(u_n, v_n), \quad gv_{n+1} = F(v_n, u_n),$$
$$gx_{n+1} = F(x_n, y_n), \quad gy_{n+1} = F(y_n, x_n),$$
$$gx_{n+1}^1 = F(x_n^1, y_n), \quad gy_{n+1}^1 = F(y_n^1, x_n^1),$$

for all $n = 0, 1, 2, \cdots$. Since $(gx, gy) = (F(x, y), F(y, x)) = (gx_1, gy_1)$ and $(F(u, v), F(v, u)) = (gu_1, gv_1)$ are comparable, then $gx \preceq gu_1$ and $gy \preceq gv_1$.

One can show that $gx \preceq gu_n$ and $gy \preceq gv_n$ for all $n$. As in Theorem 16, we conclude that $\{gu_{n+1}\} \rightarrow gx$ and $\{gv_{n+1}\} \rightarrow gy$. Analogously, we can show that $\{gu_{n+1}\} \rightarrow gx^1$ and $\{gv_{n+1}\} \rightarrow gy^1$ in $(g(X), d_g)$. Since $g(X)$ is complete and $\{gu_{n+1}\}$ converges to $gx$ and $gx^1$, we get $gx = gx^1$. Similarly $gy = gy^1$. Hence (2.19) is proved.

Since $(F, g)$ is $w$-compatible, then

$$gr = g(F(x, y)) = F(gx, gy) = F(r, s) \text{ and } gs = g(F(y, x)) = F(gy, gx) = F(s, r).$$

Hence the pair $(r, s)$ is also coupled coincidence point of $F$ and $g$. From (2.19), we have

$$gr = gx \text{ and } gs = gy.$$

Therefore

$$r = gr = F(r, s) \text{ and } s = gs = F(s, r).$$

Thus $(r, s)$ is common coupled fixed point of $F$ and $g$. The uniqueness of common coupled fixed point of $F$ and $g$ follows easily in the similar lines of the above argument.

$\square$
Example 18. Let $X = [0, 1]$ we define a partial order "≤" on $X$ as $x ≤ y$ if and only if $x ≤ y$ for all $x, y ∈ X$. Let a partial metric space $p$ on $X$ be defined by $p(x, y) = \max\{x, y\}$, for all $x, y ∈ X$, then $(X, p)$ is a complete partial metric space. Define $F : X × X → X$ as

$$F(x, y) = \frac{|x - y|}{2}$$

for all $x, y ∈ X$ and $g : X → X$ be defined by $gx = x$.

Let $ψ, α, β : [0, ∞) → [0, ∞)$ be defined by $ψ(t) = 4t$, $α(t) = 7t$ and $β(t) = \frac{7}{2}t$. Clearly $ψ$ is an altering distance function, $α$ is continuous and $β$ is lower semicontinuous, $α(0) = 0$, $β(0) = 0$ and $ψ(t) - α(t) + β(t) = \frac{t}{2} > 0$ for all $t > 0$.

Now, let $x ≤ u$ and $y ≥ v$. So we have

$$ψ(p(F(x, y), F(u, v))) = 4 \max\left\{\frac{|x - y|}{2}, \frac{|u - v|}{2}\right\}$$

$$= 2 \max\{|x - y|, |u - v|\}$$

$$= 2 \max\{x - y, y - x, u - v, v - u\}$$

$$≤ 2 \max\{x, y, u, v\}$$

$$= 2 \max\{p(gx, gy), p(gu, gv)\}$$

$$≤ 2M(x, y, u, v)$$

$$≤ 7M(x, y, u, v) - \frac{7}{2}M(x, y, u, v).$$

So

$$ψ(p(F(x, y), F(u, v))) ≤ α(M(x, y, u, v)) - β(M(x, y, u, v)).$$

Therefore, all of the conditions of Theorem 17 are satisfied. Moreover, $(0, 0)$ is the unique coupled coincidence point of $F$ and $g$.

Now we show that Theorem 14 is not applicable in this example.

For this let $(x, y) = (0, 1)$, $(u, v) = (0, 0)$ and $ϕ = β$.

Then

$$ψ(p(F(x, y), F(u, v))) = ψ(p(\frac{1}{2}, 0)) = 4 \times \frac{1}{2} = 2.$$  

$$ψ(\max\{p(x, u), p(y, v)\}) = ψ(\max\{0, 1\}) = ψ(1) = 4.$$  

$$ϕ(\max\{p(x, u), p(y, v)\}) = ϕ(\max\{0, 1\}) = β(1) = \frac{7}{2}.$$  

Thus

$$ψ(p(F(x, y), F(u, v))) > ϕ(\max\{p(x, u), p(y, v)\}) - ϕ(\max\{p(x, u), p(y, v)\}).$$

Hence, the condition (14.1) is not satisfied.

References


