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Chromatic Uniqueness of Complete Bipartite Graphs with Certain Edges Deleted II

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Abstract. For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}_2^{-s}(p,q)$ denote the set of $2-$connected bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges. In this paper, we prove that for any graph $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$, if $11 \leq s \leq q-1$ and $\Delta(G') = s-4$, where $G' = K_{p,q} - G$, then G is chromatically unique. This result extends both a theorem by Dong et al. [2] and the result in [4].

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1. Introduction

All graphs considered here are simple graphs. For a graph G, let $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$ and $P(G,\lambda)$ be the vertex set, edge set, minimum degree, maximum degree and the chromatic polynomial of G, respectively.
Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1,\lambda) \neq P(G_2,\lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be chromatically disjoint, or simply $\chi-$*disjoint*.
For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p,q)$ (resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. $2-$connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges.
For a bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let $G' = (A', B'; E')$ be the bipartite graph induced by the edge set $E' = \{xy \mid xy \notin E, x \in$

$A, y \in B$ }, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K_{p,q} - G$, where $p = |A|$ and $q = |B|$. Let $\Delta(G')$ denote the maximum degree of G'.
Dong et al. [1] have shown that any $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$, is chromatically unique if one of the following conditions holds.

(i): $5 \leq s \leq q-1$ and if $\Delta(G') = s-1$, or
(ii): $7 \leq s \leq q-1$ and if $\Delta(G') = s-2$.

In [5], we proved that for any $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$, is chromatically unique if $9 \leq s \leq q-1$ and if $\Delta(G') = s-3$. In this paper, we give a similar result by examining the chromatic uniqueness of any $G \in \mathcal{K}_2^{-s}(p,q)$, where $11 \leq s \leq q-1$ and if $\Delta(G') = s-4$. Our result is similar to that of Dong et al. in [1].

2. Preliminary Results And Notations

For a graph G and a positive integer k, a partition $\{A_1, A_2, \cdots, A_k\}$ of $V(G)$ is called a k-independent partition in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions in G. For any graph G of order n, we have (see [3]):

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1).$$

Thus, we have

Lemma 1. *If $G \sim H$, then $\alpha(G,k) = \alpha(H,k)$ for $k = 1, 2, \ldots$.*

Partition $\mathcal{K}^{-s}(p,q)$ into the following subsets:

$$\mathcal{D}_i(p,q,s) = \left\{ G \in \mathcal{K}^{-s}(p,q) \;\middle|\; \triangle(G') = i \right\}, \quad i = 1, 2, \ldots, s.$$

The following result was obtained in [1].

Theorem 2. *Let $p \geq q \geq 3$ and $1 \leq s \leq q-1$.*

(i): *$\mathcal{D}_1(p,q,s)$ is $\chi-$closed.*
(ii): *$\cup_{2 \leq i \leq (s+3)/2} \mathcal{D}_i(p,q,s)$ is $\chi-$closed for $s \geq 2$.*
(iii): *$\mathcal{D}_i(p,q,s)$ is $\chi-$closed for each i with $\lceil (s+3)/2 \rceil \leq i \leq min\{s, q-2\}$.*
(iv): *$\mathcal{D}_{q-1}(p,q,s) \cap \mathcal{K}_2^{-s}(p,q)$ is $\chi-$closed for $s = q-1$.*

For any bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, we have

$$\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2). \tag{2. 1}$$

For a bipartite graph $G = (A, B; E)$, let $\mathcal{I}(G)$ be the set of independent sets in G and let

$$\Omega(G) = \{\ Q \in \mathcal{I}(G) \mid Q \cap A \neq \emptyset, Q \cap B \neq \emptyset\ \}.$$

The following result is then obtained.

Lemma 3. *(Dong et al. [2]) For $G \in \mathcal{K}^{-s}(p,q)$,*

$$\alpha'(G,3) = |\Omega(G)| \geq 2^{\triangle(G')} + s - 1 - \triangle(G').$$

For a bipartite graph $G = (A, B; E)$, the number of $4-$independent partitions $\{\ A_1, A_2, A_3, A_4\ \}$ in G with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i = 1, 2, 3, 4$ is

$$(2^{|A|-1}-1)(2^{|B|-1}-1)+\frac{1}{3!}(3^{|A|}-3\cdot 2^{|A|}+3)+\frac{1}{3!}(3^{|B|}-3\cdot 2^{|B|}+3)$$
$$=(2^{|A|-1}-2)(2^{|B|-1}-2)+\frac{1}{2}(3^{|A|}+3^{|B|})-2.$$

Define

$$\alpha'(G,4)=\alpha(G,4)-\left\{\ (2^{|A|-1}-2)(2^{|B|-1}-2)+\frac{1}{2}(3^{|A|}+3^{|B|})-2\ \right\}.$$

Observe that for $G,H\in\mathcal{K}^{-s}(p,q)$,

$$\alpha(G,4)=\alpha(H,4)\quad\text{if and only if}\quad\alpha'(G,4)=\alpha'(H,4).$$

The following lemmas will be used to prove our main result.

Lemma 4. *(Dong et al.* [3]*) For $G=(A,B;E)\in\mathcal{K}^{-s}(p,q)$ with $|A|=p$ and $|B|=q$,*

$$\alpha'(G,4)=\sum_{Q\in\Omega(G)}(2^{p-1-|Q\cap A|}+2^{q-1-|Q\cap B|}-2)$$
$$+\left|\ \{\ \{Q_1,Q_2\}\mid Q_1,Q_2\in\Omega(G),\ Q_1\cap Q_2=\emptyset\ \}\ \right|.$$

Lemma 5. *(Dong et al.* [3]*) For a bipartite graph $G=(A,B;E)$, if uvw is a path in G' with $d_{G'}(u)=1$ and $d_{G'}(v)=2$, then for any $k\geq 2$,*

$$\alpha(G,k)=\alpha(G+uv,k)+\alpha(G-\{u,v\},k-1)+\alpha(G-\{u,v,w\},k-1).$$

By using Lemma 5we obtain the following lemma.

Lemma 6. (Roslan and Peng [5]) *For a bipartite graph $G=(A,B;E)$, if uvw, uvy and wvy are three paths in G' with $d_{G'}(u)=1$ and $d_{G'}(v)=3$, then for any $k\geq 2$,*

$$\begin{aligned}\alpha(G,k) &= \alpha(G+uv,k)+\alpha(G-\{u,v\},k-1)+\alpha(G-\{u,v,w\},k-1)+\\ &\quad \alpha(G-\{u,v,y\},k-1)+\alpha(G-\{u,v,w,y\},k-1).\end{aligned}$$

By extending Lemmas 5 and 6, we obtain the following result.

Lemma 7. *For a bipartite graph $G=(A,B;E)$, if uvw, uvy, uvt, wvy, wvt and yvt are six paths in G' with $d_{G'}(u)=1$ and $d_{G'}(v)=4$, then for any $k\geq 2$,*

$$\begin{aligned}\alpha(G,k) &= \alpha(G+uv,k)+\alpha(G-\{u,v\},k-1)+\alpha(G-\{u,v,t\},k-1)+\\ &\quad \alpha(G-\{u,v,y\},k-1)+\alpha(G-\{u,v,y,t\},k-1)+\\ &\quad \alpha(G-\{u,v,w\},k-1)+\alpha(G-\{u,v,w,t\},k-1)+\\ &\quad \alpha(G-\{u,v,w,y\},k-1)+\alpha(G-\{u,v,w,y,t\},k-1).\end{aligned}$$

Proof. Since $P(G,\lambda)=P(G+uv,\lambda)+P(G\cdot uv,\lambda)$, we have

$$\alpha(G,k)=\alpha(G+uv,k)+\alpha(G\cdot uv,k).$$

Let x be the vertex in $G\cdot uv$ produced by identifying u and v, and z the vertex in $G\cdot uv\cdot xw$ produced by identifying x and w, and s the vertex in $G\cdot uv\cdot xw\cdot zy$ produced by identifying z and y. Notice that x is adjacent to all vertices in $V(G\cdot uv\cdot xw)-\{x,w,y,t\}$, z is adjacent to all vertices in $V(G \cdot uv \cdot xw) - \{z, y, t\}$ and s is adjacent to all vertices in $V(G \cdot uv \cdot xw \cdot zy) - \{s, t\}$. Thus

$$\begin{aligned}
G \cdot uv + xw + xy + xt &= K_1 + (G - \{u, v\}), \\
(G \cdot uv + xw + xy) \cdot xy &= K_1 + (G - \{u, v, t\}), \\
(G \cdot uv + xw) \cdot xy + xt &= K_1 + (G - \{u, v, y\}), \\
(G \cdot uv + xw) \cdot xy \cdot xt &= K_1 + (G - \{u, v, y, t\}), \\
(G \cdot uv \cdot xw + zy + zt &= K_1 + (G - \{u, v, w\}), \\
(G \cdot uv \cdot xw + zy) \cdot zt &= K_1 + (G - \{u, v, w, t\}), \\
(G \cdot uv \cdot xw \cdot zy) + st &= K_1 + (G - \{u, v, w, y\}) \quad \text{and} \\
G \cdot uv \cdot xw \cdot zy \cdot st &= K_1 + (G - \{u, v, w, y, t\}).
\end{aligned}$$

We also observe that for any graph H, $\alpha(K_1+H, k) = \alpha(H, k-1)$, since $P(K_1+H, \lambda) = \lambda P(H, \lambda - 1)$. Hence

$$\begin{aligned}
\alpha(G \cdot uv, k) &= \alpha(G \cdot uv + xw, k) + \alpha(G \cdot uv \cdot xw, k) \\
&= \alpha(G \cdot uv + xw + xy + xt, k) + \alpha((G \cdot uv + xw + xy) \cdot xt, k) + \\
&\quad \alpha((G \cdot uv + xw) \cdot xy + xt, k) + \alpha((G \cdot uv + xw) \cdot xy \cdot xt, k) + \\
&\quad \alpha(G \cdot uv \cdot xw + zy + zt, k) + \alpha((G \cdot uv \cdot xw + zy) \cdot zt, k) + \\
&\quad \alpha((G \cdot uv \cdot xw \cdot zy) + st, k) + \alpha(G \cdot uv \cdot xw \cdot zy \cdot st, k) \\
&= \alpha(K_1 + (G - \{u, v\}), k) + \alpha(K_1 + (G - \{u, v, t\}), k) + \\
&\quad \alpha(K_1 + (G - \{u, v, y\}), k) + \alpha(K_1 + (G - \{u, v, y, t\}, k) + \\
&\quad \alpha(K_1 + (G - \{u, v, w\}), k) + \alpha(K_1 + (G - \{u, v, w, t\}, k) + \\
&\quad \alpha(K_1 + (G - \{u, v, w, y\}, k) + \alpha(K_1 + (G - \{u, v, w, y, t\}, k) \\
&= \alpha(G - \{u, v\}, k-1) + \alpha(G - \{u, v, t\}, k-1) + \\
&\quad \alpha(G - \{u, v, y\}, k-1) + \alpha(G - \{u, v, y, t\}, k-1) + \\
&\quad \alpha(G - \{u, v, w\}, k-1) + \alpha(G - \{u, v, w, t\}, k-1) + \\
&\quad \alpha(G - \{u, v, w, y\}, k-1) + \alpha(G - \{u, v, w, y, t\}, k-1).
\end{aligned}$$

The lemma is then obtained. □

3. Main Results

In [2], Dong et al. proved that every 2−connected graph in $\mathcal{D}_s(p, q, s)$ is $\chi-$unique. Then, Dong et al. in [1] also proved that G is $\chi-$unique for every $G \in \mathcal{D}_{s-1}(p, q, s)$, where $s \geq 5$, and that G is $\chi-$unique for every $G \in \mathcal{D}_{s-2}(p, q, s)$, where $s \geq 7$. In [5], we have shown that each graph G in $\mathcal{D}_{s-3}(p, q, s)$, where $s \geq 9$, then G is $\chi-$unique. In this section, we shall prove that for each graph G in $\mathcal{D}_{s-4}(p, q, s)$, where $s \geq 11$, G is $\chi-$unique. We first have the following lemma.

Lemma 8. *For any G in $\mathcal{D}_{s-4}(p, q, s)$, where $s \geq 9$, G' is one of the graphs in Figure 1* [8].

We now present our main result in the following theorem.

Theorem 9. *For any $G \in \mathcal{K}_2^{-s}(p, q)$, with $p \geq q \geq s + 1 \geq 12$, if $\triangle(G') = s - 4$, then G is $\chi-$unique.*

Proof. Since $s \geq 11$, $(s+3)/2 \geq s-3$ and thus by Theorem 2 ,$\mathcal{D}_{s-4}(p,q,s)$ is $\chi-$closed. By Lemma 8, if G in $\mathcal{D}_{s-4}(p,q,s)$, then G' is one of the graphs in Figure 1 [8]. Thus $\mathcal{D}_{s-4}(p,q,s)$ contain 166 graphs, which are named as V_1, V_2, ..., V_{166} (see table in [7]). We then group these graphs according to their $\alpha'(V_i,3)$ which can be calculated by using Lemma 4.

$$
\begin{aligned}
\mathcal{T}_1 &= \{\ V_1, V_2\ \}\\
\mathcal{T}_2 &= \{\ V_3, V_4\ \}\\
\mathcal{T}_3 &= \{\ V_5, V_6\ \}\\
\mathcal{T}_4 &= \{\ V_7, V_8, V_9, V_{10}\ \}\\
\mathcal{T}_5 &= \{\ V_{11}, V_{12}, V_{13}, V_{14}\ \}\\
\mathcal{T}_6 &= \{\ V_{15}, V_{16}, V_{17}, V_{18}, V_{19}, V_{20}\ \}\\
\mathcal{T}_7 &= \{\ V_{21}, V_{22}, V_{23}, V_{24}, V_{25}, V_{26}, V_{27}, V_{28}\ \}\\
\mathcal{T}_8 &= \{\ V_{29}, V_{30}, V_{31}, V_{32}, V_{33}, V_{34}, V_{35}, V_{36}\ \}\\
\mathcal{T}_9 &= \{\ V_{37}, V_{38}, \ldots, V_{54}\ \}\\
\mathcal{T}_{10} &= \{\ V_{55}, V_{56}, V_{57}, V_{58}\ \}\\
\mathcal{T}_{11} &= \{\ V_{59}, V_{60}, \ldots, V_{78}\ \}\\
\mathcal{T}_{12} &= \{\ V_{79}, V_{80}, \ldots, V_{104}\ \}\\
\mathcal{T}_{13} &= \{\ V_{105}, V_{106}, \ldots, V_{122}\ \}\\
\mathcal{T}_{14} &= \{\ V_{123}, V_{124}, \ldots, V_{142}\ \}\\
\mathcal{T}_{15} &= \{\ V_{143}, V_{144}, \ldots, V_{158}\ \}\\
\mathcal{T}_{16} &= \{\ V_{159}, V_{160}, V_{161}, V_{162}, V_{163}, V_{164}\ \}\\
\mathcal{T}_{17} &= \{\ V_{165}, V_{166}\ \}
\end{aligned}
$$

Observe that for any i, j with $1 \leq i < j \leq 17$, $\alpha'(V_{i_1},3) > \alpha'(V_{j_1},3)$ if $V_{i_1} \in \mathcal{T}_i$ and $V_{j_1} \in \mathcal{T}_j$. Thus by Lemma 8 and Equation (2. 1), $\mathcal{T}_i$ and $\mathcal{T}_j$ $(1 \leq i < j \leq 17)$ are $\chi-$disjoint and since $\mathcal{D}_{s-4}(p,q,s)$ is $\chi-$closed, each $\mathcal{T}_i$ $(1 \leq i \leq 17)$ is $\chi-$closed. Hence, for each i, to show that all graphs in $\mathcal{T}_i$ are $\chi-$unique, it suffices to show that for any two graphs, V_{i_1}, $V_{i_2} \in \mathcal{T}_i$, if $V_{i_1} \ncong V_{i_2}$, then either $\alpha'(V_{i_1},4) \neq \alpha'(V_{i_2},4)$ or $\alpha(V_{i_1},5) \neq \alpha(V_{i_2},5)$. The values of $\alpha'(V_i,4)$ by using Lemma 4.We shall establish several inequalities of the form $\alpha'(V_i,4) < \alpha'(V_j,4)$ for some i,j. Since the method used to obtain these inequalities is standard, long and rather repetitive, we shall not discuss all here. In the following, we shall only show the examples of detailed comparisons. In the first example, we compare $\alpha'(V_1,4)$ and $\alpha'(V_2,4)$ when $p > q$, and in the second example, we compute $\alpha(V_{21},5) - \alpha(V_{23},5)$ when $p = q$. The reader may refer to [6, 7] for all graphs and other detailed comparisons.

(1) V_1 and V_2 when $p > q$

$$\begin{aligned}
&\alpha'(V_1,4) - \alpha'(V_2,4)\\
&= \Bigg[\left(\sum_{i=1}^{s-4} \binom{s-4}{i} (2^{p-i-1} + 2^{q-2} - 2) \right) + 15 \cdot 2^{p-1} + 2^q + 15 \cdot 2^{q-3}\\
&\quad +11 \cdot 2^{q-6} + 15 \cdot 2^{s-5} - 75 \Bigg] - \Bigg[\left(\sum_{i=1}^{s-4} \binom{s-4}{i} (2^{q-i-1} + 2^{p-2} - 2) \right)\\
&\quad 15 \cdot 2^{q-1} + 2^p + 15 \cdot 2^{p-3} + 11 \cdot 2^{p-6} + 15 \cdot 2^{s-5} - 75 \Bigg]\\
&= \left[\sum_{i=1}^{s-4} \binom{s-4}{i} (2^{p-i-1} - 2^{q-i-1})(1 - 2^{i-1}) \right] + 15(2^{p-1} - 2^{q-1}) -\\
&\quad (2^{p-2} - 2^{q-2}) - 15(2^{p-3} - 2^{q-3}) - 11(2^{p-6} - 2^{q-6})\\
&< -15 \binom{s-4}{5} (2^{p-6} - 2^{q-6}) + 15 \cdot 2^5 (2^{p-6} - 2^{q-6})\\
&\quad -2^6(2^{p-5} - 2^{q-5}) - 15 \cdot 2^3 (2^{p-5} - 2^{q-5}) - 11(2^{p-6} - 2^{q-6})\\
&= (2^{p-6} - 2^{q-6}) \left[-15 \binom{s-4}{5} + 15 \cdot 32 - 64 - 15 \cdot 8 - 11 \right]\\
&\leq (2^{p-6} - 2^{q-6}) \left[-15 \cdot 21 + 285 \right] \left[\text{since} \quad \binom{s-4}{5} \geq \binom{7}{5} = 21 \right]\\
&= (2^{p-6} - 2^{q-5})(-30) < 0.
\end{aligned}$$

(2) V_{21} and V_{23} when $p = q$
When $p = q$, $\alpha'(V_{21}, 4) = \alpha'(V_{23}, 4)$. Thus we need to calculate $\alpha(V_{21}, 5) - \alpha(V_{23}, 5)$. Using Lemma 7,we have

$$\begin{aligned}
&\alpha(V_{21},5) - \alpha(V_{23},5)\\
&= \Bigg[\alpha(V_{21} + a_1b_1, 5) + \alpha(V_{21} - \{a_1,b_1\},4) + \alpha(V_{21} - \{a_1,b_1,e_1\},4) +\\
&\quad \alpha(V_{21} - \{a_1,b_1,d_1\},4) + \alpha(V_{21} - \{a_1,b_1,d_1,e_1\},4) +\\
&\quad \alpha(V_{21} - \{a_1,b_1,c_1\},4) + \alpha(V_{21} - \{a_1,b_1,c_1,e_1\},4) +\\
&\quad \alpha(V_{21} - \{a_1,b_1,c_1,d_1\},4) + \alpha(V_{21} - \{a_1,b_1,c_1,d_1,e_1\},4) \Bigg] -\\
&\quad \Bigg[\alpha(V_{23} + a_2b_2, 5) + \alpha(V_{23} - \{a_2,b_2\},4) + \alpha(V_{23} - \{a_2,b_2,e_2\},4) +\\
&\quad \alpha(V_{23} - \{a_2,b_2,d_2\},4) + \alpha(V_{23} - \{a_2,b_2,d_2,e_2\},4) +\\
&\quad \alpha(V_{23} - \{a_2,b_2,c_2\},4) + \alpha(V_{23} - \{a_2,b_2,c_2,e_2\},4) +\\
&\quad \alpha(V_{23} - \{a_2,b_2,c_2,d_2\},4) + \alpha(V_{23} - \{a_2,b_2,c_2,d_2,e_2\},4) \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\alpha(V_{21} - \{a_1, b_1, e_1\}, 4) - \alpha(V_{23} - \{a_2, b_2, e_2\}, 4) \Big) + \\
&\Big(\alpha(V_{21} - \{a_1, b_1, d_1\}, 4) - \alpha(V_{23} - \{a_2, b_2, d_2\}, 4) \Big) + \\
&\Big(\alpha(V_{21} - \{a_1, b_1, d_1, e_1\}, 4) - \alpha(V_{23} - \{a_2, b_2, d_2, e_2\}, 4) \Big) + \\
&\Big(\alpha(V_{21} - \{a_1, b_1, c_1\}, 4) - \alpha(V_{23} - \{a_2, b_2, c_2\}, 4) \Big) + \\
&\Big(\alpha(V_{21} - \{a_1, b_1, c_1, e_1\}, 4) - \alpha(V_{23} - \{a_2, b_2, c_2, e_2\}, 4) \Big) + \\
&\Big(\alpha(V_{21} - \{a_1, b_1, c_1, d_1\}, 4) - \alpha(V_{23} - \{a_2, b_2, c_2, d_2\}, 4) \Big) + \\
&\Big(\alpha(V_{21} - \{a_1, b_1, c_1, d_1, e_1\}, 4) - \alpha(V_{23} - \{a_2, b_2, c_2, d_2, e_2\}, 4) \Big) \\
&since \quad V_{21} + a_1b_1 \cong V_{23} + a_2b_2,\ V_{21} - \{a_1, b_1\} \cong V_{23} - \{a_2, b_2\} \\
&= \Big(\alpha'(V_{21} - \{a_1, b_1, e_1\}, 4) - \alpha'(V_{23} - \{a_2, b_2, e_2\}, 4) \Big) + \\
&\Big(\alpha'(V_{21} - \{a_1, b_1, d_1\}, 4) - \alpha'(V_{23} - \{a_2, b_2, d_2\}, 4) \Big) + \\
&\Big(\alpha'(V_{21} - \{a_1, b_1, d_1, e_1\}, 4) - \alpha'(V_{23} - \{a_2, b_2, d_2, e_2\}, 4) \Big) + \\
&\Big(\alpha'(V_{21} - \{a_1, b_1, c_1\}, 4) - \alpha'(V_{23} - \{a_2, b_2, c_2\}, 4) \Big) + \\
&\Big(\alpha'(V_{21} - \{a_1, b_1, c_1, e_1\}, 4) - \alpha'(V_{23} - \{a_2, b_2, c_2, e_2\}, 4) \Big) + \\
&\Big(\alpha'(V_{21} - \{a_1, b_1, c_1, d_1\}, 4) - \alpha'(V_{23} - \{a_2, b_2, c_2, d_2\}, 4) \Big) + \\
&\Big(\alpha'(V_{21} - \{a_1, b_1, c_1, d_1, e_1\}, 4) - \alpha'(V_{23} - \{a_2, b_2, c_2, d_2, e_2\}, 4) \Big) \\
&= \left(\sum_{i=1}^{s-4} \binom{s-4}{i} (2^{p-3-i} + 2^{q-3} - 2) - \sum_{i=1}^{s-4} \binom{s-4}{i} (2^{p-2-i} + 2^{q-4} - 2) \right) \\
&\quad + \left(\sum_{i=1}^{s-4} \binom{s-4}{i} (2^{p-3-i} + 2^{q-3} - 2) - \sum_{i=1}^{s-4} \binom{s-4}{i} (2^{p-2-i} + 2^{q-4} - 2) \right)
\end{aligned}
$$

$$+\Bigg(\sum_{i=1}^{s-4}\binom{s-4}{i}(2^{p-4-i}+2^{q-3}-2)-\sum_{i=1}^{s-4}\binom{s-4}{i}(2^{p-2-i}+2^{q-5}-2)\Bigg)$$
$$+\Bigg(\sum_{i=1}^{s-4}\binom{s-4}{i}(2^{p-3-i}+2^{q-3}-2)-\sum_{i=1}^{s-4}\binom{s-4}{i}(2^{p-2-i}+2^{q-4}-2)\Bigg)$$
$$+\Bigg(\sum_{i=1}^{s-4}\binom{s-4}{i}(2^{p-4-i}+2^{q-3}-2)-\sum_{i=1}^{s-4}\binom{s-4}{i}(2^{p-2-i}+2^{q-5}-2)\Bigg)$$
$$+\Bigg(\sum_{i=1}^{s-4}\binom{s-4}{i}(2^{p-4-i}+2^{q-3}-2)-\sum_{i=1}^{s-4}\binom{s-4}{i}(2^{p-2-i}+2^{q-5}-2)\Bigg)$$
$$+\Bigg(\sum_{i=1}^{s-4}\binom{s-4}{i}(2^{p-5-i}+2^{q-3}-2)-\sum_{i=1}^{s-4}\binom{s-4}{i}(2^{p-2-i}+2^{q-6}-2)\Bigg)$$
$$=37\Bigg\{\binom{s-3}{1}(2^{p-6}-2^{p-6})+\binom{s-4}{2}(2^{p-6}-2^{p-7})+$$
$$\binom{s-4}{3}(2^{p-6}-2^{p-8})+\cdots+\binom{s-4}{s-4}(2^{p-6}-2^{p-s-1})\Bigg\}\quad \text{since } \ p=q$$
$$>0 \quad since \ \ s\geq 11.$$

Similarly, we can show that for any two graphs V_{i_1}, $V_{i_2}\in\mathcal{T}_i$, if $V_{i_1}\ncong V_{i_2}$, then either $\alpha'(V_{i_1},4)\neq\alpha'(V_{i_2},4)$ or $\alpha(V_{i_1},5)\neq\alpha(V_{i_2},5)$ (see [6]). Hence the proof of the theorem is complete. □

As conclusion of this paper, we give the following conjecture which is true for $t=3,4,5$ and 6.

Conjecture. For any $G\in\mathcal{K}_2^{-s}(p,q)$, with $p\geq q\geq s+1\geq 2t$ $(t=3,4,5,\cdots)$, if $\triangle(G')=s-t+2$, then G is $\chi-$unique.

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[7] http://www.fsas.upm.edu.my/ yhpeng/publish/table1.pdf

[8] http://www.fsas.upm.edu.my/ yhpeng/publish/figure1.pdf