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Newton-like Methods with At least Quadratic Order of Convergence for the Computation of Fixed Points

Ioannis K. Argyros
Cameron University
Department of Mathematics Sciences
Lawton, OK 73505, USA
Email: iargyros@cameron.edu

Abstract. The well known contraction mapping principle or Banach's fixed point theorem asserts: The method for successive substitutions converges only linearly to a fixed point of an operator equation in a Banach space setting [5], [7]. In practice, if Newton's method is used one ignores the additional information about the contraction mapping information. Werner in [9] provided a local analysis for a Newton-like method of at least Q-order 3 which uses this information. Here we provide a finer local convergence analysis for the same method under weaker hypotheses which do not necessarily imply the contraction property of the mapping. A numerical example is provided to show that our results compare favorably with the ones in [9]. The semilocal convergence of the method not considered in [9] is also examined.

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique fixed point x^* of a Fréchet-differentiable operator F which is defined on a convex subset D of a Banach space X with values in X.

The contraction mapping principle or Banach's fixed point theorem [5], [7] asserts that if

$$\|F'(x)\| < 1 \text{ for all } x \in X \tag{1.1}$$

then there exists a unique fixed x^* of operator F on X. The method of succesive substitutions or Picard's iteration

$$y_{n+1} = F(y_n) \quad (y_0 \in X) \quad (n \geq 0) \tag{1.2}$$

converges only linearly to x^* (the definition of Q order for an iterative method is well known and can be found in [8, Definition 9.1.5, p. 284]). In order to increase the speed of convergence to Q order at least two, and also use the contraction mapping property of operator F Werner in [9] introduced the Newton-like method

$$x_{n+1} = x_n - M_n^{-1}(x_n - F(x_n)) \quad (x_0 \in D), \quad (x \geq 0),\ \gamma \in [0,1] \tag{1.3}$$

where, $M_n \in L(X)$ $(n \geq 0)$ the space of bounded linear operators on X, is given by

$$M_n = I - F'(\gamma x_n + (1-\gamma)F(x_n)) \quad (n \geq 0). \tag{1.4}$$

If $\gamma = 1$, we obtain Newton's method [4], [5]-[9], whereas if $\gamma = 0$ we obtain Stirling's method [1]-[3], [5]. Other choices are also possible [9]. Note that method (1.3) (as Newton's method does) requires one function evaluation and one evaluation of the Fréchet derivative F' of F per step independent of γ.

The motivation for introducing method (1.3) is due to the fact that if (1.1) holds, then $F(x_n)$ is a better approximation to x^* than x_n. Then, we can write:

$$x_n - F(x_n) = x^* - F(x^*) + \int_0^1 [I - F'(x^* + t(x_n - x^*))]dt(x_n - x^*) \tag{1.5}$$

or

$$x^* = x_n - \left\{ \int_0^1 [I - F'(x^* + t(x_n - x^*))]dt \right\}^{-1} (x_n - F(x_n)). \tag{1.6}$$

Werner noted that the choice $\gamma = \dfrac{1}{2}$ is the most appropriate choice of the free parameter γ in (1.3) leading to the midpoint rule. Werner provided a local convergence analysis for method (1.3) (see Proposition 1 in [9]) under hypothesis (1.1) when $D = X$.

Here we refine Werner's result by providing a local convergence analysis under weaker hypotheses with the following advantages: finer error estimates on $\|x_n - x^*\|$ $(n \geq 0)$, and a larger radius of convergence allowing for a wider choice of initial guesses x_0. We then provide a numerical example where our results compare favorably with the ones by Werner in [9]. Finally the semilocal convergence of method (1.3) not considered in [9] is studied.

2. Local convergence analysis of Newton-like method (1.3)

We can show the main local convergence theorem for Newton-like method (1.3):

Theorem 1. *Let x^* be a fixed point of operator F such that $\|F'(x^*)\| < 1$. Assume there exist parameters $\alpha_0 \in [0,1]$ and $L_0 \geq 0$ such that*

$$\|F(x) - F(x^*)\| \leq \alpha_0 \|x - x^*\|, \tag{2.1}$$

$$\begin{aligned} &\left\| [I - F'(x^*)]^{-1}[F'(F(x) + \delta(x - F(x))) - F'(x^*)] \right\| \leq \\ &\leq L_0 \|F(x + \delta(x - F(x)) - x^*\| \end{aligned} \tag{2.2}$$

for all $x \in D$, $\delta \in [0,1]$ and

$$\overline{U}(x^*, r) = \{x \in X \ : \| x - x^* \| \leq r\} \subseteq D, \tag{2.3}$$

where,

$$r = \frac{1}{c}, \tag{2.4}$$

$$c = a + b,$$

$$a = L_0[\frac{1}{2} + \alpha_0(1-\gamma) + 3\gamma],$$

and

$$b = L_0[(1-\gamma)\alpha_0 + \gamma].$$

Then sequence $\{x_n\}$ generated by Newton-like method (1.3) is well-defined, remains in $U(x^*, r)$ and converges to x^* provided that $x_0 \in U(x^*, r)$. Moreover the following error bounds hold for all $n \geq 0$:

$$\|x_{n+1} - x^*\| \leq \frac{a\,\|x_n - x^*\|^2}{1 - b\,\|x_n - x^*\|}. \tag{2.5}$$

Furthermore if $\alpha_0 \in [0, 1)$, x^* is the unique fixed point of F in $\overline{U}(x^*, r)$.

Proof. Let $x \in U(x^*, r)$. Then by (2.1) we have

$$\|F(x) - x^*\| = \|F(x) - F(x^*)\| \leq \alpha_0\,\|x - x^*\| \leq \|x - x^*\| < r,$$

which implies that $F(x) \in U(x^*, r)$.Note also that $F(x) + \gamma(x - F(x)) \in U(x^*, r)$, since

$$\begin{aligned} \|F(x) + \gamma(x - F(x)) - x^*\| &\leq (1 - \gamma)\,\|F(x) - F(x^*)\| + \gamma\,\|x - x^*\| \\ &\leq [(1 - \gamma)\alpha_0 + \gamma]\,\|x - x^*\| \\ &\leq \|x - x^*\| < r. \end{aligned}$$

By hypothesis $x_0 \in U(x^*, r)$. Let us assume $x_k \in U(x^*, r)$ for all $k = 0, 1, ..., n$. In view of (2.1)-(2.4) we obtain in turn

$$\begin{aligned} &\left\|[I - F'(F(x^*) + \gamma(x^* - F(x^*)))]^{-1} \cdot [I - F'(F(x^*) + \gamma(x^* - F(x^*)) \right. \\ &\left. -(I - F'(F(x_k) + \gamma(x_k - F(x_k)))]\right\| \leq \\ &\leq L_0 \left\|I - F'(x^*)]^{-1}[F(x_k) + \gamma(x_k - F(x_k)) - F(x^*) - \gamma(x^* - F(x^*))]\right\| \\ &\leq L_0 \left\|[I - F'(x^*)]^{-1}[(1 - \gamma)(F(x_k) - F(x^*)) + \gamma(x_k - x^*)]\right\| \\ &\leq L_0[(1 - \gamma)\alpha_0 + \gamma]\,\|x_k - x^*\| \leq br < 1. \end{aligned} \tag{2.6}$$

It follows from (2.6) and the Banach Lemma on invertible operators [7] that $[I - F'(F(x_k) + \gamma(x_k - F(x_k))]^{-1}$ exists, and

$$\begin{aligned} &\left\|[I - F'(F(x_k) + \gamma(x_k - F(x_k)))]^{-1}[I - F'(x^*)]\right\| \\ &\leq [1 - b\,\|x_k - x^*\|]^{-1} \leq [1 - br]^{-1}. \end{aligned} \tag{2.7}$$

We need the following estimates:

$$\begin{aligned} &\left\|[I - F'(x^*)]^{-1} \int_0^1 [F'(x^* + t(x_k - x^*)) - F'(x^* + \gamma(x_k - x^*))]dt\right\| \\ &\leq \left\|[I - F'(x^*)]^{-1} \int_0^1 [F'(x^* + t(x_k - x^*)) - F'(x^*)]dt\right\| \\ &+ \left\|I - F'(x^*)]^{-1}[F'(x^*) - F'(x^* + \gamma(x_k - x^*))]\right\| \\ &\leq L_0 \left\{\int_0^1 \|x^* + t(x_k - x^*) - x^*\|\,dt + \|x^* - x^* - \gamma(x_k - x^*)\|\right\} \\ &\leq L_0 \left(\frac{1}{2} + \gamma\right) \|x_k - x^*\|, \end{aligned} \tag{2.8}$$

and

$$\left\| [I - F'(x^*)]^{-1}[F'(x^* + \gamma(x_k - x^*)) - F'(F(x_k) + \gamma(x_k - F(x_k)))] \right\| \tag{2.9}$$
$$\leq \left\| [I - F'(x^*)]^{-1}[F'(x^* + \gamma(x_k - x^*)) - F'(x^*)] \right\|$$
$$+ \left\| [I - F'(x^*)]^{-1}[F'(x^*) - F'(F(x_k) + \gamma(x_k - F(x_k)))] \right\|$$
$$\leq L_0\{\|x^* - x^* - \gamma(x_k - x^*)\| +$$
$$\|F(x_k) + \gamma(x_k - F(x_k)) - F(x^*) - \gamma(x^* - F(x^*))\|\}$$
$$\leq L_0[\gamma + \alpha_0(1-\gamma) + \gamma] \|x_k - x^*\| .$$

Let us define, $A_k(t) = F'(x^* + t(x_k - x^*))$ and $B_k(t) = F'(F(x_k) + t(x_k - F(x_k)))$, $t \in [0,1]$. In view of (1.3) we get

$$x_{k+1} - x^* = [I - B_k(\gamma)]^{-1}[I - F'(x^*)][I - F'(x^*)]^{-1} \tag{2.10}$$
$$\{F(x_k) - F(x^*) - B_k(\gamma)(x_k - x^*)\}.$$

Using (2.1), (2.2), (2.8) and (2.9) we get

$$\left\| [I - F'(x^*)]^{-1}[F(x_k) - F(x^*) - B_k(\gamma)(x_k - x^*)] \right\| \tag{2.11}$$
$$= \left\| [I - F'(x^*)]^{-1} \int_0^1 (A_k(t) - B_k(\gamma))(x_k - x^*)dt \right\|$$
$$\leq \left\| [I - F'(x^*)]^{-1} \int_0^1 (A_k(t) - A_k(\gamma)(x_k - x^*)dt \right\|$$
$$+ \left\| \{I - F'(x^*)]^{-1}(A_k(\gamma) - B_k(\gamma))(x_k - x^*) \right\|$$
$$\leq a \|x_k - x^*\|^2 .$$

By (2.8), (2.10) and (2.11) we obtain (2.5). It follows by (2.5) and the definition of r that

$$\|x_{k+1} - x^*\| < \|x_k - x^*\| < r,$$

which shows that $x_{k+1} \in U(x^*, r)$ and $\lim_{k\to\infty} x_k = x^*$.To show uniqueness, let y^* be a fixed point of F in $\overline{U}(x^*, r)$ with $x^* \neq y^*$. Using (2.1) we get

$$\|x^* - y^*\| = \|F(x^*) - F(y^*)\| \leq \alpha_0 \|x^* - y^*\| < \|x^* - y^*\| ,$$

which contradicts $x^* \neq y^*$.That completes the proof of the theorem. □

Remark 2. The conclusions of the theorem hold under the stronger conditions

$$\left\| [I - F'(x^*)]^{-1}[F'(x^* + t(x - x^*)) - F'(x^* + \gamma(x - x^*))] \right\| \tag{2.12}$$
$$\leq L_1 |t - \gamma| \|x - x^*\|$$

and

$$\left\| [I - F'(x^*)]^{-1}[F'(x^* + \gamma(x - x^*)) - F'(F(x) + \gamma(x - F(x)))] \right\| \tag{2.13}$$
$$\leq L_2(1-\gamma) \|x - x^*\|$$

for all $x \in D$ (together with (2.2) and (2.3)). It follows from the proof of Theorem 1 that the conclusions hold provided that a is replaced by $\overline{a}$ given by

$$\overline{a} = L_1(\gamma^2 - \gamma + \frac{1}{2}) + L_2\alpha_0(1-\gamma). \tag{2.14}$$

Werner in [9] used the stronger conditions (1.1) and

$$\left\| [I - F'(x^*)]^{-1}(F'(x) - F'(y)) \right\| \leq L \|x - y\| \tag{2.15}$$

for all $x, y \in D$. To arrive at (2.5) with a, b being replaced by $\overline{\overline{a}}, b$ given by

$$\overline{\overline{a}} = \frac{L}{1-\alpha}[\frac{1}{2} + \alpha - (1+\alpha)\gamma + \gamma^2] \text{ and } 0, \tag{2.16}$$

respectively.

Clearly

$$L_0 \leq L, \ \alpha_0 \leq \alpha \tag{2.17}$$

hold in general, and $\frac{L}{L_0}, \frac{\alpha}{\alpha_0}$ can be arbitrarily large [4], [5]. Hence the error bounds are finer, and the radius of convergence larger under our (weaker) conditions. Note also that condition (2.2) can be replaced by

$$\left\| [I - F'(x^*)]^{-1}[F'(x) - F'(x^*)] \right\| \leq L_0^{'} \left\| x - x^* \right\| \tag{2.18}$$
$$\text{for all } x \in D.$$

Note also that

$$L_0^{'} \leq L \tag{2.19}$$

holds in general and $\frac{L}{L_0^{'}}$ can be arbitrarily large [4], [5].

Let us provide a numerical example, where our results compare favorably with the ones given by Werner in [9].

Example 1. *Let $X = \mathcal{C}[0,1]$, the space of continuous functions defined on interval $[0,1]$, equipped with the max–norm and let $D = \overline{U}(0,1)$.*

Define function F on D by

$$F(x)(s) = A\,x(s) + B\,s\int_0^1 \theta\,x^3(\theta)\,d\theta \tag{2.20}$$

for some given real constants $A \neq 1$ and B. Then the Fréchet–derivative F' of function F is given by

$$F'(x(w))s = A\,I\,w(s) + 3\,B\,s\int_0^1 \theta\,x^2(\theta)\,w(\theta)\,d\theta \quad \text{for all } w \in D. \tag{2.21}$$

We have $x^* = x^*(s) = 0$ is a fixed point of function F. Using (2.20) and (2.21), we can set

$$\alpha_0 = |A| + \frac{1}{2}\,|B|, \quad \alpha = |A| + \frac{3}{2}\,|B|,$$

$$L_0 = \frac{3}{2}\,|B\,(1-A)^{-1}| \quad \textit{and} \quad L = 3\,|B\,(1-A)^{-1}|.$$

Let $\delta = \gamma = 1$.

Case 1: $A = B = \frac{1}{2}$. *We get*

$$\alpha_0 = \frac{3}{4} < 1, \quad L_0 = L_0' = \frac{3}{2}, \quad L = 3 \quad \textit{and} \quad \alpha = \frac{5}{4} > 1.$$

Hence, contraction hypothesis (1.1) used by Werner in [9] *is violated. That is, there is no guarantee that sequence $\{x_n\}$ converges to $x^*(s) = 0$.*

However, by Theorem 1, we have

$$a = \frac{21}{4}, \quad b = \frac{3}{2}, \quad r = \frac{4}{27} \quad \textit{and}$$

sequence $\{x_n\}$ converges to $x^*(s)$ provided that $x_0 = x_0(s) \in \overline{U}(0,r)$.

Case 2: $A = \dfrac{1}{2}$ *and* $B = \dfrac{1}{4}$. *This time, we have*

$$\alpha_0 = \frac{5}{6}, \quad L_0 = L_0' = \frac{3}{4}, \quad L = \frac{3}{2}, \quad \alpha = \frac{7}{8},$$

$$a = \frac{21}{8}, \quad b = \frac{3}{4}, \quad \overline{\overline{a}} = 6$$

and

$$r_W = \frac{1}{6} < r_A = \frac{8}{27}.$$

That is our radius of convergence is larger which allows a wider choice of initial guesses x_0. *Moreover, our error bounds (2.5) are tighter, since* $a \leq \overline{\overline{a}}$.

3. Semilocal convergence analysis of Newton-like method (1.3)

We can show the following semilocal convergence result for Newton-like method (1.3):

Theorem 3. *Let* F *be a Fréchet-differentiable operator defined on a convex subset* D *of a Banach space* X *with values in* X. *Assume: there exist positive parameters* $p_0,\ p,\ l_0$ *and* l *such that for some* $x_0 \in D$, $\gamma \in [0,1]$ $M_0^{-1} \in L(X)$ *and for all* $x, y \in D$:

$$\|F(x) - F(x_0)\| \leq p_0 \|x - x_0\| \tag{3.1}$$

$$\left\|M_0^{-1} F'(x)\right\| \leq p, \tag{3.2}$$

$$\left\|M_0^{-1}[F'(z) - F'(w)]\right\| \leq l_0 \|z - w\| \tag{3.3}$$

for $z = z(\gamma, x) = \gamma x + (1-\gamma)F(x), w = w(\gamma) = \gamma x_0 + (1-\gamma)F(x_0)$,

$$\left\|M_0^{-1}[F'(z) - F'(v)]\right\| \leq l \|z - v\| \tag{3.4}$$

for $v = v(t) = x + t(y - x)$ *for all* $t \in [0,1]$; *for* r_0, r_1, r_2, *and* r_3 *given by*

$$r_0 = \frac{\|x_0 - F(x_0)\|}{1 - p_0}, \ r_1 = \frac{1 - (p_0 + p) + \sqrt{D_1}}{2q_0},$$

$$r_2 = \frac{1 - l(1-\gamma)\|x_0 - F(x_0)\| + \sqrt{D_2}}{2[q_0 + l(p_0(1-\gamma) + \gamma^2 - \gamma + \frac{1}{2})]},$$

$$D_1 = [1 - (p_0 + p)] - 4q_0 \|x_0 - F(x_0)\|,$$

$$r_3 = \frac{1}{q_0}\left[1 - \sqrt[3]{q\|x_0 - F(x_0\|}\right],$$

where, $q_0 = l_0[\gamma + (1-\gamma)p_0]$,

$$q_1 = l(\gamma^2 - \gamma + \frac{1}{2}), \ q_2 = l(1-\gamma), \ q = q_1 + q_2 p$$

the following hold:

$$r_0 < \min\{r_1, r_3\}, \tag{3.5}$$

$$\|x_0 - F(x_0)\| \leq \frac{1}{q}, \ \left\|M_0^{-1}[x_0 - F(x_0)]\right\| < \frac{[1 - (p_0 + p)]^2}{4q_0}, 0 \leq p_0 + p < 1 \tag{3.6}$$

or

$$r_0 < \min\{r_2, r_3\}, \tag{3.7}$$

$$\|x_0 - F(x_0)\| \leq \frac{1}{q}, \left\|M_0^{-1}[x_0 - F(x_0)]\right\| < \min\{\frac{1}{\ell(1-\gamma)}, \ r_4\} \tag{3.8}$$

where, r_4 *is the smallest root of equation*

$$c_2 r^2 + c_1 r + c_0 = 0, \tag{3.9}$$

$c_2 = \ell^2(1-\gamma)^2,\ c_1 = -2[1-\gamma+2(q_0+\ell(p_0(1-\gamma)+\gamma^2-\gamma+\frac{1}{2}))]$,
$c_0 = 1$,

$$c_1^2 \geq 4c_0c_1; \tag{3.10}$$

and

$$\overline{U}(x_0,r) \subseteq D \tag{3.11}$$

for some $\{r \in [r_0, \min\{r_1, r_3\})$ *if (3.5) and (3.6) hold or* $r \in [r_0, \min\{r_2, r_3\})$ *if (3.7)-(3.10) hold.*

Then sequence $\{x_n\}$ *generated by Newton-like method (1.3) is well defined, remains in* $\overline{U}(x_0,r)$ *for all* $n \geq 0$ *and converges to a fixed point* $x^* \in \overline{U}(x_0,r)$ *of operator F. Moreover the following estimates hold:*

$$\begin{aligned}\|x_{n+1}-F(x_{n+1})\| &\leq \frac{[q_1\|x_{n+1}-x_n\|+q_2\|x_{n+1}-F(x_n)\|]\,\|x_{n+1}-x_n\|}{1-q_0\|x_{n+1}-x_0\|} \\ &\leq \varepsilon_n\|x_n-F(x_n)\|^2,\ \ (n\geq 0)\end{aligned} \tag{3.12}$$

and

$$\|x_{n+1}-x_n\| \leq \overline{\varepsilon}_n\|x_{n-1}-F(x_{n-1})\|^2\ \ (n\geq 1), \tag{3.13}$$

where,

$$\varepsilon_n = \frac{q}{(1-q_0\|x_n-x_0\|)^2(1-q_0\|x_{n+1}-x_0\|)}\ (n\geq 0),$$

and

$$\overline{\varepsilon}_n = \frac{\varepsilon_{n-1}}{1-q_0\|x_n-x_0\|}\ (n\geq 1).$$

Furthermore if

$$r \in [r_0, r_4), \tag{3.14}$$

where,

$$r_4 = \left\{q_0+\ell\left[\gamma^2-\gamma+\frac{1}{2}+(1-\gamma)p_0\right]\right\}^{-1},$$

then the fixed point x^* *is unique in* $\overline{U}(x_0,r)$*. Finally the following estimate holds:*

$$\|x_{n+1}-x^*\| \leq \frac{\ell\left[\gamma^2-\gamma+\frac{1}{2}+(1-\gamma)p_0\right]}{1-q_0\|x_n-x_0\|}\|x_n-y^*\|^2\ \ (n\geq 0). \tag{3.15}$$

Proof. We shall first show that for all $\gamma \in [0,1]$, $x \in \overline{U}(x_0,r)$ and $\gamma x+(1-\gamma)F(x) \in \overline{U}(x_0,r)$.

We can have

$$\begin{aligned}\gamma x+(1-\gamma)F(x)-x_0 &= \gamma(x-x_0)+(1-\gamma)(F(x)-F(x_0)) \\ &\quad+(1-\gamma)(F(x_0)-x_0).\end{aligned} \tag{3.16}$$

In view of (3.1), (3.16) and the choice of $r \geq r_0$ we get

$$\begin{aligned}\|\gamma x+(1-\gamma)F(x)-x_0\| &\leq \gamma\|x-x_0\|+(1-\gamma)\|F(x)-F(x_0)\| \\ &\quad+(1-\gamma)\|F(x_0)-x_0\| \\ &\leq [\gamma+(1-\gamma)p_0]\,r+(1-\gamma)\|F(x_0)-x_0\| \leq r,\end{aligned} \tag{3.17}$$

which implies $\gamma x+(1-\gamma)F(x)$ is in $\overline{U}(x_0,r)$.

We shall next show $M(x)^{-1} \in L(X)$ for all $x \in \overline{U}(x_0, r)$. Using (3.1), (3.3) and the choice of $r \leq r_3$ we get in turn

$$\begin{aligned} & \left\| M(0)^{-1} \left[M(0) - M(x) \right] \right\| \\ & \leq \left\| M_0^{-1} \left[F'(\gamma x + (1-\gamma)F(x)) - F'(\gamma x_0 + (1-\gamma)F(x_0)) \right] \right\| \\ & \leq \ell_0 \left\| \gamma x - \gamma x_0 + (1-\gamma)(F(x) - F(x_0)) \right\| \\ & \leq \ell_0 [\gamma \left\| x - x_0 \right\| + (1-\gamma) p_0 \left\| x - x_0 \right\|] \\ & = q_0 \left\| x - x_0 \right\| \leq q_0 r < 1. \end{aligned} \tag{3.18}$$

It follows from (3.18), and the Banach Lemma on invertible operators that $M(x)^{-1} \in L(X)$ with

$$\left\| M(x)^{-1} M(0) \right\| \leq \frac{1}{1 - q_0 \left\| x - x_0 \right\|}. \tag{3.19}$$

Let us assume $x_k \in \overline{U}(x_0, r)$ for $k = 0, 1, ..., n$. We shall show (3.12), (3.13) and $x_{k+1} \in \overline{U}(x_0, r)$ hold true.

Using (1.3) we get the identity

$$\begin{aligned} & x_{k+1} - F(x_{k+1}) = \\ & = x_{k+1} - F(x_{n+1}) - M_k(x_{k+1} - x_k) - (x_k - F(x_k)) \\ & = F(x_k) - F(x_{k+1}) + F'(\gamma x_k + (1-\gamma)F(x_k))(x_{k+1} - x_k) \\ & = \int_0^1 \left[F'(x_{k+1} + t(x_k - x_{k+1})) - F'(\gamma x_k + (1-\gamma)F(x_k)) \right] (x_k - x_{k+1}) dt \end{aligned} \tag{3.20}$$

By (1.3), (3.4) and (3.19) (for $x = x_{n+1}$) and (3.20) we obtain in turn:

$$\begin{aligned} & \left\| x_{k+1} - F(x_{k+1}) \right\| \\ & \leq \frac{\ell \int_0^1 \left\| x_{k+1} + t(x_k - x_{k+1}) - \gamma x_k - (1 - \gamma F(x_k) \right\| \left\| x_{k+1} - x_k \right\| dt}{1 - q_0 \left\| x_{k+1} - x_0 \right\|} \\ & \leq \frac{\ell \left[\int_0^1 \left| t - \gamma \right| \left\| x_{k+1} - x_k \right\| dt + (1-\gamma) \left\| x_{k+1} - F(x_k) \right\| \right] \left\| x_{k+1} - x_k \right\|}{1 - q_0 \left\| x_{k+1} - x_0 \right\|} \\ & \leq \frac{\left[q_1 \left\| x_{k+1} - x_k \right\| + q_2 \left\| x_{k+1} - F(x_k) \right\| \right] \left\| x_{k+1} - x_k \right\|}{1 - q_0 \left\| x_{k+1} - x_0 \right\|}. \end{aligned} \tag{3.21}$$

We need an upper bound on the $\left\| x_{k+1} - F(x_k) \right\|$. In view of (1.3) we obtain the identity

$$\begin{aligned} & x_{k+1} - F(x_k) \\ & = x_k - F(x_k) - M_k^{-1}(x_k - F(x_k)) \\ & = M_k^{-1} \left\{ [I - F'(\gamma x_k + (1-\gamma)F(x_k))](x_k - F(x_k)) - (x_k - F(x_k)) \right\} \\ & = [M_k^{-1} M_0][M_0^{-1} F'(\gamma x_k + (1-\gamma)F(x_k))](x_k - F(x_k)). \end{aligned} \tag{3.22}$$

Using (3.2), (3.19) (for $x = x_k$) and (3.21) we obtain

$$\left\| x_{k+1} - F(x_k) \right\| \leq \frac{p \left\| x_k - F(x_k) \right\|}{1 - q_0 \left\| x_k - x_0 \right\|}. \tag{3.23}$$

In view of (1.3) we get

$$\left\| x_{k+1} - x_k \right\| \leq \frac{\left\| x_k - F(x_k) \right\|}{1 - q_0 \left\| x_k - x_0 \right\|}. \tag{3.24}$$

By combining (3.23) and (3.24) in (3.21) we obtain (3.12) and (3.13).

We must show $\|x_n - F(x_n)\| \to 0$ as $n \to \infty$, which will then imply that $\|x_{n+1} - x_n\| \to 0$ and $\|x_n - x^*\| \to 0$ as $n \to \infty$. By (3.12) is suffices to show

$$\varepsilon_n \|x_n - F(x_n)\| < 1 \tag{3.25}$$

or

$$\frac{q\,\|x_0 - F(x_0)\|}{(1 - q_0 r)^3} < 1 \tag{3.26}$$

which is true by the choice of $r \le r_3$ and the choice of $\|x_0 - F(x_0)\| < \frac{1}{q}$. We must also show $x_{k+1} \in \overline{U}(x_0, r)$. Two estimates for $\|x_{k+1} - x_0\|$ will be given.
Estimate 1. Using (3.1), (3.2), (3.5), (3.6), and (3.19). By (1.3) we get in turn

$$\begin{aligned} & x_{k+1} - x_0 \\ & = x_k - x_0 - M_k^{-1}(x_k - F(x_k)) \\ & = [M_k^{-1} M_0] M_0^{-1} [F(x_k) - F(x_0) - F'(\gamma(x_k + (1-\gamma)F(x_k))(x_k - x_0) \\ & + (F(x_0) - F(x_0))] \end{aligned} \tag{3.27}$$

and

$$\|x_{k+1} - x_0\| \le \frac{[\|F(x_0) - x_0\| + (p_0 + p)r]}{1 - q_0 r} \le r, \tag{3.28}$$

by the choice of $r \le 1$ and the choice of

$$\left\|M_0^{-1}[x_0 - F(x_0)]\right\| \le \frac{[10p_0 + p)]^2}{4q_0},\ 0 \le p_0 + p\ < 1.$$

Estimate 2. Using (3.1), (3.4), (3.19) and (3.27). We obtain that the expression inside the bracket in (3.27) composed by M_0^{-1} is bounded above (in norm) by

$$\begin{aligned} & \left\| M_0^{-1} \int_0^1 F'(x_0 + t(x_k - x_0)) - F'(\gamma x_k + (1-\gamma)F(x_k))] \right\| \|x_k - x_0\|\, dt \\ & + \left\|M_0^{-1}(F(x_0) - x_0)\right\| \\ & \le \int_0^1 \ell \,\|x_0 + t(x_k - x_0) - \gamma x_k - (1-\gamma)F(x_k)\| \,\|x_k - x_0\|\, dt \\ & + \left\|M_0^{-1}(F(x_0) - x_0\right\| \\ & \le [\|x_0 - F(x_0)\|\,(1-\gamma) + p_0(1-\gamma)\,\|x_k - x_0\| + (\gamma^2 - \gamma + \frac{1}{2}) \\ & \|x_k - x_0\|] \cdot \|x_k - x_0\| + \left\|M_0^{-1}(x_0 - F(x_0))\right\| \end{aligned} \tag{3.29}$$

That is it suffices to show

$$\begin{aligned} & \|x_{k+1} - x_0\| \\ & \le \frac{\ell\|x_0 - F(x_0)\|(1-\gamma)r + \ell[\gamma^2 - \gamma + \frac{1}{2} + p_0(1-\gamma)]r^2 + \left\|M_0^{-1}(x_0 - F(x_0))\right\|}{1 - q_0 r} \\ & \le r, \end{aligned} \tag{3.30}$$

which is true by the choice of $r \le r_2$ and the choice of

$$\left\|M_0^{-1}(x_0 - F(x_0))\right\| \le \min\left\{\frac{1}{\ell(1-\gamma)}, r_4\right\}.$$

Finally to show uniqueness, let us assume $y^* \in \overline{U}(x_0, r)$ is a fixed point of F. Using (1.3) we obtain the identity

$$\begin{aligned} & x_{k+1} - y^* \qquad (3.31)\\ & = x_k - y^* - M_k^{-1}(x_k - F(x_k)) \\ & = M_k^{-1}\left[(I - F'(\gamma x_k + (1-\gamma)F(x_k)))\right](x_k - y^* - (x_k - F(x_k))] \\ & = [M_k^{-1}M_0]M_0^{-1}\int_0^1 [F'(y^* + t(x_k - y^*)) - F'(\gamma x_k + (1-\gamma)F(x_k))](x_k - y^*)dt. \end{aligned}$$

Using (3.4), (3.19), (3.31) and the choice of $r \in [r_0, r_4)$ we obtain in turn

$$\begin{aligned} & \|x_{k+1} - y^*\| \qquad (3.32)\\ & \leq \frac{\ell\left[\int_0^1 |t-\gamma| \, \|x_k - y^*\| \, dt + (1-\gamma)(F(x_k) - F(y^*)) \, \|x_k - y^*\|\right]}{1 - q_0 r} \\ & \leq \frac{\ell\left[\gamma^2 - \gamma + \frac{1}{2} + (1-\gamma)p_0\right] \|x_k - y^*\|^2}{1 - q_0 r} \\ & < \|x_k - y^*\|, \end{aligned}$$

which shows (3.15), and $\lim_{k\to\infty} x_k = y^*$. However, we showed $\lim_{k\to\infty} x_k = x^*$. Hence, we deduce

$$x^* = y^*. \qquad (3.33)$$

That completes the proof of the theorem. □

Remark 4. It follows from theorem 3 that the Q-order of convergence for Newton-like method (1.3) is at least quadratic. Conditions (3.3) and (3.4) can be replaced by the usual stronger Lipschitz conditions where z and v are simply in D. Note also that

$$p_0 \leq p \text{ and } \ell_0 \leq \ell, \qquad (3.34)$$

hold in general, and $\frac{p}{p_0}, \frac{\ell}{\ell_0}$ can be arbitrarily large [4], [5].

REFERENCES

[1] S. Amat, S. Busquier , *Convergence and numerical analysis of a family of two-step Steffensen's methods*, Comput. and math. with Appl., **49** (2005), 13–22.

[2] I.K. Argyros, *On Stirling's method*, Tamkang J. Math., **27** (1995), 37–52.

[3] I.K. Argyros, *Stirling's method and fixed points of nonlinear operator equations in Banach space*, Publ. Inst. Math. Acad. Sin., **23** (1995), 13–20.

[4] I.K. Argyros, *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach spaces*, J. Math. Anal. Appl., **298** (2004), 374–397.

[5] I.K. Argyros, *Approximate solution of nonlinear operator equations with applications*, World Scientific Publ. Co. NJ, U.S.A., 2005.

[6] J.M. Gutiérrez, M.A. Hernàndez, M.A. Salanova, *Accesibility of solutions by Newton's method*, Intern. J. Comput. Math., **57** (1995), 239–247.

[7] L.V. Kantorovich, G.P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1982.

[8] J.M. Ortega, W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic press, New York, 1970.

[9] W. Werner, *Newton-like method for the computation of fixed points*, Comput. and Math. with Appl., **10** (1984), 77–84.