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Convolution of Salagean-type Harmonic Univalent Functions

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Abstract. A recent result of Yalcin [9] appeared in Applied Mathematics Letters (2005), concerning the convolution of two harmonic univalent functions in class $\overline{S}_H(m, n, \alpha)$ is improved.

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1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|,\ z \in D$. See Clunie and Sheil-Small [2], for more basic results on harmonic functions one may refer to the following standard introductory text book Duren [3], see also Ahuja [1] and Ponnusamy and Rasila ([6], [7]). Denote by S_H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ g(z) = \sum_{k=1}^{\infty} b_k z^k,\ |b_1| < 1. \tag{1.1}$$

The differential operator D^m was introduced by Salagean [8]. For $f = h + \overline{g}$ given by (1.1), Jahangiri et al. [4] defined the modified Salagean operator of f as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)} \tag{1.2}$$

where

$$D^m h(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k \ and \ D^m g(z) = \sum_{k=1}^{\infty} k^m b_k z^k.$$

For $0 \le \alpha < 1,\ m \in N,\ n \in N_0,\ m > n$ and $z \in U$, suppose $S_H(m,n,\alpha)$ denote the family of harmonic functions f of the form (1.1) such that

$$Re\left\{\frac{D^m f(z)}{D^n f(z)}\right\} > \alpha \tag{1.3}$$

where $D^m f$ is defined by (1.2).

Further, let the subclass $\overline{S}_H(m,n,\alpha)$ consist of harmonic functions $f_m = h + \overline{g}_m$ in $\overline{S}_H(m,n,\alpha)$ so that h and g_m are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ g_m(z) = (-1)^{m-1}\sum_{k=1}^{\infty} b_k z^k,\ a_k, b_k \ge 0. \tag{1.4}$$

The classes $S_H(m,n,\alpha)$ and $\overline{S}_H(m,n,\alpha)$ were studied by Yalcin [9].

Let us define the convolution of two harmonic functions. For harmonic functions of the form

$$f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_k \overline{z}^k$$

and

$$F_m(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} B_k \overline{z}^k$$

we define the convolution of two harmonic functions f_m and F_m as

$$(f_m * F_m)(z) = f_m(z) * F_m(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_k B_k \overline{z}^k. \tag{1.5}$$

Recently Yalcin [[9], Theorem 6] has obtained the following result for convolution of two functions in class $\overline{S}_H(m,n,\alpha)$.

Theorem A. For $0 \le \beta \le \alpha < 1$ let $f_m \in \overline{S}_H(m,n;\alpha)$ and $F_m \in \overline{S}_H(m,n;\beta)$. Then $f_m * F_m \in \overline{S}_H(m,n;\alpha) \subseteq \overline{S}_H(m,n;\beta)$.

In the present paper, motivated with the work of Kumar [5], we prove the following theorem and then we critically observed that it improves the above stated theorem of Yalcin [9].

Theorem 1. *Let the functions*

$$f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_k \overline{z}^k \qquad (a_k, b_k \ge 0)$$

and

$$F_m(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} B_k \overline{z}^k \qquad (A_k, B_k \ge 0)$$

belong to the classes $\overline{S}_H(m,n;\alpha)$ and $\overline{S}_H(m,n;\beta)$ respectively. Then $(f_m * F_m)(z) \in \overline{S}_H(2m, m+n;\alpha)$ (if m is an even integer) and $(f_m * F_m)(z) \in \overline{S}_H(2m-1, m+n-1;\alpha)$ (if m is an odd integer).

To prove this theorem we require the following lemmas. Lemma 1 and 2 are due to Yalcin [9].

Lemma 2. *A function $f_m(z)$ of the form* (1.4) *belongs to the class $\overline{S}_H(m,n;\alpha)$ if and only if*

$$\sum_{k=2}^{\infty}\frac{k^m-\alpha k^n}{1-\alpha}a_k+\sum_{k=1}^{\infty}\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}b_k\leq 1.$$

Lemma 3. $\overline{S}_H(m,n;\alpha)\subseteq\overline{S}_H(m,n;\beta)$*, if* $0\leq\beta\leq\alpha<1$.

Lemma 4. *(i) $\overline{S}_H(2m,m+n;\alpha)\subseteq\overline{S}_H(m,n;\alpha)$, (if m is an even integer).*
(ii) $\overline{S}_H(2m-1,m+n-1;\alpha)\subseteq\overline{S}_H(m,n;\alpha)$, (if m is an odd integer).

Proof. **Proof of Lemma 4(i)**
Let $f_m\in\overline{S}_H(2m,m+n;\alpha)$ then by using Lemma 2 we have

$$\sum_{k=2}^{\infty}\frac{k^{2m}-\alpha k^{m+n}}{1-\alpha}a_k+\sum_{k=1}^{\infty}\frac{k^{2m}-(-1)^{2m-(m+n)}\alpha k^{m+n}}{1-\alpha}b_k\leq 1$$

$$\Rightarrow\sum_{k=2}^{\infty}\frac{k^m(k^m-\alpha k^n)}{1-\alpha}a_k+\sum_{k=1}^{\infty}\frac{k^m(k^m-(-1)^{m-n}\alpha k^n)}{1-\alpha}b_k\leq 1.\tag{1.6}$$

Now

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{k^m-\alpha k^n}{1-\alpha}a_k+\sum_{k=1}^{\infty}\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}b_k\\&\leq\sum_{k=2}^{\infty}\frac{k^m(k^m-\alpha k^n)}{1-\alpha}a_k+\sum_{k=1}^{\infty}\frac{k^m(k^m-(-1)^{m-n}\alpha k^n)}{1-\alpha}b_k\\&\leq 1,\qquad\text{[Using (1.6)]}.\end{aligned}$$

Thus $f_m\in\overline{S}_H(m,n;\alpha)$.
The proof of Lemma 4(i) is established. □

Proof. **Proof of Lemma 4(ii)** The proof of Lemma 4(ii) is similar to that of Lemma 4(i), hence it will be omitted. □

2. Proof of the Theorem 1.

Here we only prove the Theorem 1 for the case when m is an even integer. For the case when m is an odd integer one can prove the Theorem 1 in similar way. Therefore it is omitted.

Proof. Since $f_m(z)\in\overline{S}_H(m,n;\alpha)$, then by using Lemma 2 we have

$$\sum_{k=2}^{\infty}\frac{k^m-\alpha k^n}{1-\alpha}a_k+\sum_{k=1}^{\infty}\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}b_k\leq 1.\tag{2.1}$$

Similarly $F_m(z)\in\overline{S}_H(m,n;\beta)$, we have

$$\sum_{k=2}^{\infty}\frac{k^m-\beta k^n}{1-\beta}A_k+\sum_{k=1}^{\infty}\frac{k^m-(-1)^{m-n}\beta k^n}{1-\beta}B_k\leq 1.$$

Therefore $\frac{k^m-\beta k^n}{1-\beta}A_k\leq 1,\forall\, k=2,3\ldots$

$$\frac{k^m - \beta k^n}{1-\beta} A_k \leq 1, \; \forall\, k = 2, 3 \ldots$$

and

$$\frac{k^m - (-1)^{m-n}\beta k^n}{1-\beta} B_k \leq 1, \; \forall\, k = 1, 2, 3 \ldots$$

Now, for the convolution function $f_m * F_m$ we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{k^{2m} - \alpha k^{m+n}}{1-\alpha} a_k A_k + \sum_{k=1}^{\infty} \frac{k^{2m} - (-1)^{2m-(m+n)}\alpha k^{m+n}}{1-\alpha} b_k B_k \\
&= \sum_{k=2}^{\infty} \frac{k^m(k^m - \alpha k^n)}{1-\alpha} a_k A_k + \sum_{k=1}^{\infty} \frac{k^m(k^m - (-1)^{m-n}\alpha k^n)}{1-\alpha} b_k B_k \\
&\leq \sum_{k=2}^{\infty} \left(\frac{k^m - \beta k^n}{1-\beta}\right) A_k \left(\frac{k^m - \alpha k^n}{1-\alpha}\right) a_k \\
&+ \sum_{k=1}^{\infty} \left(\frac{k^m - (-1)^{m-n}\beta k^n}{1-\beta}\right) B_k \left(\frac{k^m - (-1)^{m-n}\alpha k^n}{1-\alpha}\right) b_k \\
&\leq \sum_{k=2}^{\infty} \frac{k^m - \alpha k^n}{1-\alpha} a_k + \sum_{k=1}^{\infty} \frac{k^m - (-1)^{m-n}\alpha k^n}{1-\alpha} b_k \\
&\leq 1, \; \text{[Using (2.1)]}.
\end{aligned}$$

Therefore, we have
$(f_m * F_m)(z) \in \overline{S}_H(2m, m+n; \alpha)$ (if m is an even integer).
Similarly
$(f_m * F_m)(z) \in \overline{S}_H(2m-1, m+n-1; \alpha)$ (if m is an odd integer). □

Improvement of Yalcin's Result

In this section, we consider the following two cases and in each case, we observe that our result improves the result of Yalcin [[9], Theorem 6].
Case (i) When m is an even integer.
Case (ii) When m is an odd integer.

Here we discuss these cases one by one.
Case (i) When m is an even integer our Theorem states that $(f_m * F_m)(z) \in \overline{S}_H(2m, m+n; \alpha)$, whereas result of Yalcin gives $(f_m * F_m)(z) \in \overline{S}_H(m, n; \alpha)$. But by Lemma 3 and 4(i) we have $\overline{S}_H(2m, m+n; \alpha) \subseteq \overline{S}_H(m, n; \alpha) \subseteq \overline{S}_H(m, n; \beta)$. Therefore our result provides smaller class in comparison to the class given by Yalcin to which $(f_m * F_m)(z)$ belongs.
Case (ii) When m is an odd integer we use our result $(f_m * F_m)(z) \in \overline{S}_H(2m-1, m+n-1; \alpha)$. Since $\overline{S}_H(2m-1, m+n-1; \alpha) \subseteq \overline{S}_H(m, n; \alpha) \subseteq \overline{S}_H(m, n; \beta)$ (by Lemma 3 and 4(ii)). Our result provides better estimate in this case also.

Hence we conclude that for all values of $m \in N = \{1, 2, 3 \ldots\}$ our result improves the result of Yalcin [[9], Theorem 6].

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