Anti Fuzzy Implicative Ideals in BCK-Algebras

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Abstract. The aim of this paper is to introduce the notion of anti fuzzy implicative ideals of BCK-algebras and to investigate their properties. We give several characterizations of anti fuzzy implicative ideals. We also introduce the notion of anti Cartesian product of anti fuzzy implicative ideals, and then we study related properties.

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1. INTRODUCTION

The main problem in fuzzy mathematics is how to carry out the ordinary concepts to the fuzzy case. The difficulties lie in how to pick out the rational generalization from the large number of available approaches. It is worth noting that fuzzy ideals are different from ordinary ideals in the sense that one cannot say which BCK-algebra element belongs to the fuzzy ideal under consideration and which one does not. The concept of fuzzy sets was introduced by Zadeh [13]. Since then these ideas have been applied to other algebraic structures such as semigroups, groups, rings, modules, vector spaces and topologies. In 1991, Xi [12] applied the concept of fuzzy sets to BCK-algebras which are introduced by Imai and Iséki [5]. In [1], Biswas introduced the concept of anti fuzzy subgroups of groups. Modifying his idea, in [4], S. M. Hong and Y. B. Jun applied the idea to BCK-algebras. They introduced the notion of anti fuzzy ideals of BCK-algebras. In this paper, we introduce the notion of anti fuzzy implicative ideal of BCK-algebras, and investigate some related properties. We show that in an implicative BCK-algebra, a fuzzy subset is an anti fuzzy ideal if and only if it is an anti fuzzy implicative ideal. We show that a fuzzy subset of a BCK-algebra is a fuzzy implicative ideal if and only if the complement of this fuzzy subset is an anti fuzzy implicative ideal. Moreover, we discuss the pre-image of anti fuzzy implicative ideals. Finally, we introduce the notion of anti Cartesian product of anti fuzzy implicative ideals, and then we characterize anti fuzzy implicative ideals by it.

2. PRELIMINARIES

In this section we cite the fundamental definitions that will be used in the sequel:
**Definition 1.** [6] An algebra $(X, \ast, 0)$ of type $(2, 0)$ is called a BCK-algebra if it satisfies the following axioms for all $x, y, z \in X$:

(i) $(x \ast y) \ast (x \ast y) = 0,$
(ii) $(x \ast (x \ast y)) \ast y = 0,$
(iii) $x \ast x = 0,$
(iv) $0 \ast x = 0,$
(v) $x \ast y = 0$ and $y \ast x = 0$ imply $x = y.$

We can define a partial ordering $\leq$ on $X$ by $x \leq y$ if and only if $x \ast y = 0.$

**Proposition 2.** [6] In any BCK-algebra $X,$ the following hold for all $x, y, z \in X$:

(i) $(x \ast y) \ast z = (x \ast z) \ast y,$
(ii) $x \ast y \leq x,$
(iii) $x \ast 0 = x,$
(iv) $(x \ast z) \ast (y \ast z) \leq x \ast y,$
(v) $x \ast (x \ast (x \ast y)) = x \ast y,$
(vi) $x \leq y$ implies $x \ast z \leq y \ast z$ and $z \ast y \leq z \ast x.$

A BCK-algebra is said to be implicative if $x \ast (y \ast x) = x$ for all $x, y \in X$ (see [6,9]).

**Definition 3.** [8] A non-empty subset $I$ of a BCK-algebra $X$ is called an ideal of $X$ if it satisfies

(i) $0 \in I$
(ii) $x \ast y \in I$ and $y \in I$ imply $x \in I.$

**Definition 4.** [8] A non-empty subset $I$ of a BCK-algebra $X$ is called an implicative ideal of $X$ if it satisfies $(I_1)$ and $(I_2)$ $x \in I$ whenever $(x \ast (y \ast x)) \ast z \in I$ and $z \in I$ for all $x, y, z \in X.$

**Definition 5.** [13] Let $S$ be a non-empty set. A fuzzy subset $A$ of $S$ is a function $A : S \rightarrow [0, 1].$ Let $A$ be a fuzzy subset of $S.$ Then for $t \in [0, 1],$ the $t$-level cut of $A$ is the set $A_t = \{x \in S \mid A(x) \geq t\}$ and the complement of $A,$ denoted by $A^c,$ is the fuzzy subset of $S$ given by $A^c(x) = 1 - A(x)$ for all $x \in S$ (see [2,3,7]).

**Definition 6.** [12] A fuzzy subset $A$ of a BCK-algebra $X$ is called a fuzzy subalgebra of $X$ if $A(x \ast y) \geq \min \{A(x), A(y)\}$ for all $x, y \in X.$

**Definition 7.** [12] Let $X$ be a BCK-algebra. A fuzzy subset $A$ of $X$ is called a fuzzy ideal of $X$ if

$(F_1) \ A(0) \geq A(x),$ 
$(F_2) \ A(x) \geq \min \{A(x \ast y), A(y)\},$ for all $x, y \in X.$

**Definition 8.** [10] A fuzzy subset $A$ of a BCK-algebra $X$ is called a fuzzy implicative ideal of $X$ if it satisfies

$(F_1)$ and $(F_2) \ A(x) \geq \min \{A((x \ast (y \ast x)) \ast z), A(z)\}$ for all $x, y, z \in X.$

**Definition 9.** [4] A fuzzy subset $A$ of a BCK-algebra $X$ is called an anti fuzzy subalgebra of $X$ if

$A(x \ast y) \leq \max \{A(x), A(y)\}$ for all $x, y \in X.$

**Definition 10.** [4] A fuzzy subset $A$ of a BCK-algebra $X$ is called an anti fuzzy ideal of $X$ if

$(A_1) \ A(0) \leq A(x),$ 
$(A_2) \ A(x) \leq \max \{A(x \ast y), A(y)\},$ for all $x, y \in X.$

Definition 12. [4] Let $A$ be a fuzzy subset of a BCK-algebra. Then for $t \in [0, 1]$ the lower $t$-level cut of $A$ is the set
\[ A^t = \{ x \in X \mid A(x) \leq t \} \]

Definition 13. [4] Let $A$ be a fuzzy subset of a BCK-algebra. The fuzzification of $A^t, t \in [0, 1]$ is the fuzzy subset $\mu_{A^t}$ of $X$ defined by

Definition 14. [11] Let $f : X \to Y$ be a mapping of BCK-algebras and $A$ be a fuzzy subset of $Y$. The map $A^f$ is the inverse image of $A$ under $f$ if $A^f (x) = A(f(x)) \forall x \in X$.

3. ANTI FUZZY IMPLICATIVE IDEAL

Definition 15. A fuzzy subset $A$ of a BCK-algebra $X$ is called an anti fuzzy implicative ideal of $X$ if it satisfies
\[(A_1) \text{ and } (A_3) \quad A(x) \leq \max \{ A((x \ast (y \ast x)) \ast z) , A(z) \} \text{ for all } x, y, z \in X.\]

Example 1. (1) Every constant function $A : X \to [0, 1]$ is an anti fuzzy implicative ideal of $X$ (2) Let $X = \{0, a, b, c\}$ be a BCK-algebra with Cayley table as follows:

\[
\begin{array}{cccc}
  & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & a \\
b & b & a & 0 & b \\
c & c & c & c & 0
\end{array}
\]

Let $t_0, t_1$ be such that $t_0 \prec t_1$. Define $A : X \to [0, 1]$ by $A(0) = A(a) = A(b) = t_0$ and $A(c) = t_1$ Routine calculations give that $A$ is an anti fuzzy implicative ideal.

Proposition 16. Every anti fuzzy implicative ideal of a BCK-algebra $X$ is order preserving.

Proof. Let $A$ be an anti fuzzy implicative ideal of a BCK-algebra $X$ and let $x, y \in X$ be such that $x \leq y$. Then
\[
A(x) \leq \max \{ A((x \ast (y \ast x)) \ast z), A(y) \},
\]
\[
= \max \{ A((x \ast y) \ast (z \ast x)), A(y) \},
\]
\[
= \max \{ A(0 \ast (z \ast x)), A(y) \},
\]
\[
= \max \{ A(0), A(y) \} = A(y). \quad \square
\]

Proposition 17. Every anti fuzzy implicative ideal of a BCK-algebra $X$ is an anti fuzzy ideal.

Proof. Let $A$ be an anti fuzzy implicative ideal of a BCK-algebra $X$, so for all $x, y, z \in X$,
\[
A(x) \leq \max \{ A((x \ast (y \ast x)) \ast z), A(z) \},
\]
Putting in $y = x$, and $z = y$:
\[
A(x) \leq \max \{ A((x \ast x) \ast y), Ay \}
\]
\[
= \max \{ A(x \ast y), A(y) \}. \quad \square
\]
Combining Proposition 2.11 and 3.4 yields the following result.

Proposition 18. Every anti fuzzy implicative ideal of a BCK-algebra $X$ is an anti fuzzy subalgebra of $X$. 

Remark 19. An anti fuzzy ideal (subalgebra) of a BCK-algebra $X$ may not be an anti fuzzy implicative ideal of $X$ as shown in the following example:

**Example 2.** Let $X$ be the BCK-algebra in Example 3.2(2) and let $t_0, t_1, t_2 \in [0, 1]$ be such that $t_0 \leq t_1 \leq t_2$. Define $A : X \rightarrow A(0) = t_0, A(a) = A(b) = t_1$ and $A(c) = t_2$. Routine calculations give that $A$ is an anti fuzzy ideal (subalgebra) of $X$, but not an anti fuzzy implicative ideal of $X$ because
\[
A(a) = t_1 > \max \{A)((a \ast (b \ast a)) \ast 0), A(o)\}
= \max \{A(0), A(0)\} = t_0.
\]

**Proposition 20.** If $X$ is implicative BCK-algebra, then every anti fuzzy ideal of $X$ is an anti fuzzy implicative ideal of $X$.

**Proof.** Let $A$ be an anti fuzzy ideal of an implicative BCK-algebra $X$, so for all $x, z \in X$:
\[
A(x) \leq \max \{A(x \ast z), A(z)\},
\]
Since $X$ is an implicative, then $x \ast (y \ast x) = x$ for all $x, y \in X$. Hence
\[
A(x) \leq \max \{A((x \ast (y \ast x)) \ast z), A(z)\},
\]
This shows that $A$ is an anti fuzzy implicative ideal of $X$.

By applying Proposition 17 and 20, we have

**Theorem 21.** If $X$ is an implicative BCK-algebra, then a fuzzy subset $A$ of $X$ is an anti fuzzy ideal of $X$ if and only if it is an anti fuzzy implicative ideal of $X$.

**Proposition 22.** A fuzzy subset $A$ of a BCK-algebra $X$ is a fuzzy implicative ideal of $X$ if and only if its complement $A^c$ is an anti fuzzy implicative ideal of $X$.

**Proof.** Let $A$ be a fuzzy implicative ideal of a BCK-algebra $X$, and let $x, y, z \in X$. Then
\[
A(0) = 1 - A(0) \leq 1 - A(x) = A^c(x)
\]
and
\[
A^c(x) = 1 - A(x) \leq 1 - \min \{A((x \ast (y \ast x)) \ast z), A(z)\},
\]
\[
= 1 - \min \{1 - A^c((x \ast (y \ast x)) \ast z), 1 - A^c(z)\},
\]
\[
= \max \{A^c((x \ast (y \ast x)) \ast z), A^c(z)\}.
\]
So, $A^c$ is an anti fuzzy implicative ideal of $X$. Now let $A^c$ is an anti fuzzy implicative ideal of $X$, and let $x, y, z \in X$. Then
\[
A(0) = 1 - A^c(0) \geq 1 = A^c(x) = A(x)
\]
and
\[
A(x) - 1A^c(x) \leq 1 - \max \{A^c((x \ast (y \ast x)) \ast z), A^c(z)\},
\]
\[
= 1 - \max \{1 - A((x \ast (y \ast x)) \ast z), 1 - A(z)\},
\]
\[
= \min \{A((x \ast (y \ast x)) \ast z), A(z)\}.
\]
Thus, $A$ is a fuzzy implicative ideal of $X$.

**Theorem 23.** Let $A$ be an anti fuzzy implicative ideal of a BCK-algebra $X$. Then the set
\[
X_A = \{x \in X | A(x) = A(0) \}
\]
is an implicative ideal of $X$.

**Proof.** Clearly $0 \in X$ . Let $x, y, z \in X_A$ be such that $(x \ast (y \ast x)) \ast z \in X$ and $z \in X_A$. Then
\[
A((x \ast (y \ast x)) \ast z) = A(z) = A(0).
\]
It follows that
\[
A(x) \leq \max \{A((x \ast (y \ast x)) \ast z), A(z)\}
\]
\[
= \max \{A(0), A(0)\} = A(0).
\]
Combining Definition 15($A_1$), we get $A(x) = A(0)$ and hence $x \in X$.

**Theorem 24.** Let $A$ be a fuzzy subset $A$ of a BCK-algebra $X$. Then $A$ is an anti fuzzy implicative ideal of $X$ if and only if for each $t \in [0, 1], t \geq A(0)$, the lower $t$-level cut $A_t$ is an implicative ideal of $X$.
Proof. Let \( A \) be an anti fuzzy implicative ideal of \( X \) and let \( t \in [0,1] \) with \( t \geq A(0) \). Clearly \( 0 \in A^t \). Let \( x, y, z \in X \) be such that \((x * (y * x)) * z \in A^t \) and \( z \in A^t \). Then \( A((x * (y * x)) * z) \leq t \), \( A(z) \leq t \), hence

\[
A(x) \leq \max \{ A((x * (y * x)) * z) \mid A(z) \leq t \} \leq t .
\]

And so \( x \in A^t \). Hence \( A^t \) is an implicative ideal of \( X \).

Conversely, let \( A^t \) be an implicative ideal of \( X \), we first show \( A(0) \leq A(x) \) for all \( x \in X \).
If not, then there exists \( x_0 \in X \) such that \( A(0) \triangleleft A(x_0) \). Putting \( t_0 = \frac{1}{2} \{ A(0) + A(x_0) \} \), then

\[
0 \leq A(x_0) \triangleleft A(0) \leq 1 .
\]

Hence \( x_0 \in A^{t_0} \), so that \( A^{t_0} \neq \emptyset \). But \( A^{t_0} \) is an implicative ideal of \( X \).

Thus \( 0 \in A^{t_0} \), or \( A(0) \leq t_0 \), a contradiction. Hence \( A(0) \leq A(x) \) for all \( x \in X \). Now we prove that \( A(x) \leq \max \{ A((x * (y * x)) * z) \mid A(z) \leq t \} \) for all \( x, y, z \in X \). If not, then there exists \( x_0, y_0, z_0 \in X \) such that

\[
A(x) > \max \{ A((x_0 * (y_0 * x_0)) * z_0) \mid A(z_0) \} .
\]

Taking \( s_0 = \frac{1}{2} \{ A(x_0) + \max \{ (x_0 * (y_0 * x_0)) * z_0 \mid A(z_0) \} \} \); then \( s_0 \leq A(x_0) \) and

\[
0 \leq \max \{ A((x_0 * (y_0 * x_0)) * z_0) \} ; s_0 \leq 1 .
\]

Thus we have \( s_0 \triangleleft A(((x_0 * (y_0 * x_0)) * z_0) \) and \( s_0 \triangleleft A(z_0) \). Which imply that

\[
(x_0 * (y_0 * x_0)) * z_0 \in A^{s_0} .
\]

But \( A^{s_0} \) is an implicative ideal of \( X \). Thus \( x_0 \in A^{s_0} \) or \( A(x_0) \leq s_0 \). This is a contradiction, ending the proof.

\[\square\]

Theorem 25. If \( A \) is an anti fuzzy implicative ideal of a BCK-algebra \( X \). Then \( \mu_A \) is also an anti fuzzy implicative ideal of \( X \) where \( t \in [0,1] \), \( t \geq A(0) \).

Proof. From Theorem 24, it is sufficient to show that \( (\mu_A)^t \) is an implicative ideal of \( X \), where \( s \in [0,1] \) and \( s \geq \mu_A \). Clearly, \( 0 = (\mu_A) \). Let \( x, y, z \in X \) be such that \((x * (y * x)) * z \in (\mu_A)^s \) and \( z \in (\mu_A)^s \), hence

\[
\mu_A((x * (y * x)) * z) \leq s ,
\]

and \( \mu_A(z) \leq s \). We claim that \( x \in (\mu_A) \) or \( \mu_A(x) \leq s \). If \((x * (y * x)) * z \in A^t \) and \( z \in A^t \), then \( x \in A^t \) because \( A^t \) is an implicative ideal of \( X \). Hence.

\[
\mu_A(x) = A(x) \leq \max \{ A((x * (y * x)) * z) \mid A(z) \} , \mu_A(z) \leq s ,
\]

and so \( x \in (\mu_A)^s \). If \((x * (y * x)) * z \notin A^t \) or \( z \in A^t \), then \( \mu_A((x * (y * x)) * z) = 0 \) or \( \mu_A(z) = 0 \) then clearly \( \mu_A(x) \leq s \), and so \( x \in (\mu_A)^s \). Therefore \( (\mu_A)^s \) is an implicative ideal of \( X \).

\[\square\]

Theorem 26. Theorem 3.13. Let \( f : X \rightarrow Y \) be a homomorphism of BCK-algebras. If \( A \) is an anti fuzzy implicative ideal of \( Y \), then \( A^f \) is an anti fuzzy implicative ideal of \( X \).

Proof. Since \( A \) is an anti fuzzy implicative ideal of \( Y \), then \( A(0^t) \leq A(f(x)) \) for any \( x \in X \), and so, \( A^f(0^t) = A(f(0)) = A(0^t) \leq A(f(x)) = A^f(x) \). For any \( x, y, z \in X \), we have

\[
A^f(x) = A(f(x)) \leq \max \{ A((f(x) * (f(y) * f(x))) * f(z)) \mid A(f(z)) \} ,
\]

and so \( x \in (\mu_A)^s \). Then \( \mu_A((x * (y * x)) * z) = 0 \) or \( \mu_A(z) = 0 \) then clearly \( \mu_A(x) \leq s \), and so \( x \in (\mu_A)^s \). Therefore \( (\mu_A)^s \) is an implicative ideal of \( X \).

This completes the proof.

\[\square\]
Theorem 3.14. Let \( f : X \rightarrow Y \) be an epimorphism of BCK-algebras. If \( A^f \) is an anti fuzzy implicative ideal of \( X \), then \( A \) is an anti fuzzy implicative ideal of \( Y \).

**Proof.** Let \( y \in Y \), there exists \( x \in X \) such that \( f(x) = y \). Then
\[
A(y) = A(f(x)) = A^f(x) \geq A^f(0) = A(0'),
\]

Let \( x', y', z' \in Y \). Then there exist \( x, y, z \in X \) such that \( f(x) = x', f(y) = y' \) and \( f(z) = z' \). It follows that
\[
A(x') = A(f(x)) = A^f(x) \leq \max \{ A^f((x * (y * x)) * z), A^f(z) \},
\]
\[
= \max \{ A(f((x * (y * x)) * z)), A(f(z)) \},
\]
\[
= \max \{ A((f(x) * (f(y) * f(x))) * f(z)), A(f(z)) \},
\]
\[
= \max \{ A((x' * (y' * x')) * z'), A(z') \}.
\]

Hence \( A \) is an anti fuzzy implicative ideal of \( Y \).

4. **ANTI CARTESIAN PRODUCT OF ANTI FUZZY IMPLICATIVE IDEALS**

**Definition 28.** Let \( \lambda \) and \( \mu \) be the fuzzy subsets in a set \( X \). The anti Cartesian product \( \lambda \times \mu : X \times X \rightarrow [0, 1] \) is defined by \( (\lambda \times \mu)(x, y) = \max \{ \lambda(x), \mu(y) \} \) for all \( x, y \in X \).

**Theorem 29.** If \( \lambda \) and \( \mu \) are anti fuzzy implicative ideals of a BCK-algebra \( X \), then \( \lambda \times \mu \) is an anti fuzzy implicative ideal of \( X \times X \).

**Proof.** Let \( x, x' \in X \)
\[
(\lambda \times \mu)(0, 0) = \max \{ \lambda(0), \mu(0) \}
\]
\[
\leq \max \{ \lambda(x), \mu(x') \} = (\lambda \times \mu)(x, x')
\]
For any \( (x, x'), (y, y'), (z, z') \in X \times X \) we have
\[
(\lambda \times \mu)(x, x') = \max \{ \lambda(x), \mu(x') \}
\]
\[
\leq \max \{ (\lambda(x) * (y * x)) * z), (\lambda(z)) \}, \max \{ \lambda(x' * (y' * x')) * z'), (\lambda(z')) \}
\]
\[
= \max \{ (\lambda(x' * (y' * x')) * z'), (\lambda(z')) \}, \max \{ (\lambda(x) * (y * x)) * z), (\lambda(z)) \}
\]
\[
\leq \max \{ (\lambda(x) * (y * x)) * z), (\lambda(z)) \}, \max \{ (\lambda(x') * (y' * x')) * z'), (\lambda(z')) \}
\]
\[
= \max \{ (\lambda(x' * (y' * x')) * z'), (\lambda(z')) \}, (\lambda(x) * (y * x)) * z), (\lambda(z)) \}
\]

Hence \( \lambda \times \mu \) is an anti fuzzy implicative ideal of \( X \times X \).

**Theorem 30.** Let \( \lambda \) and \( \mu \) be fuzzy subsets in a BCK-algebra \( X \) such that \( \lambda \times \mu \) is an anti fuzzy implicative ideal of \( X \times X \). Then:

(i) either \( \lambda(x) \geq \lambda(0) \) or \( \mu(x) \leq \mu(0) \) for all \( x \in X \);

(ii) if \( \lambda(x) \geq \lambda(0) \) for all \( x \in X \), then either \( \lambda(x) \geq \mu(0) \) or \( \mu(x) \geq \mu(0) \);

(iii) if \( \mu(x) \geq \mu(0) \) for all \( x \in X \), then either \( \lambda(x) \geq \lambda(0) \) or \( \mu(x) \geq \lambda(0) \).

**Proof.** (i) Suppose that there exist \( x, y \in X \) such that \( \lambda(x) \| \mu(y) \). Then
\[
(\lambda \times \mu)(x, y) = \max \{ \lambda(x), \mu(y) \} \geq \max \{ \lambda(0), \mu(0) \} = (\lambda \times \mu)(0, 0).
\]
This is a contradiction and we obtain (i).

(ii) Assume that there exist \( x, y \in X \) such that \( \lambda(x) \| \mu(y) \). Then
\[
(\lambda \times \mu)(x, y) = \max \{ \lambda(x), \mu(y) \} \geq (\lambda \times \mu)(0, 0),
\]
which is a contradiction. Hence (ii) holds.

(iii) Similar to (ii).

**Theorem 31.** Let \( \lambda \) and \( \mu \) be fuzzy subsets in a BCK-algebra \( X \) such that \( \lambda \times \mu \) is an anti fuzzy implicative ideal of \( X \times X \). Then either \( \mu \) or \( \lambda \) is an anti fuzzy implicative ideal of \( X \).
Proof. By Theorem 30(i), without loss of generality we assume that \( \mu(x) \geq \mu(0) \) for all \( x \in X \). From (iii) it follows that either \( \lambda(x) \geq \lambda(0) \) or \( \mu(x) \geq \lambda(0) \). If \( \mu(x) \geq \lambda(0) \) for all \( x \in X \), then \( (x, x') \in X \times X \), since \( \lambda \times \mu \) is an anti fuzzy implicative ideal of \( X \times X \). We have

\[
(\lambda \times \mu)(x, x') = \max\{ (\lambda \times \mu)(0, x), (\mu(x) \leq \lambda(0)) \}
\]

Putting \( x = y = z = 0 \), then

\[
\mu(x') = (\lambda \times \mu)(0, x') \leq \max\{ (\lambda \times \mu)(0, (x' \ast (y' \ast x'))) \ast z', (x' \ast (y' \ast x')) \ast z' \}, \quad (\lambda \times \mu)(0, z')
\]

This proves that \( \mu \) is an anti fuzzy implicative ideal of \( X \). The second part is similar. This completes the proof.

5. Conclusion

We discussed the notion of anti fuzzy implicative ideals of \( BCK \)-algebras and gave several characterizations. Also, we introduced the notion of anti cartesian product of anti fuzzy implicative ideals. Our definitions probably can be applied in other kinds of anti ideals of \( BCK \)-algebras.

References