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An Improved Error Analysis for the Secant Method Under the Gamma Condition

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Abstract. We provide sufficient convergence conditions for the Secant method for approximating a locally unique solution of an operator equation in a Banach space. The main hypothesis is a type of gamma condition first introduced in [9] for the study of Newton's method. Our sufficient convergence condition reduces to the one obtained in [12] for Newton's method although in general it can be weaker. A numerical example is also provided.

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1. 1}$$

where F is a Fréchet-differentiable operator mapping a convex subset D of a Banach space X into a Banach space Y.

The most popular methods for generating sequences approximating x^* are undoubtedly Newton's method

$$y_{n+1} = y_n - F'(y_n)^{-1}F(y_n) \quad (n \geq 0), \quad (y_0 \in D), \tag{1. 2}$$

and the Secant method

$$x_{n+1} = x_n - [x_{n-1}, x_n]^{-1}F(x_n) \quad (n \geq 0), \quad (x_{-1}, x_0 \in D). \tag{1. 3}$$

The advantages and disadvantages of using the Secant method over Newton's method are well known [1]–[14].
Here, $F'(x)$, $[x, y] \in L(X, Y)$ the space of bounded linear operators, by $[x, y]$ we mean $[x, y; F]$, and the divided difference of order one at (x, y) satisfying

$$[x, y](x - y) = F(x) - F(y) \tag{1. 4}$$

for all $x, y \in D$ with $x \neq y$ [4], [6], [9].

There is an extensive literature on methods (1.2) and (1.3). A survey of such results can be found in [1]–[9], [14], and the references there.

It turns out that so far there are two ways of studying method (1.2): Newton–Kantorovich-type local and semilocal convergence results depending on a domain containing the initial guess x_0 and Lipschitz conditions on $F'(x)$ [4], [6], [9]; Smale-type theorems that require information only at x_0 and the analyticity of F [11]–[14].

Moreover, Wang [12] introduced the weaker than Smale's gamma γ-condition and successfully applied it to Newton and Newton-type methods. Yakoubson [14] extended Smale's work for the Secant method using a strong analyticity assumption on operator F.

The results mentioned above are based on the assumption that the sequence

$$\left\| \frac{F'(x_0)^{-1}F^{(n)}(x_0)}{n!} \right\| \quad (n \geq 2), \tag{1. 5}$$

is bounded above by

$$\gamma(F, x_0) = \sup_{k\geq 2} \left\| \frac{F'(x_0)^{-1}F^{(n)}(x_0)}{n!} \right\|^{\frac{1}{n-1}}. \tag{1. 6}$$

However, this kind of assumption may not be reasonable. Particularly, for some concrete and special operators appearing in connection with the Durand–Kerner method, it is really so [8].

Here we provide a convergence analysis for the Secant method using an even weaker version of Wang's gamma condition (see (2.1)). It turns out that even in the special case when method (1.3) reduces to (1.2) our error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ are finer than the ones in [12] and the information on the location of the solution x^* at least as precise. Note also that these advantages are obtained under the same computational cost. Numerical examples are also provided.

2. Semilocal Convergence Analysis of Method (1.3)

Let $x_0 \in X$ and $r > 0$. We denote by $U(x_0, r) = \{x \in X\colon \|x - x_0\| < r\}$.

We introduce the (γ_0, γ) condition:

Definition 2.1. Suppose:

$$0 < \gamma_0 \leq \gamma. \tag{2. 1}$$

We say F satisfies the gamma (γ_0, γ) condition at $x_0 \in D$ in $\overline{U}(x_0, r) \subseteq D$ if operator F is Fréchet-differentiable at $x = x_0$, $F'(x_0)^{-1} \in L(Y, X)$ such that for all $r < \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma_0}$, $x, y, w \in \overline{U}(x_0, r)$

$$\|F'(x_0)^{-1}([x,y] - [y,w])\| \leq \int_0^1 \int_0^1 \frac{2\gamma[t\|x-y\| + (1-t)\|y-w\|]dsdt}{[1 - \gamma\|s(tx + (1-t)y) + (1-s)(ty + (1-t)w) - x_0\|]^3}, \tag{2. 2}$$

and

$$\begin{aligned}
&\|F'(x_0)^{-1}([x,y]-F'(x_0))\| \\
&\le \int_0^1\int_0^1 \frac{2\gamma_0\|x_0-tx-(1-t)y\|dsdt}{[1-s\gamma_0\|x_0-tx-(1-t)y\|]^3} \\
&= \int_0^1 [1-\gamma_0\|x_0-tx_{k-1}-(1-t)x_k\|]^{-2}-1. \qquad (2.\ 3)
\end{aligned}$$

Example 2.2. Let us provide a class of operators that satisfies both (2.2) and (2.3). For simplicity, we set $\gamma_0=\gamma$, and assume F is twice Fréchet-differentiable on $\overline{U}(x_0,r)$ satisfying:

$$\|F'(x_0)^{-1}F''(x)\| \le \frac{2\gamma}{(1-\gamma\|x-x_0\|)^3}\,. \qquad (2.\ 4)$$

Note that condition (2.4) used in [12] requires the existence of the second Fréchet-derivative, whereas we only require the existence of the first derivative. It is known that $\gamma(F,x_0)\le\gamma$ [11], [13], [14], which is the γ-motivation for our study. Moreover, assume divided difference $[x,y]$ is given by

$$[x,y]=\int_0^1 F'[y+t(x-y)]dt \qquad (2.\ 5)$$

for all $x,y\in\overline{U}(x_0,r)\subseteq D$, which holds in many interesting cases [7], [8]. Then using (2.4), we can have in turn:

$$\begin{aligned}
&\|F'(x_0)^{-1}([x,y]-[y,w])(y-w)\| \\
&= \Big\|\int_0^1\int_0^1 F''[s(tx+(1-t)y)+(1-s)(ty+(1-t)w]ds \\
&\qquad \cdot [t(x-y)+(1-t)(y-w)]dt(y-w)\Big\| \\
&\le \int_0^1\int_0^1 \frac{2\gamma(t\|x-y\|+(1-t)\|y-w\|)\|y-w\|dsdt}{[1-\gamma\|s(tx+(1-t)y)+(1-s)(ty+(1-t)w)-x_0\|]^3}, \qquad (2.\ 6)
\end{aligned}$$

which justifies condition (2.2). Moreover using again (2.4) we can obtain

$$\begin{aligned}
&\|F'(x_0)^{-1}([x,y]-F'(x_0)) = \Big\|F'(x_0)^{-1}\int_0^1 [F'(tx+(1-t)y)-F'(x_0)]dt\Big\| \\
&= \Big\|\int_0^1\int_0^1 F'(x_0)^{-1}F''[((1-s)x_0 \\
&\qquad + s(tx+(1-t)y))][x_0-ty-(1-t)x]dsdt\Big\| \\
&\le \int_0^1\int_0^1 \frac{2\gamma\|x_0-tx-(1-t)y\|dsdt}{[1-s\gamma\|x_0-tx-(1-t)y\|]^3}\,, \qquad (2.\ 7)
\end{aligned}$$

which justifies condition (2.3).
It is convenient for us to define scalar function f, and scalar sequences $\{r_n\}$, $\{s_n\}$,

$\{t_n\}$, for some $\alpha \geq 0$, $a \geq 0$, $b \geq 0$ by

$$\begin{aligned} f(t) &= \frac{\alpha}{\gamma} - t + \frac{\gamma t^2}{1-\gamma t}, \quad t \neq \frac{1}{\gamma}, && (2.\ 8) \\ r_{-1} &= -a, \quad r_0 = 0, \quad r_1 = b, \\ r_{n+1} &= r_n - \\ & \int_0^1 \int_0^1 \frac{2\gamma[t(r_{n-1}-r_{n-2}) + (1-t)(r_n - r_{n-1})]dsdt}{[1-\gamma s(tr_{n-2} + (1-t)r_{n-1}) - \gamma(1-s)(tr_{n-1} + (1-t))r_n]^3} \\ & \quad \times (1-t)r_n]^{-2}\}g(r_{n-1}, r_n)(r_n - r_{n-1})dt \ (n \geq 1), && (2.\ 9) \\ s_{-1} &= -a, \quad s_0 = 0, \quad s_1 = b, \\ s_{n+1} &= s_n - f(s_n)g(s_{n-1}, s_n) \ \ (n \geq 0), && (2.\ 10) \end{aligned}$$

and

$$\begin{aligned} t_{-1} &= -a, \quad t_0 = 0, \\ t_{n+1} &= t_n - \frac{t_n - t_{n-1}}{f(t_n) - f(t_{n-1})} f(t_n), && (2.\ 11) \end{aligned}$$

where, function g is given by:

$$g(r,s) = \frac{(1-\gamma_0 r)(1-\gamma_0 s)}{2(1-\gamma_0 r)(1-\gamma_0 s) - 1} \quad \text{for all } r, s \in \left[0, \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma_0}\right). \qquad (2.\ 12)$$

We need the following lemma on majorizing sequence $\{t_n\}$.

Lemma 2.3. *Assume*:

$$\alpha = b\gamma \frac{1+2a\gamma}{1+a\gamma} \leq 3 - 2\sqrt{2}. \qquad (2.\ 13)$$

Then sequence $\{t_n\}$ generated by (2.11) is monotonically increasing and converges to the smallest root

$$t^* = \frac{1+\alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma} \qquad (2.\ 14)$$

of equation $f(t) = 0$, with the largest root being

$$t^{**} = \frac{1+\alpha + \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma}. \qquad (2.\ 15)$$

Moreover, the following estimate holds for

$$q = \frac{1-\gamma t^{**}}{1-\gamma t^*}, \quad q_0 = \lambda \frac{t^*}{t^{**}}, \quad q_1 = q\frac{t^*-b}{t^{**}-b},$$

and p_n be the Fibonacci sequence:

$$t^* - t_n = \begin{cases} e_n(t^{**} - t^*), & \alpha < 3 - 2\sqrt{2} \\ h_n^{-1}, & \alpha = 3 - 2\sqrt{2}, \end{cases} \quad (n \geq 0), \qquad (2.\ 16)$$

where,

$$e_n = \frac{q_0^{p_{n-2}} q_1^{p_{n-1}}}{q - q_0^{p_{n-2}} q_1^{p_{n-1}}}, \qquad (2.\ 17)$$

and

$$h_n = \frac{\gamma p_{n-1}}{1-\gamma t^*} + \frac{p_{n-1}}{t^* - b} + \frac{p_{n-2}}{t^*}. \qquad (2.\ 18)$$

Proof. We shall show estimates:

$$t_k < t_{k+1}, \tag{2. 19}$$

and

$$t_k < t^* \tag{2. 20}$$

hold true for all $k \geq 0$. Estimates (2.19) and (2.20) hold true by the initial conditions for $k = 0$. Let us assume that they hold true for $k = 0, 1, \ldots, n-1$ for $n \geq 1$ a fixed natural number.

In view of the induction hypotheses and (2.11), we can obtain in turn for $[s, t] = [s, t; f]$:

$$\begin{aligned} t^* - t_n &= t^* - t_{n-1} + [t_{n-2}, t_{n-1}, f]^{-1}(f(t_{n-1}) - f(t^*)) \\ &= [t_{n-2}, t_{n-1}]^{-1}([t_{n-2}, t_{n-1}] - [t_{n-1}, t^*])(t^* - t_{n-1}) \\ &= -[t_{n-2}, t_{n-1}]^{-1}(t^* - t_{n-1})(t^* - t_{n-2})[t_{n-1}, t_{n-2}, t^*], \end{aligned} \tag{2. 21}$$

where by $[s, t, u]$ we mean $[s, t, u; f]$ the divided difference of order two of scalar function f at the points s, t and u.

It follows that there exist $\beta_0 \in (t_{n-2}, t_{n-1})$, and $\beta \in (t_{n-2}, t^*)$

$$[t_{n-2}, t_{n-1}] = f'(\beta_0) < 0 \tag{2. 22}$$

and

$$[t_{n-1}, t_{n-2}, t^*] = \frac{f''(\beta)}{2} > 0, \tag{2. 23}$$

since

$$-1 < f'(t) < 0, \tag{2. 24}$$

and

$$f''(t) = \frac{2\gamma}{(1 - \gamma t)^3} > 0, \tag{2. 25}$$

for $t \in \left[0, \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma}\right)$, which together with (2.21) imply (2.20) for $n = k$.

Using (2.11) we can write

$$t_{n+1} - t_n = (t^* - t_n)[t_{n-1}, t_n]^{-1}[t^*, t_n] > 0, \tag{2. 26}$$

which implies (2.19) for $n = k$. That completes the induction for estimates (2.19) and (2.20). It follows that sequence $\{t_n\}$ converges to t^*.

In view of (2.14), (2.15) and (2.21), we can easily see that

$$\begin{aligned} q\frac{t^* - t_{n+1}}{t^{**} - t_{n+1}} &= q\frac{t^* - t_n}{t^{**} - t_n} q\frac{t^* - t_{n-1}}{t^{**} - t_{n-1}} \\ &= q_0^{p_{n-1}} q_1^{p_n} \quad (n \geq 0). \end{aligned} \tag{2. 27}$$

Clearly, if $\alpha < 3 - 2\sqrt{2}$, then $t^* \neq t^{**}$. It then follows from (2.28) that the first part of estimate (2.16) holds true. Otherwise, set $\lambda_n = \gamma(t^* - t_n)$ and $\mu_n = \sqrt{2}\lambda_n$. It then follows from (2.21) that

$$t^* - t_{n+1} = \frac{\gamma(t^* - t_n)(t^* - t_{n-1})}{[1 - 2(1 - \gamma t_{n-1})(1 - \gamma t_n)](1 - \gamma t^*)} \quad (n \geq 0), \tag{2. 28}$$

from which it follows that

$$\lambda_{n+1} = \frac{\lambda_n \lambda_{n-1}}{\lambda_{n-1} + \lambda_n + \sqrt{2}\lambda_{n-1}\lambda_n} \quad (n \geq 0), \tag{2. 29}$$

and

$$\mu_{n+1} = \frac{\mu_n \mu_{n-1}}{\mu_{n-1} + \mu_n + \mu_{n-1}\mu_n} \quad (n \geq 0), \tag{2. 30}$$

or

$$\frac{1}{\mu_n} = \frac{p_{n-2}}{\mu_0} + \frac{p_{n-1}}{\mu_1} + p_n - 1 \quad (n \geq 0), \tag{2. 31}$$

by the definition of the Fibonacci sequence ($p_{-2} = 1$, $p_{-1} = 0$, $p_{n+1} = p_n + p_{n-1}$ $(n \geq -1)$). It then follows by the definition of λ_n that the second part of estimate (2.16) also holds true.

That completes the proof of Lemma 2.3.

Corollary 2.4. *If:*

(a) $a < 3 - 2\sqrt{2}$, *then for all* $n \geq 0$

$$0 \leq t^* - t_n \leq \frac{q_0^{p_n}}{q - q_0^{p_n}}(t^{**} - t^*) \leq \frac{t^{**} - t^*}{q - q_0}\left(q_0^{\frac{1}{\sqrt{5}}}\right)^{\left(\frac{1+\sqrt{5}}{2}\right)^n}. \tag{2. 32}$$

(b) $a = 3 - 2\sqrt{2}$, *then for all* $n \geq 1$

$$0 \leq t^* - t_n \leq \frac{t^* - b}{p_{n-1}} \leq \sqrt{5}(t^* - b)\left(\frac{2}{1+\sqrt{5}}\right)^{n-1}. \tag{2. 33}$$

Proof. The result follows immediately from estimate (2.16) and the fact that

$$p_n \geq \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n \quad (n \geq 0). \tag{2. 34}$$

Remark 2.5. (a) For $F = f$, $D = \left(-\infty, \frac{1}{\gamma}\right)$, $\gamma = \gamma_0$, and $X = Y = \mathbf{R}$, x_n becomes t_n and x^* is t^*. That is estimate (2.16) is sharp. Note also that f satisfies (2.5).

(b) In the special case when $x_{-1} = x_0$ condition (2.13) reduces to Wang's [12] sufficient convergence condition for Newton's method

$$\alpha = b\gamma \leq 3 - 2\sqrt{2}. \tag{2. 35}$$

(c) If we set $X = Y = \mathbf{R}$, then it can easily be seen that condition (2.5) is satisfied. Other examples which satisfy (2.5) can be found in [7], [8].

Using induction on n it follows immediately from the definitions of sequences $\{r_n\}$, $\{s_n\}$, $\{t_n\}$ that the following relationship holds between them:

Lemma 2.6. *If* $\gamma_0 < \gamma$, *and (2.13) holds true, then*

$$r_n < s_n < t_n \quad (n > 1), \tag{2. 36}$$

$$0 < r_{n+1} - r_n < s_{n+1} - s_n < t_{n+1} - t_n \quad (n > 1), \tag{2. 37}$$

$$0 \leq r^* - r_n \leq s^* - s_n \leq t^* - t_n \quad (n \geq -1), \tag{2. 38}$$

and

$$r^* \leq s^* \leq t^*, \tag{2. 39}$$

where, $r^* = \lim_{n\to\infty} r_n$, *and* $s^* = \lim_{n\to\infty} s_n$.

Note that if $\gamma_0 = \gamma$ (2.37)–(2.40) hold true as equalities.

Remark 2.6. In view of (2.37)–(2.40), one hopes that sequences $\{r_n\}$ and $\{s_n\}$ may converge under conditions weaker than (2.14). Such conditions already exist in the literature. We refer the reader to [5, 4, 6] where we provided sufficient convergence conditions for sequences more general than $\{r_n\}$ and $\{s_n\}$.

However, we do not pursue this here. Instead we provide the main semilocal convergence theorem for the Secant method (1.3), under the (γ_0, γ) condition:

Theorem 2.7. *Let operator F satisfy the (γ_0, γ) condition at $x_0 \in D$ in*

$$\overline{U\left(x_0, \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma_0}\right)} \subseteq D,$$

let $x_{-1}, x_0 \in D$ with $\|x_0 - x_{-1}\| \leq a$, and

$$\|[x_{-1}, x_0]^{-1} F(x_0)\| \leq b. \tag{2. 40}$$

Furthere, assume condition (2.13) holds true.

Then, sequence $\{x_n\}$ generated by Secant method (1.3) is well defined, remains in $\overline{U}(x_0, r^*)$ for all $n \geq 0$, and converges to a unique solution x^* of equation $F(x) = 0$ in $\overline{U}(x_0, r^*)$.

Moreover, the following estimates hold for all $n \geq -1$

$$\|x_{n+1} - x_n\| \leq r_{n+1} - r_n, \tag{2. 41}$$

and

$$\|x_n - x^*\| \leq r^* - r_n. \tag{2. 42}$$

Furthermore, if there exists $R \in \left(r^*, \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma_0}\right]$ satisfying

$$\int_0^1 [1 - \gamma(tR + (1-t)r^*)]^{-2} dt = 2, \tag{2. 43}$$

then the solution x^* is unique in $U(x_0, R)$.

Proof. We shall show:

$$\|x_{k+1} - x_k\| \leq r_{k+1} - r_k, \tag{2. 44}$$

and

$$\overline{U}(x_{k+1}, r^* - r_{k+1}) \subseteq \overline{U}(x_k, r^* - r_k) \tag{2. 45}$$

hold for all $k \geq -1$.

For every $z \in \overline{U}(x_1, r^* - r_1)$

$$\|z - x_0\| \leq \|x - x_1\| + \|x_1 - x_0\| \leq r^* - r_1 + r_1 = r^* - r_0$$

implies $z \in \overline{U}(x_0, r^* - r_0)$. We also have that (2.41) holds, and

$$\|x_1 - x_0\| = \|[x_{-1}, x_0]^{-1} F(x_0)\| = b.$$

Therefore (2.45) and (2.46) hold for $k = -1, 0$. Let us assume $x_1, x_2, \ldots, x_k$ are well defined and (2.45), (2.46) hold true for $n = 0, 1, \ldots, k-1$, where $k \geq 1$ is a fixed natural number.

We shall establish the existence of $[x_{k-1}, x_k]^{-1}$ which will also imply that x_{k+1} is well defined. Using condition (2.3) for $x = x_{k-1}$ and $y = x_k$, and the induction hypotheses we obtain

$$\begin{aligned} &\|F'(x_0)^{-1}(F'(x_0)-[x_{k-1},x_k])\| \\ &\quad\le \int_0^1\int_0^1 \frac{2\gamma_0\|x_0-tx_{n-1}-(1-t)x_k\|dsdt}{[1-s\gamma_0\|x_0-tx_{k-1}-(1-t)x_k\|]^3} \\ &\quad= \int_0^1 [1-\gamma_0\|x_0-tx_{k-1}-(1-t)x_k\|]^{-2}dt-1 \\ &\quad\le \int_0^1 [1-\gamma_0(r_k+t(r_{k-1}-r_k))]^{-2}dt-1 \\ &\quad= \frac{1}{(1-\gamma_0 r_{k-1})(1-\gamma_0 r_k)}-1<\frac{1}{(1-\gamma_0 r^*)^2}-1\le 1. \end{aligned} \tag{2. 46}$$

It follows from (2.46) and the Banach Lemma on invertible operators [5], [9] that $[x_{k-1},x_k]^{-1}$ exists, and

$$\begin{aligned} \|[x_{k-1},x_k]^{-1}F'(x_0)\| &\le \left[1-\left(\frac{1}{(1-\gamma_0 r_{k-1})(1-\gamma_0 r_k)}-1\right)\right]^{-1} \\ &= \frac{(1-\gamma_0 r_{k-1})(1-\gamma_0 r_k)}{2(1-\gamma_0 r_{k-1})(1-\gamma_0 r_k)-1}=g(r_{k-1},r_k). \end{aligned} \tag{2. 47}$$

In view of (1.3), condition (2.2) for $x=x_{k-2}$, $y=x_{k-1}$, and $w=x_k$ gives:

$$\begin{aligned} &\|F'(x_0)^{-1}F(x_k)\|=\|F'(x_0)^{-1}([x_{k-2},x_{k-1}]-[x_{k-1},x_k])(x_{k-1}-x_k)\| \\ &\le \int_0^1\int_0^1 \frac{2\gamma[t\|x_{k-2}-x_{k-1}\|+(1-t)\|x_{k-1}-x_k\|]\|x_k-x_{k-1}\|dsdt}{[1-\gamma\|s(tx_{k-2}+(1-t)x_{k-1})+(1-s)(tx_{k-1}+(1-t)x_k)-x_0\|]^3} \\ &\le \int_0^1\int_0^1 \frac{2\gamma[t(r_{k-1}-t_{k-2})+(1-t)(r_k-r_{k-1})](r_k-r_{k-1})dsdt}{[1-\gamma s(tr_{k-2}+(1-t)r_{k-1})-\gamma(1-s)(tr_{k-1}+(1-t)r_k]^3} \\ &= h(r_{k-1},r_k). \end{aligned} \tag{2. 48}$$

By (1.3), (2.10), (2.47) and (2.48) we get:

$$\begin{aligned} \|x_{k+1}-x_k\| &\le \|[x_{k-1},x_k]^{-1}F'(x_0)\|\,\|F'(x_0)^{-1}F(x_k)\| \\ &\le g(r_{k-1},r_k)h(r_{k-1},r_k)=r_{k+1}-r_k, \end{aligned} \tag{2. 49}$$

which shows (2.45) for all $k\ge -1$.

We also have that for every $z\in \overline{U}(x_{k+1},x^*-r_{k+1})$ we get

$$\|z-x_k\|\le\|z-x_{k+1}\|+\|x_{k+1}-x_k\|\le r^*-r_{k+1}+r_{k+1}-r_k=r^*-r_k.$$

That is,

$$z\in\overline{U}(x_k,r^*-r_k), \tag{2. 50}$$

which implies (2.46). The induction for (2.45) and (2.46) is now complete.

Lemma 2.6 imply that sequence $\{x_n\}$ is Cauchy (since $\{r_n\}$ is Cauchy (since $\{r_n\}$ is a Cauchy sequence) in a Banach space X and as such it converges to some $x^*\in\overline{U}(x_0,r^*)$ (since $\overline{U}(x_0,r^*)$ is a closed set). By letting $n\to\infty$ in (1.3) (or $k\to\infty$ in (2.48)) we obtain $F(x^*)=0$.

We shall show uniqueness of the solution x^* first in $\overline{U}(x_0,r^*)$. Let y^* be a solution of equation $F(x)=0$ in $\overline{U}(x_0,r^*)$. Set $L=[x^*,y^*]$. In view of (2.3), we get

$$\begin{aligned}
&\|F'(x_0)^{-1}(F'(x_0) - L)\| \\
&\le \int_0^1 [1 - \gamma_0(t\|x_0 - y^*\| + (1-t)\|x^* - x_0\|)]^{-2} dt - 1 \quad (2.\ 51)\\
&\le \int_0^1 [1 - \gamma_0(tr^* + (1-t)r^*)]^{-2} dt - 1 \\
&= (1 - \gamma_0 r^*)^{-2} - 1 < 1. \quad (2.\ 52)
\end{aligned}$$

It follows from (2.52) and the Banach Lemma on invertible operators that L^{-1} exists. Thus from the identity

$$F(x^*) - F(y^*) = [x^*, y^*](x^* - y^*), \quad (2.\ 53)$$

we deduce $x^* = y^*$.

If $R \in \left(r^*, \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma_0}\right]$ satisfies (2.44) and y^* is a solution of equation $F(x) = 0$ in $U(x_0, R)$, then as in (2.51) we get

$$\|F'(x_0)^{-1}[F'(x_0) - L]\| < \int_0^1 [1 - \gamma_0(tR + (1-t)r^*]^{-2} dt - 1 = 1. \quad (2.\ 54)$$

Hence, again we deduce $x^* = y^*$.

That completes the proof of the theorem.

Remark 2.8. In view of Lemma 2.6 $\{s_n\}$, s^* or $\{t_n\}$, t^* can replace $\{r_n\}$, r^* respectively in Theorem 2.7. Note that we could have used easier $\{t_n\}$, t^* in Theorem 2.7 but we wanted to leave the results as uncluttered as possible using the finer possible majorizing sequence $\{r_n\}$.

We now complete this study with numerical examples.

Example 2.9. Let $X = Y = \mathbf{R}$, $\gamma_0 = \gamma = \alpha > 0$, $D = \left[0, \frac{1}{\gamma}\right)$, and define function f on D by

$$f(t) = 1 - t + \frac{\gamma t^2}{1 - \gamma t}. \quad (2.\ 55)$$

We shall use the Secant method (1.3) to find the smallest positive zero of equation $f(t) = 0$. Let $t_{-1} = -.000001$, and $t_0 = 0$. Using (2.17) we can have for $\alpha = \frac{1}{2}(3 - 2\sqrt{2}) = .0857864 = \alpha_0$ and $\alpha = \frac{3}{4}(3 - 2\sqrt{2}) = .1286797 = \alpha_1$ the following table:

Table 1: Numerical Values for $t^* - t_n$

n	α_0	α_1
0	1.119	1.232
1	1.188×10^{-1}	2.322×10^{-1}
2	1.522×10^{-2}	5.891×10^{-2}
3	2.618×10^{-4}	4.362×10^{-3}
4	5.937×10^{-7}	9.145×10^{-5}
5	2.324×10^{-11}	1.463×10^{-7}

Example 2.10. Let $X = C[0,1]$, the space of all functions v, continuous on the interval $[0,1]$, with norm

$$\|v\| = \max_{0 \le s \le 1} |v(s)|, \quad D = \overline{U}(0,1), \quad \lambda \in \mathbf{R}, \ K(s,t)$$

a continuous function of two variables $s, t \in [0,1]$, and $h(s)$ a continuous function on $[0,1]$. Consider nonlinear integral equation

$$v(s) = \lambda v(s) \int_0^1 K(s,t)v(t)dt + h(s). \tag{2. 56}$$

Equations like (2.56) appear in connection with radiative transfer, neutron transport, and in the kinetic theory of gasses [2], [3], [10].

In order for us to solve equation (2.56), we define operator T on D by

$$T(v(s)) = \lambda v(s) \int_0^1 K(s,t)v(t)dt + h(s) - x(s). \tag{2. 57}$$

Let us consider some special cases of interest:

Case 1 (Chandrasekhar's equation [2], [3], [10]). Set $\lambda = \frac{1}{4}$, $K(s,t) = \frac{s}{s+t}$, $s + t \neq 0$, and $h(s) = 1$. Choose $v_0(s) = 1$, and $v_{-1}(s) = 1.0000001$. Let us also denote by δ an upper bound on $\|T'(v_0(s))^{-1}\|$. That is,

$$\|T'(v_0(s))^{-1}\| \leq \delta. \tag{2. 58}$$

We can have:

$$\|T'(v_0(s))^{-1}T(v_0(s))\| \leq \delta\|T(v_0(s))\| \leq \delta|\lambda| \ln 2 = b, \tag{2. 59}$$

and

$$\|T''(v(s))\| \leq 2|\lambda| \max_{0\leq s\leq 1} \left| \int_0^1 \frac{s}{s+t} dt \right| \leq 2|\lambda| \ln 2, \tag{2. 60}$$

since,

$$\max_{0\leq s\leq 1} \left| \int_0^1 \frac{s}{s+t} dt \right| = \ln 2. \tag{2. 61}$$

Condition (2.4) certainly holds if

$$2\delta|\lambda| \ln 2 \leq 2\gamma.$$

Hence, we can set $\gamma = \delta|\lambda| \ln 2$. Using the choices above we get

$$b = \gamma = .265197108.$$

Hypothesis (2.13) is satisfied, since

$$\alpha = .070329508 < 3.2\sqrt{2} = .17157287.$$

Hence, the conclusions of Theorem 2.7 can apply, since any solution $v^*(s)$ of equation $F(v(s)) = 0$, satisfies (2.56).

Case 2. Let $D = U(0, 1-c)$ for some $c \in [0,1)$, set $h(s) = v^3(s) - c + 1$, and $v_0(s) = 1$. As above it can easily be seen that we can set for

$$d = \max_{0\leq s\leq 1} \left| \int_0^1 K(s,t)dt \right| < \infty :$$

$$b = [1 - c + d|\lambda|]\delta, \quad \gamma = [2 - c + d|\lambda|]\delta, \tag{2. 62}$$

and

$$\gamma_0 = \frac{1}{2}[3 - c + 2d|\lambda|]\delta. \tag{2. 63}$$

In view of (2.62), and (2.63) we have:

$$\gamma_0 < \gamma \quad \text{for all } d, \delta, \lambda \in R \text{ and } c \in [0,1). \tag{2. 64}$$

It also follows from the above choices of b and γ that for v_{-1}, c close enough to v_0, 1 respectively, and λ sufficiently small condition (2.13) holds true. That is as in Case 1, the conclusions of Theorem 2.7 apply. Note however that in this case finer sequence $\{s_n\}$ than $\{t_n\}$ can be used as a majorizing sequence for Secant method (1.3) (see also Lemma 2.6).

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