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On the Midpoint Method for Solving Generalized Equations

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Abstract. We approximate a locally unique solution of a generalized equations in a Banach space setting using a new midpoint methods (see (1.2) and (3.10)). An existence–convergence theorem and a radius of convergence are given under Lipschitz and center–Lipschitz conditions on the first order Fréchet derivative and Lipschitz–like continuity property of set–valued mappings. We show that our method (1.2) is locally quadratically convergent using a fixed points theorem [10]. Motivated by optimization considerations [3], [4] related to the resolution on nonlinear equations, a smaller ratio and a larger radius of convergence are also provided. Our methods extend the midpoint method related to the resolution of nonlinear equations [7].

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the generalized equation

$$0 \in F(x) + G(x), \tag{1.1}$$

where F is a continuous function defined in a neighborhood V of the solution x^* included in a Banach space X with values in itself, and G is a set–valued map from X to its subsets with closed graph.

Many problems in mathematical economics, variational inequalities and other fields can be formulated as in equation (1.1) [14]–[17].
We consider a new midpoint method for $x_0 \in V$ being the initial guess and all $k \geq 0$

$$\left\{ \begin{array}{l} 0 \in F(x_k) + \nabla F(x_k)\ (y_k - x_k) + G(y_k) \\ 0 \in F(y_k) + \nabla F(\dfrac{x_k + y_k}{2})\ (x_{k+1} - y_k) + G(x_{k+1}), \end{array} \right. \tag{1.2}$$

where $\nabla F(x)$ is the first order Fréchet derivative of F at x.

For $G = \{0\}$, the mildpoint method was introduced in [1]–[5], [7] to solve nonlinear equations:

$$\left\{ \begin{array}{l} y_k = x_k - (\nabla F(x_k))^{-1}\ F(x_k) \\ x_{k+1} = x_k - \left(\nabla F(\dfrac{x_k + y_k}{2})\right)^{-1}\ F(x_k). \end{array} \right. \tag{1.3}$$

In [7] the convergence of order three of iterative method (1.3) is studied under Kantorovich–type assumptions. A special Lipschitz–type condition on ∇F is used in [1] to obtain a Kantorovich–type convergence theorem. In [5] a midpoint two–step method is introduced to solve nonlinear equations under mild Newton–Kantorovich–type assumptions; the obtained results are extended the case in wich the underlying operator may be differentiable. Ezquerro et al. [11] presented a convergence result of method (1.3) using a new type of recurrence relations for this method. Hernández and Salanova [12] investigated a modified midpoint method by changing an evaluation of ∇F at $z_k = \dfrac{x_k + y_k}{2}$ in method (1.3) by an evaluation of operator F at the same point.

The purpose of this paper is to study the convergence analysis of method (1.2) under Lipschitz–type conditions on the first order Fréchet derivative and Lipschitz–like continuity of set–valued mappings.

The structure of this paper is the following. In section 2, we collect a number of basic definitions and recall a fixed points theorem for set–valued maps. In section 3, we show the existence and the quadratically convergence of the sequence defined by (1.2). Finally, we give some remarks on our method using some ideas related to nonlinear equations [3], [4].

2. Preliminaries and assumptions

In order to make the paper as self–contained as possible we reintroduce some results on fixed point theorem [3]–[10]. We let $\mathcal{Z}$ be a metric space equipped with the metric ρ. For $A \subset \mathcal{Z}$, we denote by dist $(x, A) = \inf\,\{\rho(x, y),\ y \in A\}$ the distance from a point x to A. The excess e from A to the set $C \subset \mathcal{Z}$ is given by $e(C, A) = \sup\,\{\text{dist}\,(x, A),\ x \in C\}$. Let $\Lambda : X \rightrightarrows Y$ be a set–valued map, we denote by $gph\,\Lambda = \{(x, y) \in X \times Y,\ y \in \Lambda(x)\}$ and $\Lambda^{-1}(y) = \{x \in X,\ y \in \Lambda(x)\}$ is the inverse of Λ. We call $B_r(x)$ the closed ball centered at x with radius r.

Definition 1. (see [6], [13], [16]) A set–valued Λ is said to be pseudo–Lipschitz around $(x_0, y_0) \in gph\,\Lambda$ with modulus M if there exist constants a and b such that

$$e(\Lambda(y') \cap B_a(y_0), \Lambda(y'')) \leq M \parallel y' - y'' \parallel, \quad \text{for all } y' \text{ and } y'' \text{ in } B_b(x_0). \tag{2.4}$$

We need the following fixed point theorems.

Lemma 2. (see [10]) *Let $(Z, \| \,.\, \|)$ be a Banach space, let ϕ a set–valued map from Z into the closed subsets of Z, let $\eta_0 \in Z$ and let r and λ be such that $0 \leq \lambda < 1$ and*
(a) $\text{dist }(\eta_0, \phi(\eta_0)) \leq r(1-\lambda)$,
(b) $e(\phi(x_1) \cap B_r(\eta_0), \phi(x_2)) \leq \lambda \ \| x_1 - x_2 \|, \ \ \forall x_1, x_2 \in B_r(\eta_0)$,
then ϕ has a fixed–point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single–valued, then x is the unique fixed point of ϕ in $B_r(\eta_0)$.

We suppose that, for every point x in a open convex neighborhood V of x^*, $\nabla F(x)$ exist. We will make the following assumptions:

($\mathcal{H}0$) The first order Fréchet derivative ∇F is L–Lipschitz on V. That is

$$\| \nabla F(x) - \nabla F(y) \| \leq L \ \| x - y \| \ \text{ for all } x, y \in V. \tag{2.5}$$

It follows from (2.5) that there exists $L_0 \in [0, L]$ such that

$$\| \nabla F(x) - \nabla F(x^*) \| \leq L_0 \ \| x - x^* \| \ \text{ for all } x \in V. \tag{2.6}$$

($\mathcal{H}1$) $[F(x^*) + \nabla F(x^*)(.-x^*) + G(.)]^{-1}$ is M–pseudo–Lipschitz around $(0, x^*)$.

Before stating the main result on this study, we need to introduce some notations. First, for $k \in I\!N$ and (y_k), (x_k) defined in (1.2), let us define the set–valued mappings $Q, \psi_k, \phi_k : X \rightrightarrows X$ by the following

$$Q(.) := F(x^*) + \nabla F(x^*)(.-x^*) + G(.); \ \psi_k(.) := Q^{-1}(Z_k(.)); \ \phi_k(.) := Q^{-1}(W_k(.)) \tag{2.7}$$

where Z_k and W_k are defined from X to X by

$$\begin{aligned} Z_k(x) &:= F(x^*) + \nabla F(x^*)(x - x^*) - F(y_k) - \nabla F(\frac{x_k + y_k}{2}) \ (x - y_k) \\ W_k(x) &:= F(x^*) + \nabla F(x^*)(x - x^*) - F(x_k) - \nabla F(x_k) \ (x - x_k) \end{aligned} \tag{2.8}$$

3. Local convergence analysis for method (1.2)

We show the main local convergence result for method (1.2):

Theorem 3. *We suppose that assumptions ($\mathcal{H}0$) and ($\mathcal{H}1$) are satisfied. For every constant $C > C_0 = \dfrac{3\,M\,L}{2}$, there exist $\delta > 0$ such that for every starting point x_0 in $B_\delta(x^*)$ (x_0 and x^* distinct), and a sequence (x_k) defined by (1.2) which satisfies*

$$\| x_{k+1} - x^* \| \leq C \, \| x_k - x^* \|^2 \,. \tag{3.9}$$

Remark 4. (a) Theorem 3 remains valid if one replaces the algorithm (1.2) by the following method

$$\left\{ \begin{array}{l} 0 \in F(x_k) + \nabla F(x_k) \ (y_k - x_k) + G(y_k) \\ 0 \in F(x_k) + \nabla F(\dfrac{x_k + y_k}{2}) \ (x_{k+1} - x_k) + G(x_{k+1}). \end{array} \right. \tag{3.10}$$

(b) The results of this paper seem also true for a general assumption: F is defined in a neghborhood V of the solution x^* included in a Banach space X with values in another Banach space Y, and G is a set–valued map from X to its subsets of Y with closed graph.

The proof of Theorem 3 is by induction on k. We need to give two results. In the first, we prove the existence of starting point y_0 for x_0 in V. In the second, we state a result which the starting point (x_0, y_0). Let us mention that y_0 and x_1 are a fixed points of ϕ_0 and ψ_0 respectively if and only if $0 \in F(x_0)+\nabla F(x_0)\ (y_0-x_0)+G(y_0)$ and $0 \in F(y_0) + \nabla F(\frac{x_0+y_0}{2})\ (x_1 - y_0) + G(x_1)$ respectively.

Proposition 5. *Under the assumptions of Theorem 3, there exists $\delta > 0$ such that for every starting point x_0 in $B_\delta(x^*)$ (x_0 and x^* distinct), the set–valued map ϕ_0 has a fixed point y_0 in $B_\delta(x^*)$, and satisfying*

$$\| y_0 - x^* \| \leq C \| x_0 - x^* \|^2 . \tag{3.11}$$

Proof. By hypothesis $(\mathcal{H}1)$ there exist positive numbers M, a and b such that

$$e(Q^{-1}(y') \cap B_a(x^*), Q^{-1}(y'')) \leq M \| y' - y'' \|,\ \forall y', y'' \in B_b(0). \tag{3.12}$$

Fix $\delta > 0$ such that

$$\delta < \delta_0 = \min\left\{a\ ,\ \sqrt{\frac{2\ b}{3\ L}}\ ,\ \frac{1}{C}\right\}. \tag{3.13}$$

The main idea of the proof of Proposition 5 is to show that both assertions (a) and (b) of Lemma 2 hold; where $\eta_0 := x^*$, ϕ is the function ϕ_0 defined in (2.7) and where r and λ are numbers to be set. According to the definition of the excess e, we have

$$\operatorname{dist}(x^*, \phi_0(x^*)) \leq e\Big(Q^{-1}(0) \cap B_\delta(x^*), \phi_0(x^*)\Big). \tag{3.14}$$

Moreover, for all point x_0 in $B_\delta(x^*)$ (x_0 and x^* distinct) we have

$$\begin{aligned} \| W_0(x^*) \| &= \| F(x^*) - F(x_0) - \nabla F(x_0)\ (x^* - x_0) \| \\ &= \| \int_0^1 (\nabla F(x_0 + t(x^* - x_0)) - \nabla F(x_0))\ (x^* - x_0)\ dt \| . \end{aligned}$$

In view of assumption $(\mathcal{H}0)$ we obtain

$$\| W_0(x^*) \| \leq \frac{L}{2} \| x^* - x_0 \|^2 . \tag{3.15}$$

Then (3.13) yields, $W_0(x^*) \in B_b(0)$.
Using (3.12) we have

$$\begin{aligned} e\Big(Q^{-1}(0) \cap B_\delta(x^*), \phi_0(x^*)\Big) &= e\Big(Q^{-1}(0) \cap B_\delta(x^*), Q^{-1}[W_0(x^*)]\Big) \\ &\leq \frac{M\ L}{2} \| x^* - x_0 \|^2 . \end{aligned} \tag{3.16}$$

By inequality (3.14), we get

$$\operatorname{dist}(x^*, \phi_0(x^*)) \leq \frac{M\ L}{2} \| x^* - x_0 \|^2 . \tag{3.17}$$

Since $C > C_0$, there exists $\lambda \in [0, 1[$ such that $C(1-\lambda) \geq C_0$ and

$$\operatorname{dist}(x^*, \phi_0(x^*)) \leq C\ (1-\lambda)\ \| x_0 - x^* \|^2 . \tag{3.18}$$

By setting $r := r_0 = C\ \| x_0 - x^* \|^2$ we can deduce from the inequality (3.18) that the assertion (a) in Lemma 2 is satisfied.
Now, we show that condition (b) of Lemma 2 is satisfied.

By (3.13) we have $r_0 \le \delta \le a$. Using ($\mathcal{H}0$) we have for $x \in B_\delta(x^*)$ the following estimates

$$
\begin{array}{rcl}
\| W_0(x) \| & = & \| F(x^*) + \nabla F(x^*)\ (x - x^*) - F(x_0) - \nabla F(x_0)\ (x - x_0) \| \\
& \le & \| F(x^*) - F(x_0) - \nabla F(x_0)\ (x^* - x_0) \| + \\
& & \| (\nabla F(x^*) - \nabla F(x_0))\ (x - x^*) \| \\
& \le & \dfrac{L}{2} \| x^* - x_0 \|^2 + L_0\ \| x^* - x_0 \|\ \| x - x^* \| \\
& \le & \dfrac{3\,L}{2}\ \delta^2.
\end{array}
\tag{3.19}
$$

Then by (3.13) we deduce that for all $x \in B_\delta(x^*)$ we have $W_0(x) \in B_b(0)$. Then it follows that for all $x', x'' \in B_{r_0}(x^*)$, we have

$$
e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) \le e(\phi_0(x') \cap B_\delta(x^*), \phi_0(x'')),
$$

which yields by (3.12) and ($\mathcal{H}0$):

$$
\begin{array}{rcl}
e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) & \le & M\ \| W_0(x') - W_0(x'') \| \\
& \le & M\ \| (\nabla F(x_0) - \nabla F(x^*))\ (x'' - x') \| \\
& \le & M\ L_0\ \delta\ \| x'' - x' \|\ .
\end{array}
\tag{3.20}
$$

Without loss generality we may assume that $\delta < \dfrac{\lambda}{M\,L_0}$ and thus condition (b) of Lemma 2 is satisfied. Since both conditions of Lemma 2 are fulfilled, we can deduce the existence of a fixed point $y_0 \in B_{r_0}(x^*)$ for the map ϕ_0. This finishes the proof of Proposition 5. □

Proposition 6. *Under the assumptions of Theorem 3, there exist $\delta > 0$ such that for every starting point x_0 in $B_\delta(x^*)$ and y_0 given by Proposition 5 (x_0 and x^* distinct), and the set–valued map ψ_0 has a fixed point x_1 in $B_\delta(x^*)$ satisfying*

$$
\| x_1 - x^* \| \le C \| x_0 - x^* \|^2,
\tag{3.21}
$$

where the constant C is given by Theorem 3.

Proof. The proof of Proposition 6 is the same one as that of Proposition 5. The choice of δ is the same one given by (3.13). The inequality (3.14) is valid if we replace ϕ_0 by ψ_0. Moreover, for all point x_0 in $B_\delta(x^*)$ (x_0 and x^* distinct), we have

$$
\begin{array}{rcl}
\| Z_0(x^*) \| & = & \| F(x^*) - F(y_0) - \nabla F(\frac{x_0 + y_0}{2})\ (x^* - y_0) \| \\
& = & \| \displaystyle\int_0^1 (\nabla F(y_0 + t(x^* - y_0)) - \nabla F(\frac{x_0 + y_0}{2}))\ (x^* - y_0)\ dt \|
\end{array}
$$

In view of assumption ($\mathcal{H}0$) and Proposition 5 we get

$$
\begin{array}{rcl}
\| Z_0(x^*) \| & \le & L\ (\frac{1}{2} \| y_0 - x_0 \| + \frac{1}{2} \| y_0 - x^* \|) \| y_0 - x^* \| \\
& \le & \dfrac{L}{2}\ (2 \| y_0 - x^* \| + \| x_0 - x^* \|)\ \| y_0 - x^* \| \\
& \le & \dfrac{L\,C}{2}\ (2\,C\ \| x_0 - x^* \|^2 + \| x_0 - x^* \|)\ \| x^* - x_0 \|^2 \\
& \le & \dfrac{L}{2}\ (2\,C^2\,\delta^2 + C\,\delta)\ \| x^* - x_0 \|^2\ .
\end{array}
\tag{3.22}
$$

By (3.13) and (3.22) we have

$$
\| Z_0(x^*) \| \quad \le \quad \frac{3\,L}{2}\ \| x_0 - x^* \|^2\ .
\tag{3.23}
$$

Then (3.13) yields, $Z_0(x^*) \in B_b(0)$. Setting $r := r_0 = C \ \parallel x_0 - x^* \parallel^2$, we can deduce from the assertion (a) in Lemma 2 is satisfied.
By (3.13) we have $r_0 \leq \delta \leq a$, and moreover for $x \in B_\delta(x^*)$ we have

$$\begin{array}{rcl} \parallel Z_0(x) \parallel & = & \parallel F(x^*) + \nabla F(x^*)\ (x - x^*) - F(y_0) - \nabla F(\dfrac{x_0 + y_0}{2})\ (x - y_0) \parallel \\ & \leq & \parallel F(x^*) - F(y_0) - \nabla F(x^*)\ (x^* - y_0) \parallel + \\ & & \parallel (\nabla F(x^*) - \nabla F(\dfrac{x_0 + y_0}{2}))\ (x - y_0) \parallel . \end{array} \tag{3.24}$$

Using assumption $(\mathcal{H}0)$ we obtain

$$\parallel Z_0(x) \parallel \quad \leq \quad \frac{5\,L_0}{2}\ \ \delta^2 \tag{3.25}$$

A slight change in the end of proof of Proposition 5 shows that the condition (b) of Lemma 2 is satisfied. The existence of a fixed point $x_1 \in B_{r_0}(x^*)$ for the map ψ_0 is ensured. This finishes the proof of Proposition 6. □

Proof of Theorem 3. Keeping $\eta_0 = x^*$ and setting $r := r_k = C \parallel x^* - x_k \parallel^2$, the application of Proposition 5 and Proposition 6 to the map ϕ_k and ψ_k respectively gives the existence of a fixed points y_k and x_{k+1} for ϕ_k and ψ_k respectively which is an elements of $B_{r_k}(x^*)$. This last fact implies the inequality (3.9), which is the desired conclusion. □

Remark 7. (a) It follows from the proof of Proposition 5 that constants C_0 and δ_0 can be replaced by the more precise

$$\overline{C_0} = \frac{M\,L}{2}, \tag{3.26}$$

and

$$\overline{\delta_0} = \min\left\{a\ ,\ \sqrt{\frac{2\ b}{L + 2\ L_0}}\ ,\ \frac{1}{C}\right\}, \tag{3.27}$$

respectively. Note that

$$\overline{C_0} \leq C_0, \tag{3.28}$$

and

$$\delta_0 \leq \overline{\delta_0}. \tag{3.29}$$

(b) The constant δ_0 in the proof of Proposition (6) can be given by :

$$\overline{\overline{\delta_0}} = \min\left\{a\ ,\ \sqrt{\frac{2\ b}{3\ L_0}}\ ,\ \frac{1}{C}\ ,\ 1\right\}. \tag{3.30}$$

Indeed by adding and substracting $\nabla F(x^*)\ (x^* - y_0)$ inside the first norm in the computation of $\parallel Z_0(x^*) \parallel$ we arrive at an estimate corresponding (3.23) :

$$\parallel Z_0(x^*) \parallel \quad \leq \quad \frac{3\,L_0}{2}\ \parallel x_0 - x^* \parallel^2 . \tag{3.31}$$

Hence $\overline{\overline{\delta_0}}$ can be replace δ_0 in the proof of Proposition 6. This modification is usefull when $\overline{\overline{\delta_0}} > \delta_0$. These observations are important in computational mathematics since the allow a smaller ratio C and a larger radius of convergence [3], [4].

Remark 8. The sequence (y_n) given by algorithm (1.2) is also quadratically convergent to a solution x^* of (1.1) (see [9]). Note that the midpoint method for nonlinear equations was shown by us to be of order three (see [1]–[5], [7], [8]). However we had to introduce Lipschitz conditions on the second Fréchet derivative $\nabla^2 F$. Here we simply used hypotheses on ∇F only. In a future paper using the Ostrowski representation for F given in [8] we will recover the third order of convergence of method (1.2).

Application 9. (see [14])
Let K be a convex set in $I\!R^n$, P is a topological space and φ is a function from $P \times K$ to $I\!R^n$, the "perturbed" variational inequality problem consists of seeking k_0 in K such that

$$\text{For each } k \in K, \quad (\varphi(p,k_0); k-k_0) \ \geq \ 0 \tag{3.32}$$

where $(.;.)$ is the usual scalar product on $I\!R^n$ and p is fixed parameter in P. Let $\mathcal{I}_K$ be a convex indicator function of K and ∂ denotes the subdifferential operator. Then the problem (3.32) is equivalent to problem

$$0 \in \varphi(p,k_0) + \partial \mathcal{I}_K(k_0). \tag{3.33}$$

The problem (3.32) is equivalent to (3.33) which is a generalized equation in the form (1.1). Consequently, we can approximate the solution k_0 of (3.32) using our methods (1.2) and (3.10).

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