Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 40 (2008) pp. 71-81

Sum Intuitionistic Fuzzy Closure Spaces

A. M. Zahran
Department of Mathematics
Faculty of Science(Assuit)
Al-Azhar University
Assiut, Egypt
Email: zahran15@hotmail.com

M. Azab Abd-Allah
Department of Mathematics
Faculty of Science
Assiut University
Assiut, Egypt
Email: mazab57@yahoo.com

Kamal El-Saady and A. Ghareeb
Department of Mathematics
Faculty of Science(Qena)
South Valley University
Qena, Egypt
Email: el-saady@lycos.com
Email: ghareeba@svu.edu.eg

Abstract. We will prove the existence of final intuitionistic fuzzy topological spaces and final intuitionistic fuzzy closure spaces. From this fact, we can define intuitionistic quotient spaces of their spaces and sum of intuitionistic fuzzy closure spaces. In this paper, additivity of two kinds of intuitionistic fuzzy closure spaces are studied.

AMS (MOS) Subject Classification Codes: 03E72, 54A40, 54B15
Key Words: Intuitionistic Fuzzy Closure Spaces, Intuitionistic Fuzzy Topological Space, Sum Intuitionistic Fuzzy Topological Spaces, Sum Intuitionistic Fuzzy Closure Spaces.

1. Introduction

Šostak [19] introduced the fundamental concept of a fuzzy topological structure as an extension of both crisp topology and Chang fuzzy topology [5]. Later on he has developed the theory of fuzzy topological spaces in [20, 21]. In [16], Ramadan gave a similar definition, namely "smooth topological space". It has been developed in many direction [6,10-14].

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [2-4]. Recently, Çoker and his colleagues [8, 9] introduced the notion of intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Samanta and Mondal [17, 18] introduced the notion of intuitionistic gradation of openness as a generalization of intuitionistic fuzzy topological spaces [9] and smooth topological spaces.

In this paper, we will prove the existence of final intuitionistic fuzzy topological spaces and final intuitionistic fuzzy closure spaces. From this fact, we will define intuitionistic quotient spaces of their spaces. Moreover, the additivity of two kinds of intuitionistic fuzzy closure spaces are studied.

Throughout this paper, let X be a nonempty set, $I = [0,1]$, $I_\circ = (0,1]$ and $I_1 = [0,1)$. For $\alpha \in I$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. The family of all fuzzy sets on X denoted by I^X. Notions and notations not described in this paper are standard and usual.

2. Intuitionistic Fuzzy Topological Spaces

Definition 1. [18] An intuitionistic gradation of openness (IGO, for short) on X is an ordered pair $(\tau, \tau*)$ of functions from I^X to I such that:

(IGO1): $\tau(\lambda) + \tau*(\lambda) \leq 1$, for each $\lambda \in I^X$,
(IGO2): $\tau(\underline{0}) = \tau(\underline{1}) = 1$, $\tau*(\underline{0}) = \tau*(\underline{1}) = 0$,
(IGO3): $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau*(\lambda_1 \wedge \lambda_2) \leq \tau*(\lambda_1) \vee \tau*(\lambda_2)$, for each $\lambda_i \in I^X$ and $i \in \{1, 2\}$,
(IGO4): $\tau(\bigvee_{i\in\Delta} \lambda_i) \geq \bigwedge_{i\in\Delta} \tau(\lambda_i)$ and $\tau*(\bigvee_{i\in\Delta} \lambda_i) \leq \bigvee_{i\in\Delta} \tau*(\lambda_i)$, for each family $\{\lambda_i \in I^X | i \in \Delta\}$.

The triplet $(X, \tau, \tau*)$ is called an intuitionistic fuzzy topological space (ifts, for short). τ and $\tau*$ may be interpreted as gradation of openness and gradation of nonopenness, respectively. Let $(\tau_1, \tau*_1)$ and $(\tau_2, \tau*_2)$ be IGO's on X. We say $(\tau_1, \tau*_1)$ is finer than $(\tau_2, \tau*_2)$ ($(\tau_2, \tau*_2)$ is coarser than $(\tau_1, \tau*_1)$) if $\tau_2(\lambda) \leq \tau_1(\lambda)$ and $\tau*_2(\lambda) \geq \tau*_1(\lambda)$ for all $\lambda \in I^X$.

Definition 2 ([18])**.** Let $(\tau, \tau*)$ be an IGO on X and the functions $\mathcal{F}, \mathcal{F}^* : I^X \to I$ defined by $\mathcal{F}(\mu) = \tau(\underline{1} - \mu)$ and $\mathcal{F}^*(\mu) = \tau * (\underline{1} - \mu)$ for all $\mu \in I^X$. Then $(\mathcal{F}, \mathcal{F}^*)$ is called an intuitionistic gradation of closedness (IGC, for short) on X.

Definition 3 ([15])**.** A function $\mathcal{C} : I^X \times I_\circ \times I_1 \to I^X$ is called an *intuitionistic fuzzy closure operator* if for each $\lambda,\ \mu \in I^X$, $r \in I_\circ$ and $s \in I_1$ with $r + s \leq 1$, the operator $\mathcal{C}$ satisfies the following conditions:

(C1) $\mathcal{C}(\underline{0}, r, s) = \underline{0}$.

(C2) $\lambda \leq \mathcal{C}(\lambda, r, s)$.

(C3) if $\lambda \leq \mu$, then $\mathcal{C}(\lambda, r, s) \leq \mathcal{C}(\mu, r, s)$.

(C4) $\mathcal{C}(\lambda, r, s) \vee \mathcal{C}(\mu, r, s) = \mathcal{C}(\lambda \vee \mu, r, s)$.

(C5) $\mathcal{C}(\lambda, r, s) \leq \mathcal{C}(\lambda, r_1, s_1)$ if $r \leq r_1$ and $s \geq s_1$ with $r_1 + s_1 \leq 1$.

The pair $(X, \mathcal{C})$ is called an intuitionistic fuzzy closure space. An intuitionistic fuzzy closure space $(X, \mathcal{C})$ is called *topological* if

(C6) $\mathcal{C}(\mathcal{C}(\lambda, r, s), r, s) = \mathcal{C}(\lambda, r, s)$, for each $\lambda \in I^X$ and $r \in I_\circ$, $s \in I_1$ with $r + s \leq 1$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be intuitionistic intuitionistic fuzzy closure operators on X. We say that $\mathcal{C}_1$ is finer than $\mathcal{C}_2$ ($\mathcal{C}_2$ is coarser than $\mathcal{C}_1$) iff $\mathcal{C}_1(\lambda, r, s) \leq \mathcal{C}_2(\lambda, r, s)$, for each $\lambda \in I^X$ and $r \in I_\circ$, $s \in I_1$ with $r + s \leq 1$.

Theorem 4 ([15]). *Let $(X, \tau, \tau*)$ be an ifts. Then for each $r \in I_\circ$, $s \in I_1$, $\lambda \in I^X$ we define an operator $\mathcal{C}_{\tau,\tau*} : I^X \times I_\circ \times I_1 \to I^X$ as follows*

$$\mathcal{C}_{\tau,\tau*}(\lambda, r, s) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu, \tau(\underline{1} - \mu) \geq r, \tau*(\underline{1} - \mu) \leq s\}.$$

Then $\mathcal{C}_{\tau,\tau*}$ is an intuitionistic fuzzy closure operator.

Theorem 5 ([15]). *Let $(X, \mathcal{C})$ be an intuitionistic fuzzy closure space. Define the functions $\tau_{\mathcal{C}}, \tau*_{\mathcal{C}} : I^X \to I$ by*

$$\tau_{\mathcal{C}}(\lambda) = \bigvee\{r \in I_\circ \mid \mathcal{C}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\},$$

$$\tau*_{\mathcal{C}}(\lambda) = \bigwedge\{s \in I_1 \mid \mathcal{C}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\}.$$

Then:
(1) $(\tau_{\mathcal{C}}, \tau*_{\mathcal{C}})$ is an IGO on X.

(2) We have $\mathcal{C} = \mathcal{C}_{\tau_{\mathcal{C}},\tau*_{\mathcal{C}}}$ iff $(X, \mathcal{C})$ satisfies the following conditions:

(a) It is a topological intuitionistic fuzzy closure space.

(b) If $r_1 = \bigvee\{r \in I_\circ \mid \mathcal{C}(\lambda, r, s) = \lambda\}$ and $s_1 = \bigwedge\{s \in I_1 \mid \mathcal{C}(\lambda, r, s) = \lambda\}$, then $\mathcal{C}(\lambda, r_1, s_1) = \lambda$.

Definition 6 ([15]). Let $(X, \tau_1, \tau*_1)$ and $(Y, \tau_2, \tau*_2)$ be ifts's. A function $f : X \to Y$ is called an intuitionistic fuzzy continuous if $\tau_1(f^{-1}(\mu)) \geq \tau_2(\mu)$ and $\tau*_1(f^{-1}(\mu)) \leq \tau*_2(\mu)$ for all $\mu \in I^Y$. Equivalently, $\mathcal{F}_1(f^{-1}(\mu)) \geq \mathcal{F}_2(\mu)$ and $\mathcal{F}_1^*(f^{-1}(\mu)) \leq \mathcal{F}_2^*(\mu)$ for all $\mu \in I^Y$.

Definition 7 ([1]). Let $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ be two intuitionistic fuzzy closure spaces. A function $f : (X, \mathcal{C}_1) \to (Y, \mathcal{C}_2)$ is said to be a $\mathcal{C}$-map if for all $\lambda \in I^X$, $r \in I_\circ$, $s \in I_1$ with $r + s \leq 1$,

$$f(\mathcal{C}_1(\lambda, r, s)) \leq \mathcal{C}_2(f(\lambda), r, s).$$

Theorem 8 ([1]). *Let $(X, \tau_1, \tau*_1)$ and $(Y, \tau_2, \tau*_2)$ be ifts's. A function $f : (X, \tau_1, \tau*_1) \to (Y, \tau_2, \tau*_2)$ is an intuitionistic fuzzy continuous iff $f : (X, \mathcal{C}_{\tau_1,\tau*_1}) \to (Y, \mathcal{C}_{\tau_2,\tau*_2})$ is a $\mathcal{C}$-map.*

Using Theorem 8, we can easily prove the following corollary:

Corollary 9 ([1]). *Let $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ be intuitionistic fuzzy closure spaces. If $f : (X, \mathcal{C}_1) \to (Y, \mathcal{C}_2)$ is $\mathcal{C}$-map, then $f : (X, \mathcal{T}_{\mathcal{C}_1}, \mathcal{T}^*_{\mathcal{C}_1}) \to (Y, \mathcal{T}_{\mathcal{C}_2}, \mathcal{T}^*_{\mathcal{C}_2})$ is an intuitionistic fuzzy continuous map.*

Definition 10. An intuitionistic fuzzy topological property ξ is called an additive, if for any family of intuitionistic fuzzy topological spaces $\{(X_i, \tau_i, \tau*_i)\}_{i\in\Gamma}$ with property ξ, then the sum $(X, \tau, \tau*)$ also has property ξ

Definition 11. An intuitionistic fuzzy closure space $(X, \mathcal{C})$ is said to be:

(r, s)-fuzzy-T_0: If for all x, $y \in X$ such that $x \neq y$, there exists $\lambda \in I^X$ such that $\mathcal{C}(\lambda, r, s) = \lambda$ and $\lambda(x) \neq \lambda(y)$,where $r \in I_\circ$, $s \in S_1$.

(r, s)-fuzzy-T_1: If $\mathcal{C}(\chi_{\{x\}}, r, s) \leq \chi_{\{x\}}$, for each $x \in X$, where $r \in I_\circ$, $s \in S_1$ and $\chi_{\{x\}}$ is the characteristic function of x.

3. Final Intuitionistic Fuzzy Topological Spaces

Theorem 12. *Let Y be a set and $\{(X_i, \tau_i, \tau*_i)\}_{i\in\Gamma}$ be a family of intuitionistic fuzzy topological spaces. Let $f_i\colon X_i \to Y$ be a function for each $i \in \Gamma$. Define the functions $\tau, \tau*\colon I^Y \to I$ by*

$$\tau(\lambda) = \bigwedge_{i\in\Gamma} \tau_i(f_i^{-1}(\lambda)),\ \ \tau*(\lambda) = \bigvee_{i\in\Gamma} \tau*_i(f_i^{-1}(\lambda)).$$

Then:

(1) $(\tau, \tau*)$ is the finest intuitionistic fuzzy topology on Y for which each f_i is intuitionistic fuzzy continuous.

(2) $f\colon (Y, \tau, \tau*) \to (Z, \tau_Z, \tau*_Z)$ is an intuitionistic fuzzy continuous map iff each $fof_i\colon (X.\tau_i, \tau*_i) \to (Z, \tau_Z, \tau*_Z)$ is intuitionistic fuzzy continuous.

Proof. (1) (IGO1) and (IGO2) are trivial from the definition of τ, $\tau*$.

(IGO3)

$$\begin{aligned} \tau(\lambda \wedge \mu) &= \tau_i(f_i^{-1}(\lambda) \wedge f_i^{-1}(\mu)) \\ &\geq \tau_i(f_i^{-1}(\lambda) \wedge \tau_i(f_i^{-1}(\mu))) \\ &\geq \tau(\lambda) \wedge \tau(\mu). \end{aligned}$$

and

$$\begin{aligned} \tau*(\lambda \wedge \mu) &= \tau*_i(f_i^{-1}(\lambda) \wedge f_i^{-1}(\mu)) \\ &\leq \tau*_i(f_i^{-1}(\lambda) \vee \tau*_i(f_i^{-1}(\mu))) \\ &\leq \tau*(\lambda) \vee \tau*(\mu), \end{aligned}$$

which is a contradiction. Hence $\tau(\lambda \wedge \mu) \geq \tau(\lambda) \wedge \tau(\mu)$, $\tau*(\lambda \wedge \mu) \leq \tau*(\lambda) \vee \tau*(\mu)$.

(IGO4) We will try to prove that $\tau(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigvee_{i\in\Gamma} \tau(\lambda_i)$, $\tau*(\bigvee_{i\in\Gamma} \lambda_i) \leq \bigvee_{i\in\Gamma} \tau*(\lambda_i)$. Suppose $\tau(\bigvee_{i\in\Gamma} \lambda_i) \ngeq \bigvee_{i\in\Gamma} \tau(\lambda_i)$ and $\tau*(\bigvee_{i\in\Gamma} \lambda_i) \nleq \bigvee_{i\in\Gamma} \tau*(\lambda_i)$. From the definition of $(\tau, \tau*)$, there exists $i \in \Gamma$ such that

$$\tau(\bigvee_{i\in\Gamma} \lambda_i) \leq \tau_i(f^{-1}(\bigvee_{i\in\Gamma} \lambda_i)) < \bigvee_{i\in\Gamma} \tau(\lambda_i)$$

and

$$\tau*(\bigvee_{i\in\Gamma} \lambda_i) \geq \tau*_i(f^{-1}(\bigvee_{i\in\Gamma} \lambda_i)) > \bigvee_{i\in\Gamma} \tau*(\lambda_i).$$

But, we have

$$\begin{aligned} \tau_i(f^{-1}(\bigvee_{i\in\Gamma} \lambda_i)) &= \tau_i(\bigvee_{i\in\Gamma} f^{-1}(\lambda_i)) \\ &\geq \bigvee_{i\in\Gamma} \tau(f^{-1}(\lambda_i)) \\ &\geq \bigvee_{i\in\Gamma} \tau(\lambda_i). \end{aligned}$$

and

$$\begin{aligned} \tau*_i(f^{-1}(\bigvee_{i\in\Gamma}\lambda_i)) &= \tau*_i(\bigvee_{i\in\Gamma} f^{-1}(\lambda_i)) \\ &\leq \bigvee_{i\in\Gamma}\tau*(f^{-1}(\lambda_i)) \\ &\leq \bigvee_{i\in\Gamma}\tau*(\lambda_i), \end{aligned}$$

which is a contradiction. Hence $\tau(\bigvee_{i\in\Gamma}\lambda_i) \geq \bigvee_{i\in\Gamma}\tau(\lambda_i)$, $\tau*(\bigvee_{i\in\Gamma}\lambda_i) \leq \bigvee_{i\in\Gamma}\tau*(\lambda_i)$ for any family $\{\lambda_i \mid i \in \Gamma\} \subseteq I^Y$.

Secondly, since $\tau(\lambda) \leq \tau_i(f_i^{-1}(\lambda))$ and $\tau*(\lambda) \geq \tau*_i(f_i^{-1}(\lambda))$ for each $i \in \Gamma$, each f_i is intuitionistic fuzzy continuous map.

Finally, we will show that $(\tau, \tau*)$ is the finest intuitionistic fuzzy topology on Y for which each f_i is intuitionistic fuzzy continuous map. If $f_i\colon (X_i, \tau_i, \tau*_i) \to (Y, \tau^\star, \tau*^\star)$ is intuitionistic fuzzy continuous, we have $\tau^\star(\nu) \leq \tau_i^\star(f_i^{-1}(\nu))$ and $\tau*^\star(\nu) \geq \tau*_i(f_i^{-1}(\nu))$, for each $i \in \Gamma$, $\nu \in I^Y$. By using the definition of τ, $\tau*$, it follows $\tau^\star(\nu) \leq \tau(\nu)$ and $\tau*^\star(\nu) \geq \tau*(\nu)$ for all $\nu \in I^Y$.

(2) ($\Rightarrow$.) Trivial.

($\Leftarrow$.) Since $fof_i\colon (X, \tau_i, \tau*_i) \to (Z, \tau_Z, \tau*_Z)$ is an intuitionistic fuzzy continuous, we have for each $\mu \in I^Z$,

$$\tau_Z(\mu) \leq \tau_i((fof_i)^{-1}(\mu)) = \tau_i(f_i^{-1}(f^{-1}(\mu))).$$

and

$$\tau*_Z(\mu) \geq \tau*_i((fof_i)^{-1}(\mu)) = \tau*_i(f_i^{-1}(f^{-1}(\mu))).$$

By using the definition of τ, $\tau*$, it follows $\tau_Z(\mu) \leq \tau(f^{-1}(\mu))$ and $\tau*_Z(\mu) \geq \tau*(f^{-1}(\mu))$ for each $\mu \in I^Z$. Hence $f\colon (Y, \tau, \tau*) \to (Z, \tau_Z, \tau*_Z)$ is intuitionistic fuzzy continuous map.

□

Definition 13. The structure $(\tau, \tau*)$ defined in Theorem 12 is called the final intuitionistic fuzzy topology on Y associated with the families $\{(X_i, \tau_i, \tau*_i)\}_{i\in\Gamma}$ and $(f_i)_{i\in\Gamma}$.

Corollary 14. (Sum Intuitionistic fuzzy topological spaces)
Let $\{(X_i, \tau_i, \tau*_i)\}_{i\in\Gamma}$ be a family of intuitionistic fuzzy topological spaces, for different $i, j \in \Gamma$. X_i and X_J be disjoint, $X = \cup_{i\in\Gamma} X_i$. Let $id_i\colon X_i \to X$ be the identity map for which $i \in \Gamma$. Define functions $\tau, \tau*\colon I^Y \to I$ by

$$\tau(\lambda) = \bigwedge_{i\in\Gamma}\tau_i(id_i^{-1}(\lambda)),\ \tau*(\lambda) = \bigvee_{i\in\Gamma}\tau*_i(id_i^{-1}(\lambda)).$$

Then:

(1) $(\tau, \tau*)$ is the finest intuitionistic fuzzy topology on X for which each id_i is intuitionistic fuzzy continuous.

(2) $f\colon (X, \tau, \tau*) \to (Z, \tau_Z, \tau*_Z)$ is an intuitionistic fuzzy continuous map iff each $foid_i\colon (X, \tau_i, \tau*_i) \to (Z, \tau_Z, \tau*_Z)$ is intuitionistic fuzzy continuous map.

Corollary 15. *Let Y be a set and $(X, \tau, \tau*)$ be an intuitionistic fuzzy topological space. Let $f\colon X \to Y$ be a surjective function. Define the functions $\tau^f, \tau*^f : I^Y \to I$ by*

$$\tau^f(\lambda) = \tau(f^{-1}(\lambda)),\ \tau*^f(\lambda) = \tau*(f^{-1}(\lambda))$$

Then:

(1) $(\tau^f, \tau*^f)$ is the finest intuitionistic fuzzy topology on X which f is intuitionistic fuzzy continuous.

(2) $g\colon (Y, \tau^f, \tau*^f) \to (Z, \tau_Z, \tau*_Z)$ is an intuitionistic fuzzy continuous iff $gof\colon (X, \tau, \tau*) \to (Z, \tau_Z, \tau*_Z)$ is intuitionistic fuzzy continuous.

Definition 16. Let $(X, \tau, \tau*)$ be an intuitionistic fuzzy topological space and Y a set. Let $f\colon X \to Y$ be a surjective function. The final intuitionistic fuzzy topological spaces τ^f on Yassociated the $(X, \tau, \tau*)$ and f is called the quotient intuitionistic fuzzy topological space and the function f is called fuzzy quotient map.

Theorem 17. *Let $(X, \tau_1, \tau*_1)$ and $(Y, \tau_2, \tau*_2)$ be intuitionistic fuzzy topological spaces. Let $f\colon (X, \tau_1, \tau*_1) \to (Y, \tau_2, \tau*_2)$ is a surjective intuitionistic fuzzy continuous function .*

(1) If f is an intuitionistic open function, then f is intuitionistic fuzzy quotient function.

(2) If f is an intuitionistic closed function, then f is intuitionistic fuzzy quotient function.

Proof. we only show that $\tau_2 = \tau^f$. By using Corollary 15 , we have $\tau(\lambda) \leq \tau^f(\lambda)$ and $\tau*_2(\lambda) \geq \tau*^f(\lambda)$ for all $\lambda \in I^Y$. Conversely, we have

$$\begin{aligned} \tau^f(\lambda) &= \tau_1(f^{-1}(\lambda)) \\ &\leq \tau_2(f(f^{-1}(\lambda))) \\ &= \tau_2(\lambda). \end{aligned}$$

and

$$\begin{aligned} \tau*^f(\lambda) &= \tau*_1(f^{-1}(\lambda)) \\ &\geq \tau*_2(f(f^{-1}(\lambda))) \\ &= \tau*_2(\lambda). \end{aligned}$$

(2) Trivial. □

4. Final Intuitionistic Fuzzy Closure Spaces

Theorem 18. *Let Y be a set and $\{(X_i, \mathcal{C}_i)\}_{i\in\Gamma}$ be a family of intuitionistic fuzzy closure spaces. Let $f_i\colon X_i \to Y$ be a surjective function for each $i \in \Gamma$. Define the function $\mathcal{C}\colon I^Y \times I_\circ \times I_1 \to I^Y$ by*

$$\mathcal{C}_i(\lambda, r, s) = \bigvee_{i\in\Gamma} f_i(\mathcal{C}_i(f_i^{-1}(\lambda), r, s)).$$

Then:

(1) $\mathcal{C}$ is the finest intuitionistic fuzzy closure operator on Y for which each f_i is $\mathcal{C}$-map.

(2) $f\colon (Y,\mathcal{C}) \to (Z,\mathcal{C}_Z)$ is a $\mathcal{C}$-map iff each $fof_i\colon (X,\mathcal{C}_i) \to (Z,\mathcal{C}_Z)$ is $\mathcal{C}$-map.

Proof. (1) Firstly, we will show that $\mathcal{C}$ is intuitionistic fuzzy closure operator on Y. (C1), (C3) and (C4) are easily proved from the definition of $\mathcal{C}$. For (C2) we have

$$\begin{aligned}\mathcal{C}(\lambda,r,s) &= \bigvee_{i\in\Gamma} f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)) \\ &\geq f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)) \\ &\geq f_i(f_i^{-1}(\lambda)) = \lambda.\end{aligned}$$

Secondly, we have

$$\begin{aligned}\mathcal{C}(f_i(\lambda),r,s) &= \bigvee_{i\in\Gamma} f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)) \\ &\geq f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)) \\ &\geq f_i(\mathcal{C}_i(\lambda,r,s)).\end{aligned}$$

Hence $f_i\colon (X_i,\mathcal{C}_i) \to (Y,\mathcal{C})$ is $\mathcal{C}$-map.

Finally, we will show that $\mathcal{C}$ is the finest intuitionistic fuzzy closure operator on Y for which each f_i is $\mathcal{C}$-map. If $f_i\colon (X_i,\mathcal{C}_i) \to (Y,\mathcal{C}^*)$ is $\mathcal{C}$-map for each $i \in \Gamma$, then we have, for each $\lambda_i \in I^{X_i}$ and $r \in I_\circ$, $s \in I_1$, $f_i(\mathcal{C}_i(\lambda_i,r,s)) \leq \mathcal{C}^*(f_i(\lambda_i),r,s)$. It follows that

$$\begin{aligned}\mathcal{C}(\lambda,r,s) &= \bigvee_{i\in\Gamma} f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)) \\ &\leq \bigvee_{i\in\Gamma} \mathcal{C}^*(f_i(f_i^{-1}(\lambda),r,s)) \\ &= \mathcal{C}^*(\lambda,r,s).\end{aligned}$$

(2) ($\Rightarrow$). Trivial.

($\Leftarrow$). Let $fof_i\colon (X_i,\mathcal{C}_i) \to (Z,\mathcal{C}_Z)$ be $\mathcal{C}$-map, then we have

$$(fof_i)(\mathcal{C}_i(\lambda_i,r,s) \leq \mathcal{C}_Z(fof_i(\lambda_i),r,s)$$

It follows that

$$\begin{aligned}f(\mathcal{C}(\lambda,r,s)) &= f(\bigvee_{i\in\Gamma} f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s)))) \\ &= \bigvee_{i\in\Gamma} f(f_i(\mathcal{C}_i(f_i^{-1}(\lambda),r,s))) \\ &\leq \bigvee_{i\in\Gamma} \mathcal{C}_Z(f(f_i(f_i^{-1}(\lambda))),r,s) \\ &= \mathcal{C}_Z(f(\lambda),r,s).\end{aligned}$$

□

From Theorem 18, we can state the following definition.

Definition 19. The structure $\mathcal{C}$ is called the final intuitionistic fuzzy closure operator on Y associated the families $\{(X_i, \mathcal{C}_i)\}$ and $(f_i)_{i\in\Gamma}$.

Corollary 20. (Sum intuitionistic fuzzy closure spaces) *Let $\{(X_i, \mathcal{C}_i)\}_{i\in\Gamma}$ be a family of intuitionistic fuzzy closure spaces, for different i, $j \in \Gamma$, X_i and X_j be disjoint, $X = \cup_{i\in\Gamma} X_i$. Let $id\colon X_i \to X$ be the identity map for each $i \in \Gamma$. Define the function $\mathcal{C}\colon I^X \times I_\circ \times I_1 \to I^X$ by*

$$\mathcal{C}(\lambda, r, s) = \bigvee_{i\in\Gamma} id_i(\mathcal{C}_i(id_i^{-1}(\lambda), r, s)).$$

Then:

(1) $\mathcal{C}$ is the finest intuitionistic fuzzy closure operator on X for which each id_i is $\mathcal{C}$-map.

(2) $f\colon (Y, \mathcal{C}) \to (Z, \mathcal{C}_Z)$ is a $\mathcal{C}$-map iff each $f o id_i\colon (X, \mathcal{C}_i) \to (Z, \mathcal{C}_Z)$ is $\mathcal{C}$-map.

Definition 21. Let $(X, \mathcal{C})$ be an intuitionistic fuzzy closure space and Y a set. Let $f\colon X \to Y$ be a surjective function. Define the function $\mathcal{C}^f\colon I^Y \times I_\circ \times I_1 \to I^Y$ by

$$\mathcal{C}^f(\lambda, r, s) = f(\mathcal{C}(f^{-1}(\lambda), r, s)).$$

The $(Y, \mathcal{C}^f)$ induced by f is called the intuitionistic fuzzy quotient space of $(X, \mathcal{C})$ and the function f is called an intuitionistic fuzzy quotient map.

Theorem 22. *Let Y be a set and $\{(X_i, \tau_i, \tau{*}_i)\}_{i\in\Gamma}$ be a family of intuitionistic fuzzy topological spaces. let $f\colon X_i \to Y$ be surjective function for each $i \in \Gamma$ and $\{(X_i, \mathcal{C}_{\tau_i, \tau*_i})\}_{i\in\Gamma}$ a family of intuitionistic fuzzy closure spaces induced by $\{(X_i, \tau_i, \tau{*}_i)\}_{i\in\Gamma}$. Define the functions τ and $\tau*$ on Y as Theorem 18 and the function $\mathcal{C}\colon I^Y \times I_\circ \times I_1 \to I^Y$ by*

$$\mathcal{C}(\lambda, r, s) = \bigvee_{i\in\Gamma} f_i(\mathcal{C}_{\tau_i, \tau*_i}(f_i^{-1}(\lambda), r, s)).$$

Then:

(1) $\mathcal{C}$ is finer than $\mathcal{C}_{\tau,\tau*}$ induced by $(\tau, \tau*)$.

(2) $(\tau_{\mathcal{C}}, \tau{*}_{\mathcal{C}}) = (\tau, \tau*)$.

Proof. (1) Since $f_i\colon (X, \tau_i, \tau{*}_i) \to (Y, \tau, \tau*)$ is intuitionistic fuzzy continuous for each $i \in \Gamma$, by Theorem 8, $f_i\colon (X_i, \mathcal{C}_{\tau_i,\tau*_i}) \to (Y, \mathcal{C}_{\tau,\tau*})$ is a $\mathcal{C}$-map for each $i \in \Gamma$. From Theorem 18, $\mathcal{C}$ is finer than $\mathcal{C}_{\tau,\tau*}$.

(2) First, we will show that for each $i \in \Gamma$, $f_i\colon (X_i, \tau_i, \tau{*}_i) \to (Y, \tau_{\mathcal{C}}, \tau{*}_{\mathcal{C}})$ is intuitionistic fuzzy continuous. Suppose there exists $\lambda \in I^Y$ such that $\tau{*}_{\mathcal{C}}(\lambda) \not\leq \tau{*}_i(f_i^{-1}(\lambda))$ and $\tau_{\mathcal{C}}(\lambda) \not\leq \tau_i(f_i^{-1}(\lambda))$. Then there exists $r_o \in I_\circ$, $s_o \in I_1$ with $\mathcal{C}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda$ such that $\tau{*}_{\mathcal{C}}(\lambda) \geq r_\circ > \tau{*}_i(f_i^{-1}(\lambda))$ and $\tau_{\mathcal{C}}(\lambda) \leq s_\circ < \tau_i(f_i^{-1}(\lambda))$. On the other hand, we have

$$\begin{aligned}
\underline{1} - \lambda &= \mathcal{C}(\underline{1} - \lambda, r, s) \\
&= \bigvee_{i\in\Gamma} f_i(\mathcal{C}_{\tau_i,\tau*_i}(f_i^{-1}(\underline{1} - \lambda), r, s)) \\
&\geq f_i^{-1}(\mathcal{C}_i(\underline{1} - f_i^{-1}(\lambda), r_\circ, s_\circ).
\end{aligned}$$

It implies

$$\begin{array}{rcl} f_i^{-1}(\lambda) & = & f_i^{-1}(\underline{1}-\lambda) \\ & \geq & f_i^{-1}(f_i(\mathcal{C}_{\tau,\tau*}(\underline{1}-f_i^{-1}(\lambda),r_\circ,s_\circ))) \\ & \geq & \mathcal{C}_{\tau_i,\tau*_i}(f_i^{-1}(\lambda),r_\circ,s_\circ). \end{array}$$

But we have $\mathcal{C}_{\tau_i,\tau*_i}(f_i^{-1}(\lambda),r_\circ,s_\circ) = f_i^{-1}(\lambda)$. Since $\tau_{\mathcal{C}_{\tau_i}} = \tau_i$ and $\tau*_{\mathcal{C}_{\tau*_i}} = \tau*_i$, we have $\tau_i(f_i^{-1}(\lambda)) \geq r_\circ$ and $\tau*_i(f_i^{-1}(\lambda)) \leq s_\circ$, which is a contradiction. Hence $f_i\colon (X_i,\tau_i,\tau*_i) \to (Y,\tau_{\mathcal{C}},\tau*_{\mathcal{C}})$ is intuitionistic fuzzy continuous.

Secondly, since $(\tau,\tau*)$ is the final intuitionistic fuzzy topology on Y, by Theorem 12, we have $\tau_{\mathcal{C}}(\lambda) \leq \tau(\lambda)$ and $\tau*_{\mathcal{C}}(\lambda) \geq \tau*(\lambda)$ for all $\lambda \in I^Y$. Conversely, since $\tau_{\mathcal{C}_{\tau,\tau*}} = \tau$ and $\tau*_{\mathcal{C}_{\tau,\tau*}} = \tau*$, we only show that $\tau_{\mathcal{C}_{\tau,\tau*}}(\lambda) \leq \tau_{\mathcal{C}}(\lambda)$ for all $\lambda \in I^Y$. Suppose there exists $\lambda \in I^Y$ such that $\tau_{\mathcal{C}_{\tau,\tau*}}(\lambda) \not\geq \tau_{\mathcal{C}}(\lambda)$ and $\tau*_{\mathcal{C}_{\tau*}}(\lambda) \not\leq \tau*_{\mathcal{C}}(\lambda)$. Then there exist $r_\circ \in I_\circ$, $s_\circ \in I_1$ with $\mathcal{C}_{\tau,\tau*}(\underline{1}-\lambda,r_\circ,s_\circ) = \underline{1}-\lambda$ such that $\tau_{\mathcal{C}_{\tau,\tau*}}(\lambda) \geq r_\circ > \tau_{\mathcal{C}}(\lambda)$ and $\tau*_{\mathcal{C}_{,\tau\tau*}}(\lambda) \leq s_\circ < \tau*_{\mathcal{C}}(\lambda)$. On the other hand, we have

$$\underline{1}-\lambda = \mathcal{C}_{\tau,\tau*}(\underline{1}-\lambda,r_\circ,s_\circ) \geq \mathcal{C}(\underline{1}-\lambda,r_\circ,s_\circ).$$

Hence $\mathcal{C}(\underline{1}-\lambda,r_\circ,s_\circ) = \underline{1}-\lambda$. So, $\tau_{\mathcal{C}}(\lambda) \geq r_\circ$ and $\tau*_{\mathcal{C}}(\lambda) \leq s_\circ$, which is a contradiction. □

Example 23. *Let $X=\{a,b\}$, $Y=\{x\}$ be sets. Define $\tau,\tau*: I^X \to I$ as follows:*

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{1},\underline{0}; \\ \frac{1}{2}, & \text{if } \lambda = \underline{1}-a_{0.5} \text{ or } \underline{1}-b_{0.7}; \\ \frac{1}{2}, & \text{if } \lambda = \underline{1}-(a_{0.5}\vee b_{0.7}); \\ 0, & \text{otherwise.} \end{cases}$$

$$\tau*(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{1},\underline{0}; \\ \frac{1}{2}, & \text{if } \lambda = \underline{1}-a_{0.5} \text{ or } \underline{1}-b_{0.7}; \\ \frac{1}{2}, & \text{if } \lambda = \underline{1}-(a_{0.5}\vee b_{0.7}); \\ 1, & \text{otherwise.} \end{cases}$$

From Theorem 4, we obtain
$\mathcal{C}_{\tau,\tau*}: I^X \times I_\circ \times I_1 \to I^X$ as follows:

$$\mathcal{C}_{\tau,\tau*}(\lambda,r,s) = \begin{cases} \underline{0}, & \text{if } \lambda=\underline{0},\ r\in I_\circ,\ s\in I_1; \\ a_{0.5}, & \text{if } \underline{0}\neq\lambda\leq a_{0.5},\ 0<r\leq\frac{1}{2},\ \frac{1}{2}<s\leq 1; \\ b_{0.7}, & \text{if } \underline{0}\neq\lambda\leq b_{0.7},\ 0<r\leq\frac{1}{2},\ \frac{1}{2}<s\leq 1; \\ a_{0.5}\vee b_{0.7}, & \text{if } \lambda\leq a_{0.5}\vee b_{0.7},\ \lambda\not\leq a_{0.5},\ \lambda\not\leq b_{0.7}, \\ & 0<r\leq\frac{1}{2},\ \frac{1}{2}<s\leq 1; \\ \underline{1}, & \text{otherwise.} \end{cases}$$

From Corollary 15, we have the quotient space τ^f, $\tau*^f$ on Y of $(X,\tau,\tau*)$ as follows:

$$\tau^f(\nu) = \tau(f^{-1}(\nu)) = \begin{cases} 1, & \text{if } \nu=\underline{0},\underline{1}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\tau*^f(\nu) = \tau*(f^{-1}(\nu)) = \begin{cases} 0, & \text{if } \nu=\underline{0},\underline{1}, \\ 1, & \text{otherwise.} \end{cases}$$

From Theorem 4, we have

$$\mathcal{C}_{\tau^f,\tau*^f}(\nu,r,s)=\begin{cases}\underline{0}, & \text{if } \nu=\underline{0},\ r\in I_\circ,\ s\in I_1,\\ \underline{1}, & \text{otherwise.}\end{cases}$$

Since $\mathcal{C}^f(\nu,r,s)=f(\mathcal{C}_{\tau,\tau*}(f^{-1}(\nu),r,s))$ *from Theorem 22, we have*

$$\mathcal{C}^f(\nu,r,s)=\begin{cases}\underline{0}, & \text{if } \nu=\underline{0},\ r\in I_\circ,\ s\in I_1,\\ x_{0.7}, & \text{if } \underline{0}\neq\nu\leq x_{0.5},\ 0<r\leq\frac{1}{2},\ \frac{1}{2}<s\leq 1;\\ \underline{1}, & \text{otherwise.}\end{cases}$$

Hence $\mathcal{C}_f$ *is finer than* $\mathcal{C}_{\tau^f,\tau*^f}$ *and* $\mathcal{C}_{\tau^f,\tau*^f}\neq\mathcal{C}^f$*. Moreover,* $\mathcal{C}^f$ *is topological from Theorem 4. Since*

$$x_{0.7}=\mathcal{C}^f(x_{0.5},\frac{1}{3},\frac{1}{3})\neq\mathcal{C}^f(\mathcal{C}^f(x_{0.5},\frac{1}{3},\frac{1}{3}),\frac{1}{3},\frac{1}{3})=1,$$

an intuitionistic fuzzy closure operator $\mathcal{C}^f$ *is not topological. From Theorem 5, we have*

$$\tau_{\mathcal{C}^f}(\nu)=\begin{cases}1, & \text{if } \nu=\underline{0},\ or\underline{1};\\ 0, & \text{otherwise.}\end{cases},\qquad \tau_{\mathcal{C}^f}(\nu)=\begin{cases}0, & \text{if } \nu=\underline{0},\ or\underline{1};\\ 1, & \text{otherwise..}\end{cases}$$

Hence $(\tau_{\mathcal{C}^f},\tau*_{\mathcal{C}^f})=(\tau^f,\tau*^f)$.

Theorem 24. *Let* $\{(X_i,\mathcal{C}_i)\}_{i\in\Gamma}$ *be a family of pairwise disjoint* (r,s)*-fuzzy-*T_0 *intuitionistic fuzzy closure spaces. Then their sum intuitionistic fuzzy closure space* $(X,\mathcal{C})$ *is also* (r,s)*-fuzzy-*T_0.

Proof. (1) $x,y\in X_i$, $i\in\Gamma$. Since $(X_i,\mathcal{C}_i)$ is $(r,s)-T_0$, there exists $\lambda\in I^{X_i}$ such that $\mathcal{C}_i(\lambda,r,s)=\lambda$ and $\lambda(x)\neq\lambda(y)$, since $\lambda\in I^X$. By using corollary 20, we have $\mathcal{C}(\lambda,r,s)=\lambda$.

(2) $x\in X_i$ and $y\in X_j$, $i,j\in\Gamma$, $i\neq j$. Let $\lambda\in I^{X_i}$, it can be easily checked that $\mathcal{C}(\lambda,r,s)=\lambda$ such that $\lambda(x)\neq\lambda(y)$, where $\mathcal{C}_i(\lambda,r,s)=\lambda$. Hence the sum $(X,\mathcal{C})$ is (r,s)-fuzzy-T_0. □

Theorem 25. *Let* $\{(X_i,\mathcal{C}_i)\}_{i\in\Gamma}$ *be a family of pairwise disjoint* (r,s)*-fuzzy-*T_1 *intuitionistic fuzzy closure spaces. Then their sum intuitionistic fuzzy closure space* $(X,\mathcal{C})$ *is also* (r,s)*-fuzzy-*T_1.

Proof. Let $x\in X=\cup X_i$, then $x\in X_{i_0}$ for some $i_o\in\Gamma$. But $(X_{i_0},\mathcal{C}_{i_0})$ is (r,s)-fuzzy-T_0, $\mathcal{C}_{i_0}(\chi_{\{x\}},r,s)\leq\chi_{\{x\}}$. Since $\chi_{\{x\}}\in I^X$. By using Corollary 20, $\mathcal{C}(\chi_{\{x\}},r,s)\leq\chi_{\{x\}}$. Hence the sum $(X,\mathcal{C})$ is (r,s)-fuzzy-T_1 space. □

References

[1] M. Azab Abd-allah, *Intuitionistic fuzzy closure and topological spaces*(Submitted).

[2] K. Atanassov, *Intuitionistic fuzzy sets, Fuzzy Sets and Systems*, **20(1)** (1986), 87-96.

[3] K. Atanassov, *New operators defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **61(2)** (1993), 131-142.

[4] K. Atanassov and S. Stoeva, *Intuitionistic fuzzy sets* , Proc. of Polish Symp. on Interval and Fuzzy Mathematics, Poznan, August, (1983), 23-26.

[5] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182-190 .

[6] K. C. Chattopadhyay, R. N. Hazra and S. K. Samanta, *Gradation of openness: Fuzzy topology*, Fuzzy Sets and Systems, **49(2)** (1992), 237-242.

[7] K. C. Chattopadhyay, R. N. Hazra and S. K. Samanta, *Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness*, Fuzzy Sets and systems, **54** (1993), 207-212.

[8] D. Çoker, *An introduction of Double fuzzy topological spaces*, Fuzzy Sets and Systems, **88** (1997), 81-89.

[9] D. Çoker and M. Demirci, *An introduction to intuitionistic fuzzy topological spaces in Šostak's sense*, Busefal, **67** (1996), 67-76.

[10] U. Höhle and A. P. Šostak, *A general theory of fuzzy topological spaces*, Fuzzy Sets and Systems, **73** (1995), 131-149.

[11] U. Höhle and A. P. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, The Handbooks of Fuzzy sets series, Volume 3, Kluwer Academic Publishers, Dordrecht (Chapter 3), 1999.

[12] Y. C. Kim, *Smooth fuzzy closure and topological spaces*, Kangweon-Kyungki Math. Jour., **7(1)** (1999), 11-25.

[13] Y.C. Kim, *Initial L-fuzzy closure spaces*, Fuzzy Sets and Systems, **133** (2003), 277-297.

[14] T. Kubiak and A. P. Šostak, *Lower set-valued fuzzy topologies*, Quaestions Math., **20(3)** (1997), 423-429.

[15] E. P. Lee and Y. B. Im, *Mated fuzzy topological spaces*, J. of fuzzy logic and intelligent systems (Korea), **11(2)** (2001), 161-165.

[16] A. A. Ramadan, *Smooth topological spaces*, Fuzzy Sets and Systems, **48** (1992), 371-375.

[17] S. K. Samanta and T. K. Mondal, *Intuitionistic gradation of openness: Intuitionistic fuzzy topology*, Busefal , **73** (1997), 8-17.

[18] S. K. Samanta and T. K. Mondal, *On intuitionistic gradation of openness*, Fuzzy Sets and Systems , **131** (2002), 323-336.

[19] A. P. Šostak, *On a fuzzy topological structure*, Suppl. Rend. Circ. Matem. Palerms ser **11** (1985), 89-103.

[20] A. P. Šostak, *On the neighbourhood structure of a fuzzy topological space*, Zb. Rodova Univ. Nis, ser Math, **4** (1990), 7-14.

[21] A. P. Šostak, *Basic structures of fuzzy topology*, J. of Math. Sciences, **78**, N6, (1996), 662-701.