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Rotational Effects On Rayleigh Wave Speed In Orthotropic Medium

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Abstract. A rotational effect on rayleigh wave speed in orthotropic materials is studied. A formula for the wave speed is derived. Rayleigh wave speed for some rotating and non-rotating orthotropic materials is calculated.

1. Introduction

In 1885, Rayleigh[5] studied the surface waves (called the rayleigh after his name) which propagate along the plane surface of elastic solid. After that a number of researches [4, 3, 6, 10, 8, 11, 2] studied the Rayleigh wave speed by using different techniques in different kind of materials. Recently Pham and Ogden [9] discussed the Rayleigh wave speed in orthotropic elastic solids. In this article we have extended the work of Pham and Ogden [9] and derived the formula for Rayleigh wave speed in rotating orthotropic materials with and without rotational effect is studied.

2. Boundary Value Problem & Secular Equation

Consider the semi-infinite stress-free surface of orthotropic material. We choose the rectangular co-ordinate system in such a way that $x_3 - axis$ is normal to the boundary and the material occupies region $x_3 \leq 0$. By following Pham and Ogden [9] we consider the plane harmonic waves in $x_1 - direction$ in $x_1x_3 - plane$ with displacement components (u_1, u_2, u_3) such that
Generalized Hook's law gives

$$\left.\begin{array}{l} \sigma_{11} = c_{11}u_{1,1} + c_{13}u_{3,3} \\ \sigma_{33} = c_{13}u_{1,1} + c_{33}u_{3,3} \\ \sigma_{13} = c_{55}(u_{1,3} + u_{3,1}) \end{array}\right\} \qquad (2.\ 1)$$

where the elastic constants $c_{11}, c_{33}, c_{13}, c_{55}$ satisfy the inequalities

$$c_{ii} > 0, i = 1,3,4 \ , c_{11}c_{33} - c_{13}^2 > 0 \tag{2. 2}$$

which are the necessary and sufficient conditions for the straign energy of the material to be positive definite. If a homogeneous elastic body is rotating about an axis, we may choose x_3-axis, with a constant angular velocity $\mathbf{\Omega}$ then equations of motion for infinitesimal deformation may be written as follows [7]

$$\sigma_{ij,j} = \rho\{\ddot{u_i} + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\varepsilon_{ijk}\Omega_j \dot{u_k}\} \tag{2. 3}$$

where $\mathbf{\Omega} = \Omega(0,0,1)$
The Eqs. (2.3), for the problem may be written as

$$\left.\begin{array}{l} \sigma_{11,1} + \sigma_{13,3} = \rho(\ddot{u_1} - \Omega^2 u_1) \\ \sigma_{31,1} + \sigma_{33,3} = \rho\ddot{u_3} \end{array}\right\} \tag{2. 4}$$

In view of (2.1), Eqs. (2.4) can be written as

$$\left.\begin{array}{l} c_{11}u_{1,11} + c_{13}u_{3,31} + c_{55}(u_{1,33} + u_{3,13}) = \rho(\ddot{u_1} - \Omega^2 u_1) \\ c_{55}(u_{1,31} + u_{3,11}) + c_{13}u_{1,13} + c_{33}u_{3,33} = \rho\ddot{u_3} \end{array}\right\} \tag{2. 5}$$

The boundary conditions of zero traction are

$$\sigma_{3i} = 0 \ , i = 1,3 \ \textit{on the plane} \ x_3 = 0 \tag{2. 6}$$

Usual requirements that the displacements and the stress components decay away from the boundary implies

$$u_i \to 0, \ \sigma_{ij} \to 0 \ (i.j = 1,3) \ as \ x_3 \to \infty \tag{2. 7}$$

Considering the harmonic waves propagating in x-direction, by following Pham and Ogden [9]we write;

$$u_j = \phi_j(kx_3)exp(ik(x_1 - ct)) \ ; \ j = 1,3 \tag{2. 8}$$

where k is the wave number and c is the wave speed and $\phi_j, j = 1,3$ are the functions to be determined. Substituting (2.8) into (2.5) gives

$$\begin{aligned} c_{55}k^2\phi_1'' + ik(c_{55} + c_{13})\phi_3' + \{k^2(\rho c^2 - c_{11}) + \rho\Omega^2\}\phi_1 = 0, \\ c_{33}\phi_3'' + ik(c_{55} + c_{13})\phi_1' + (\rho c^2 - c_{55})\phi_3 = 0. \end{aligned} \tag{2. 9}$$

In terms of $\phi_j; j = 1,3$ after taking into account (2.1) and (2.8) the boundary conditions (2.6) give

$$\begin{aligned} ic_{13}\phi_1 + c_{33}\phi_3' = 0 \\ \phi_1' + i\phi_3 = 0 \ \textit{on the plane} \ x_3 = 0. \end{aligned} \tag{2. 10}$$

while from (2.7) we have

$$\phi_j, \phi_j' \to 0 \ as \ x_3 \to -\infty \tag{2. 11}$$

Laplace transform of (2.9) by using (2.10) we have

$$\begin{aligned} \{k^2(c_{55}s^2 + \rho c^2 - c_{11)+\rho\Omega^2}\}\overline{\phi_1}(s) + ik^2(c_{13} + c_{55})s\overline{\phi_3}(s) = \\ c_{55}k^2\{s\phi_1(0) + \phi_1'(0)\} + ik^2(c_{13} + c_{55})\phi_3(0) \end{aligned}$$

$$\begin{aligned} i(c_{13} + c_{55})s\overline{\phi_1}(s) + (c_{33}s^2 - c_{55} + \rho c^2)\overline{\phi_3}(s) \\ = i(c_{13} + c_{55})\phi_1(0) + c_{33}\{s\phi_3(0) + \phi_3'(0)\} \end{aligned} \tag{2. 12}$$

From (2.12) we have

$$\overline{\phi_1}(s) = \frac{\begin{vmatrix} c_{55}k^2\{s\phi_1(0)+\phi_1'(0)\}+ik^2(c_{13}+c_{55})\phi_3(0) & ik^2(c_{13}+c_{55})s \\ i(c_{13}+c_{55})\phi_1(0)+c_{33}\{s\phi_3(0)+\phi_3'(0)\} & (c_{33}s^2-c_{55}+\rho c^2) \end{vmatrix}}{k^2c_{33}c_{55}s^4+[k^2\{(c_{13}+c_{55})^2+c_{33}(\rho c^2-c_{11})+c_{55}(\rho c^2-c_{55})\} +c_{33}\rho\Omega^2]s^2+(\rho c^2-c_{55})\{k^2(\rho c^2-c_{11})+\rho\Omega^2\}} \quad (2.\ 13)$$

Let ${s_1}^2$, ${s_2}^2$ be the roots of quadratic equation in s^2 (where s_1, s_2 must have positive real parts) of the denominator,

$$k^2c_{33}c_{55}s^4+[k^2\{(c_{13}+c_{55})^2+c_{33}(\rho c^2-c_{11})+c_{55}(\rho c^2-c_{55})\}+c_{33}\rho\Omega^2]s^2 +(\rho c^2-c_{55})\{k^2(\rho c^2-c_{11})+\rho\Omega^2\}=0 \quad (2.\ 14)$$

By considering (2.11) the inverse Laplace transform of $\overline{\phi_1}(s)$ gives

$$\phi_1(y)=A_1exp[s_1y]+A_2exp[s_2y] \quad (2.\ 15)$$

where $y=kx_3$. By using (2.15), (2.9) and (2.11) we have

$$\phi_3(y)=\alpha_1A_1exp[s_1y]+\alpha_2A_2exp[s_2y] \quad (2.\ 16)$$

where

$$\alpha_j=\frac{i[k^2\{c_{55}{s_j}^2+(\rho c^2-c_{11})\}+\rho\Omega^2]}{k^2(c_{13}+c_{55})s_j}\ ,j=1,2$$

As ${s_1}^2,{s_2}^2$ are the roots of (2.14), therefore, we must have

$${s_1}^2+{s_2}^2=-\frac{[k^2\{(c_{13}+c_{55})^2+c_{33}(\rho c^2-c_{11})+c_{55}(\rho c^2-c_{55})\}+c_{33}\rho\Omega^2]}{k^2c_{33}c_{55}}$$

$${s_1}^2{s_2}^2=\frac{(\rho c^2-c_{55})\{k^2(\rho c^2-c_{11})+\rho\Omega^2\}}{k^2c_{33}c_{55}} \quad (2.\ 17)$$

Substituting (2.15) and (2.16) into (2.10) we get

$$(ic_{13}+c_{33}\alpha_1s_1)A_1+(ic_{13}+c_{33}\alpha_2s_2)A_2=0$$
$$(s_1+i\alpha_1)A_1+(s_2+i\alpha_2)A_2=0 \quad (2.\ 18)$$

For non-trivial solution of (2.18), the determinant of the coefficients must vanish i.e

$$(ic_{13}+c_{33}\alpha_1s_1)(s_2+i\alpha_2)-(ic_{13}+c_{33}\alpha_2s_2)(s_1+i\alpha_1)=0 \quad (2.\ 19)$$

Substituting the value of α_1 and α_2 from (2.16) into (2.19) and simplifying we have the following equation

$$(\rho c^2-c_{55})[k^2{c_{13}}^2+c_{33}\{k^2(\rho c^2-c_{11})+\rho\Omega^2\}] -k\rho c^2\sqrt{c_{33}c_{55}}\sqrt{\{k^2(\rho c^2-c_{11})+\rho\Omega^2\}(\rho c^2-c_{55})}=0 \quad (2.\ 20)$$

The Eq. (2.20) may be written as

$$\sqrt{\frac{c_{33}}{c_{55}}\frac{\frac{\rho c^2}{c_{11}}-\frac{c_{55}}{c_{11}}}{\frac{\rho c^2}{c_{11}}-1+\frac{\rho\Omega^2}{k^2c_{11}}}}[\frac{{c_{13}}^2}{c_{11}c_{33}}+\frac{\rho c^2}{c_{11}}-1+\frac{\rho\Omega^2}{k^2c_{11}}]-\frac{\rho c^2}{c_{11}}=0 \quad (2.\ 21)$$

The Eq. (2.21) is the required Rayleigh wave speed formula for orthotropic materials.

3. Rayleigh wave speed in some rotating and non-rotating orthotropic materials

Value of Ω may be chosen at random for convenience we take

$$\frac{\Omega^2}{k^2} = \frac{c_{11}}{\rho}.$$

Therefore, the above equation becomes

$$\sqrt{\frac{c_{33}}{c_{55}} \frac{\frac{\rho c^2}{c_{11}} - \frac{c_{55}}{c_{11}}}{\frac{\rho c^2}{c_{11}}}} [\frac{{c_{13}}^2}{c_{11}c_{33}} + \frac{\rho c^2}{c_{11}}] - \frac{\rho c^2}{c_{11}} = 0 \tag{3. 1}$$

and can be written as

$$\begin{aligned} &\rho^3 c_{33}(c_{33} - c_{55})c^6 + \rho^2 c_{33}(2{c_{13}}^2 - c_{33}c_{55})c^4 \\ &\quad + \rho {c_{13}}^2({c_{13}}^2 - 2c_{33}c_{55})c^2 - c_{55}{c_{13}}^4 = 0 \end{aligned} \tag{3. 2}$$

Now using the computer software Mathematica and the following Table [1]
We have

Table 1

Materials	Stiffness$(10^{10} N/m^2)$				Density (Kg/m^3)
	c_{11}	c_{13}	c_{33}	c_{55}	ρ
Iodic acid HIO_3	3.01	1.11	4.29	2.06	4.64
Bariumsodium niobate $Ba_2NaNb_5O_{15}$	23.9	5.00	13.5	6.60	5.30

It is evident from (3.2) that there will be six values of c, but we have taken those

Table 2. For Rotating Materials

Materials	Speed(Km/s)
Iodic acid	82.41
Bariumsodium Niobate	120.97

values which satisfy Eq. (3.1)

Similarly for other values of Ω we can find Rayleigh wave speed in the given materials.

If $\Omega = 0$(stationary case), then the Eq. (2.21) becomes

$$\sqrt{\frac{c_{33}}{c_{55}} \frac{\frac{\rho c^2}{c_{11}} - \frac{c_{55}}{c_{11}}}{\frac{\rho c^2}{c_{11}} - 1}} [\frac{{c_{13}}^2}{c_{11}c_{33}} + \frac{\rho c^2}{c_{11}} - 1] - \frac{\rho c^2}{c_{11}} = 0 \tag{3. 3}$$

which may be written as

$$\rho^3 c_{33}(c_{33}-c_{55})c^6+\rho^2 c_{33}\{2{c_{13}}^2-c_{33}c_{55}-c_{11}(2c_{33}-c_{55})\}c^4$$
$$+\rho({c_{13}}^2-c_{11}c_{33})({c_{13}}^2-c_{11}c_{33}-2c_{33}c_{55})c^2-c_{55}({c_{13}}^2-c_{11}c_{33})^2=0 \quad (3.\ 4)$$

Again using the Table [1] and computer software Mathematica we have from (3.4) that there are three waves which propagate in the non-rotating material with distinct velocities as shown in the following

Thus tremendous rotational effects on the Rayleigh wave speed can be seen from

Table 3. For Non-Rotating Materials

Materials	Speed(Km/s)
Iodic acid	53.44, 80.94, 125.37
Bariumsodium Niobate	102.55, 213.59, 296.44

the last two Tables.

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