VOLUME XXVIII (1995)

THE PUNJAB UNIVERSITY

JOURNAL OF MATHEMATICS

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THE PUNJAB
LAHORE-54590
PAKISTAN

Notice to Contributors

1. The Journal is meant for publication of research papers and review articles covering state of the art in a particular area of mathematical science.

2. Manuscripts should be typewritten and in a form suitable for publication. As far as possible, the use of complicated notations should be avoided. Figures, drawn on separate sheets of white paper in Black Ink, should be of a size suitable for inclusion in the Journal. The contributors are required to provide the disk containing the file of the paper composed in Latex or scientific word.

3. Reference should be given at the end of the paper and be referred to by numbers in serial order on square brackets, e.g. [3]. Reference should be typed as follows:

Reference to Paper:

Hoo, S.C. : BCI- algebras with conditions, **Math. Japonica, 32,** No. 5 (1987) 749-756.

Reference to Book:

Mitchell, B.: **Theory of categories**, New York: Academic Press, 1965.

4. Contributions and other correspondence should be addressed to Dr. A. Majeed, Mathematics Department, Punjab University, New Campus, Lahore-20, Pakistan.

5. The decision to accept or reject a paper for publication in the Journal rests fully with the Editorial Board.

6. Authors, whose papers will be published in the Journal, will be supplied 30 free reprints of their papers and a copy of the issue containing their contributions.

7. The Journal which is published annually will be supplied free of cost in exchange with other Journals of Mathematics.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 1 – 9

A GENERALIZATION OF THE KURATOWSKI CLOSURE-COMPLEMENT PROBLEM

M.S. Moslehian

Department of Mathematics,
Ferdowsi University of Mashhad
P.O. Box 1159-91775
Mashhad, Iran.

N. Tavallaii

Department of Mathematics,
Ferdowsi University of Mashhad
P.O. Box 1159-91775
Mashhad, Iran.

ABSTRACT

By considering the interval (0, 1) on the real line it is easy to show that it is not possible, in general, to obtain boundary of a given set by using the complementation, he closure and the interior operations on that set. Therefore one can generalize the Kuratowski closure-complement problem in special sense. In this paper, we will show that if any pair of operations among closure, interior, boundary and complementations be chosen, then by using only two of the operations on any given set X we obtain that X belongs to a specific finite family. In addition, for any pair of these operations we will give necessary and sufficient condition that the related family puses the largest cardinal number.

1980 Mathematical Subject Classificaton (1985 Revision) Primary 54B99.

KEY WORDS AND PHRASES: Closure, Interior, Boundary, Relative topology and Connected topological space.

INTRODUCTION

Let S be a topological space and b, c, i and k denote the boundary, the closure, the interior and the complementation, respectively. (See [1] for definition and properties of these operations). If X is a non-empty subset of S and α and β are any pair of the operations, then $X\alpha$ and $X\alpha\beta$ denote the image of X and $X\alpha$ respectively under α and β. We will show that if α and β are any pair of the operations, then the family $AB(X) = BA(X) = \{X, X\alpha, X\beta,$

$X\alpha\alpha$, $X\alpha\beta$, $X\beta\alpha$, $X\beta\beta$, $X\alpha\alpha\alpha$, $X\alpha\alpha\beta$, $X\alpha\beta\alpha$, $X\alpha\beta\beta$, $X\beta\alpha\alpha$, $X\beta\alpha\beta$, $X\beta\beta\alpha$, $X\beta\beta\beta$, ...}; which will be formed from a given set by using only the operations α and β, iterated in any order; consists of the sets which will be specified. We also give necessary and sufficient conditions, in every case, for $AB(X)$ to have the greatest cardinal number. Moreover, for all α and β an example is given of a subset X of a topological space S such that $AB(X)$ have the greatest cardinal number. In the case $\alpha = \beta$, the problem will have the obvious solution. Therefore we assume that $\alpha \neq \beta$. We consider six cases:

Case (1): $\alpha = c$ and $\beta = k$

This is the same as the Kuratowski closure-complement problem [3]. With due attention to $Xckckckc = Xckc$ and $Xkckckckc = Xkckc$, we have $CK(X) = \{X, Xk, Xc, Xkc, Xck, Xkck, Xckc, Xkckc, Xckck, Xkckck, Xckckc, Xkckckc, Xckckck, Xkckckck\}$

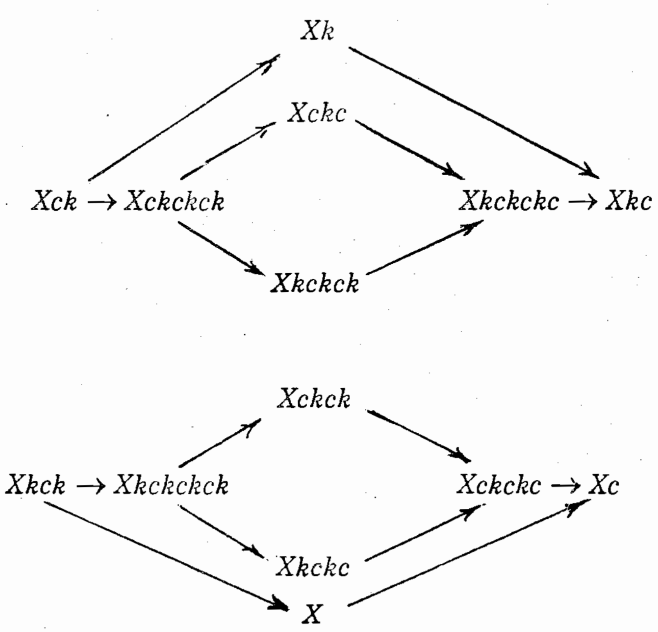


Fig. 1 - diagram of the family $CK(X)$

(In this diagram and future diagrams the arrows denote inclusion relations)

By considering the above diagram, we can easily see $CK(X) = CK(XK)$. In the following theorem, necessary and sufficient conditions on X in order for $CK(X)$ to consist of precisely fourteen distinct sets are mentioned.

Theorem 1

Let X be a subset of the arbitrary topological space S. Then X is a 14-set iff the following five conditions hold:

(A) $Xbi = Xkbi = Xci \backslash Xic \neq \phi$

(B) $X \cap Xckckck = X \backslash Xcic \neq \phi$

(C) $Xk \cap Xkckckck = Xici \backslash X \neq \phi$

(D) $Xcib = Xcic \backslash Xci \neq \phi$

(E) $Xkcib = Xic \backslash Xici \neq \phi$

Moreover, the conditions are independent, i.e., in general, no four imply the fifth.

Proof. [4]

It is not hard to see that the subset $X = (0,1) \cup (1,2) \cup Q(2,3) \cup \{4\}$ of R with ordinary topology is a 14-set. ($Q(2, 3)$ denotes the set of rationals in (2, 3)) [4]. Also $X = \{1, 3, 6\}$ in finite topological space $S = \{1,2,3,4,5,6,7\}$ with base $\{\phi, S, \{1\}, \{7\}, \{1, 2\}, \{6, 7\}, \{3, 5\}\}$ is a 14-set ([2], [5]).

Case (2): $\alpha = c$ and $\beta = i$

With due attention to $Xicic = Xic$ and $Xcici = Xci$, we have $CI(X) = \{X, Xi, Xc, Xic, Xci, Xici, Xcic\}$

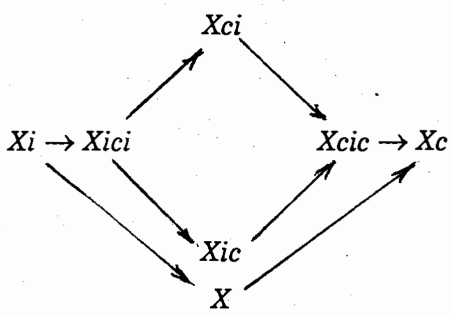


Fig. 2 - diagram of the family $CI(X)$

We know Xi is equal to $Xkck$, hence $CK(X) = CI(X) \cup CI(XK)$. By using the above relation, it is easy to show that $CI(X)$ consists of precisely seven sets iff $CK(X)$ consists of precisely fourteen sets, i.e. iff X is a 14-set. Therefore every 14-set of a topological space is an example of a set X that $|CI(X)| = 7$.

Case (3): $\alpha = i$ and $\beta = k$

By using $CK(X)$ and $Xi = Xkck$, we can show that $IK(X) = \{X, Xk, Xi, Xki, Xik, Xkik, Xiki, Xkiki, Xikik, Xkikik, Xikiki, Xkikiki, Xikikik, Xkikikik\} = CK(X)$.

Therefore the diagram of family $IK(X)$ is the same as the diagram of $CK(X)$.

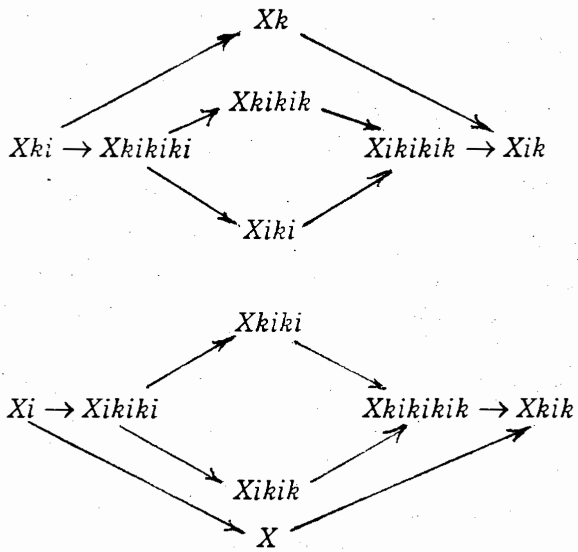


Fig. 3 - diagram of the family $KI(X)$

It is obvious that $IK(X)$ consists of precisely fourteen sets iff X is a 14-set. Hence, every 14-set of a topological space is an example of a set X that $|IK(X)| = 14$.

Case (4): $\alpha = b$ and $\beta = k$

By using the relations $Xbbb = Xbb$, $Xbkb = Xbb$, $Xkb = Xb$, we have $BK(X) = \{X, Xk, Xb, Xbk, Xbb, Xbbk\}$.

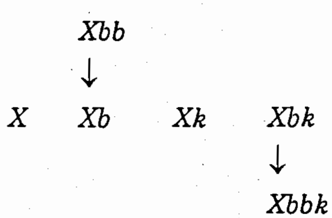


Fig. 4 - diagram of the family $BK(X)$

Theorem 2

Let X be a subset of a topological space S. Then $BK(X)$ consists of precisely six sets iff the following two conditions hold:

(A) $Xb \neq S$

(B) $Xbi \neq \phi$

Moreover, the conditions are independent.

Proof

First, suppose that $BK(X)$ consists of precisely six sets. Since $Xbbk \neq Xbk$ and $Xbbk = Xbi \cup Xbk$, so that $Xbi \neq \phi$. If $Xb = S$, then $Xbk = \phi$ and hence $Xbb = Xbk$. Thus $Xb \neq S$.

Second, we shall show that the two conditions are sufficient. Suppose that $Xbi \neq \phi$ and $Xb \neq S$, then X is neither open nor closed (since if X is open, then $X = Xi$ and therefore $Xbi = Xibi = \phi$, thus we will have a contradicton; similarly, Xk isn't open and hence X isn't closed, since $Xkbi = Xbi \neq \phi$). Now, we can show that the sets X, Xk, Xb, Xbk, Xbb and $Xbbk$ are pairwise distinct. It is obvious that $Xb \neq Xbk$, $Xbb \neq Xbbk$ and $X \neq Xk$; also $X \neq Xb$ since Xb is closed but X isn't closed. Similarly we have $X \neq Xbk$, $X \neq Xbb$, $X \neq Xbbk$, $Xk \neq Xb$, $Xk \neq Xbk$, $Xk \neq Xbb$ and $Xk \neq Xbbk$. On the other hand, $Xb = Xbi \cup Xbb$ and $Xbi \neq \phi$ so that $Xb \neq Xbb$. Also, if $Xb = Xbbk$ then $Xbk = Xbb \subseteq Xb$ and hence $Xbk = \phi$ or $Xb = S$; thus $Xb \neq Xbbk$. But $Xbk \neq Xbb$ since $Xb \neq Xbbk$. Also $Xbk \neq Xbbk$ since $Xb \neq Xbb$.

The sets $X_1 = Q$ and $X_2 = (0, 1)$ on the real line under its usual topology, show that the conditions (A) and (B) are independent.

Corollary 3

Suppose that X is a 14-set, then $BK(X)$ consists of precisely six sets.

Proof

$Xbi \neq \phi$ since X is a 14-set. Also $Xcib \neq \phi$. On the other hand if $Xb = S$, then $Xc = S$, hence $Xcib = \phi$; thus $Xb \neq S$. Now, by the previous theorem, $BK(X)$ consists of precisely six distinct sets.□

By the above corollary, every 14-set is an example of a set X that $|BK(X)| = 6$.

Case (5): $\alpha = b$ and $\beta = i$

With due attention to $Xibb = Xib$, $Xibi = \phi$, $Xbbi = \phi$ and $Xbbb = Xbb$ we have $BI(X) = \{\phi, X, Xi, Xb, Xib, Xbi, Xbb, Xbib\}$.

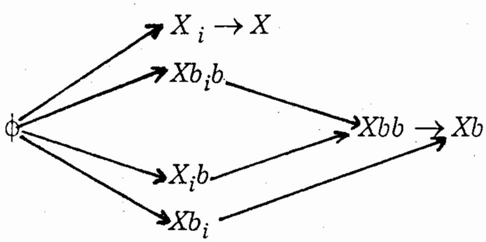


Fig. 5- diagram of the family $BI(X)$

Theorem 4

Let X be a subset of a topological space S. Then the following conditions on X are necessary and sufficient for $BI(X)$ consists of precisely eight distinct sets:

(A) $Xib \neq \phi$

(B) $Xbib \neq \phi$

(C) $Xbb - Xbib \neq \phi$

(D) $Xbb - Xib = \phi$

(E) $Xib \,\Delta\, Xbib = (Xib \cup Xbib) - (Xib \cap Xbib) \neq \phi$

Moreover, the conditions are independent.

Proof

It is obvious the mentioned conditions are necessary. To show these conditions are sufficient, suppose that X satisfies the above conditions. We will show that the sets ϕ, X, Xi, Xb, Xbi, Xib, Xbb and $Xbib$ are pairwise distinct. If $X = \phi$ then $Xib = \phi$. Thus $X \neq \phi$. It is easily shown that $Xi \neq \phi$, $Xb \neq \phi$, $Xbi \neq \phi$, $Xib \neq \phi$ and $Xbib \neq \phi$. Also, $Xbb \neq \phi$ since $Xib \subseteq Xbb$. Moreover, $Xbi \neq \phi$ and hence, as it was already shown, X is neither open nor closed. Therefore $X \neq Xi$, $X \neq Xb$, $X \neq Xbi$, $X \neq Xib$, $X \neq Xbib$ and $X \neq Xbb$. Also, if $Xi = Xb$ then $Xib = Xbib$ and hence $Xib \,\Delta\, Xbib = \phi$; thus $Xi \neq Xb$. Since $Xi \cap Xbi = \phi$, it follows that $Xi \neq Xbi$. If $Xi = Xib$, then $Xi = Xibi = \phi$. Thus we have $Xi \neq Xib$. Similarly we can show that $Xi \neq Xbib$ and

$Xi \neq Xbb$. If $Xb = Xbi$, then Xbi is open and closed, so that $Xbib = \phi$. Thus we have $Xb \neq Xbi$. If $Xb = Xib$; then $Xbi = Xibi = \phi$ and so that $Xb \neq Xib$. Similarly we can prove that $Xb \neq Xbib$, $Xb \neq Xbb$, $Xbi \neq Xib$, $Xbi \neq bib$ and $Xbi \neq Xbb$. By the hypothesises $Xbb - Xib \neq \phi$ and $Xbb - Xbib \neq \phi$, so that $Xbb \neq Xib$ and $Xbb \neq Xbib$. Finally, if $Xib = Xbib$, then $Xib \Delta Xbib = \phi$. Thus $Xib \neq Xbib$.

Now, consider the subset $X_1 = \{1\} \cup Q(2, 3)$, $X_2 = \{0\} \cup (2, 3)$, $X_3 = Q(0, 1) \cup (1, 2) \cup Q(2, 3)$, $X_4 = (0, 1) \cup Q(1, 2) \cup (2, 3)$ of the real line under its usual topology, and also consider the subset $X_5 = Q(-2, -1) \cup (-1, 1) \cup Q(1, 2) \cup \{3\}$ of the space $S = R - \{-2, 2\}$ under its relative Euclidean topology. Then by using the set X_i we can easily show that the i-th condition is independent of the other conditions.□

If $X = (1, 2) \cup Q(3, 4)$ is a subset of the real line under its usual topology then $Xi = (1, 2)$, $Xb = \{1, 2\} \cup [3, 4]$, $Xbi = (3, 4)$, $Xib = \{1, 2\}$, $Xbib = \{3, 4\}$, $Xbb = \{1, 2, 3, 4\}$. Therefore $BI(X)$ consists of eight distinct sets. It is posisble that X is a 14-set but $BI(X)$ consists of less than eight distinct sets. An example of such a set is the set $X = (0, 1) \cup (1, 2) \cup Q(2, 3) \cup \{4\}$ in the topological space $S = R - \{2, 3\}$ with relative Euclidean topology.

Case(6): $\alpha = b$ and $\beta = c$

With due attention to $Xbc = Xb$ and $Xcbb = Xcb$, we have $BC(X) = \{X, Xb, Xc, Xbb, Xcb\}$.

$$\begin{array}{c} Xcb \rightarrow Xbb \rightarrow Xb \rightarrow Xc \\ \hphantom{Xcb \rightarrow Xbb \rightarrow Xb \rightarrow{}} \uparrow \\ \hphantom{Xcb \rightarrow Xbb \rightarrow Xb \rightarrow{}} X \end{array}$$

Fig. 6 - diagram of the family $BC(X)$

Theorem 5

Let X be a subset of a topological space S. Then $BC(X)$ consists of precisely five distinct sets iff the following two conditions hold:

(A) $Xbi \neq \phi$

(B) $Xci \cap Xbb \neq \phi$

Moreover, the conditions are independent.

Proof

Suppose that $BC(X)$ consists of preciseey five sets. Then $Xb \neq Xbb$ and hence $Xbi \neq \phi$ since $Xb = Xbi \cup Xbb$, $Xcb \subseteq Xbb$, so that if $Xbb = \phi$ then $Xcb = Xbb$. Thus $Xbb \neq \phi$. Since $Xbb \neq Xcb$, there is some element x belonging to Xbb such that $x \notin Xcb$. But $Xbb \subseteq Xbc \subseteq Xcc$ so that $x \in Xcc$. Also we know that $Xcc = Xci \cup Xcb$; therefore $x \notin Xcb$ and so that $x \in Xci$, hence $x \in Xbb \cap Xci$. Thus $Xbb \cap Xci \neq \phi$.

Now, suppose that $Xbi \neq \phi$ and $Xbb \cap Xci \neq \phi$. We shall show that the five sets X, Xb, Xc, Xbb and Xcb are pairwise distinct. X is neither open nor closed since $Xbi \neq \phi$. Therefore $X \neq Xb$, $X \neq Xc$, $X \neq Xcb$ and $X \neq Xbb$. If $Xb = Xc$ then $Xci = Xbi$, but $Xci \cap Xbb \neq \phi$; hence we have $Xbi \cap Xbb \neq \phi$, that it is impossible. Thus $Xb \neq Xc$. Also $Xb \neq Xcb$ since otherwise $Xbi = Xcbi = Xcbbi = (Xc)bbi = \phi$, that it is contradict to the hypothesis (A). Since $Xbi \neq \phi$ and $Xb = Xbi \cup Xbb$ we have $Xb \neq Xbb$. If $Xc = Xcb$ then $Xci = Xcbi = \phi$ and hence $Xci \cap Xbb = \phi$; thus $Xc \neq Xcb$. Similarly, it is proved that $Xc \neq Xbb$. Finally if $Xcb = Xbb$ then $Xci \cap Xbb \neq \phi$ and therefore $Xci \cap Xcb \neq \phi$ which is impossible; thus $Xcb \neq Xbb$.

Moreover, by using the set $X_1 = Q(2, 3)$ and $X_2 = (0,1) \cup (2,3)$ on R under Euclidean topology, it is easily shown that the two conditions are independent.□

Corollary 6

If X is a 14-set, then $BC(X)$ consists of precisely five sets.

Proof

$Xbi \neq \phi$ and $Xici - X \neq \phi$ since X is a 14-set. If $x \in Xici - X$ then $x \in Xici \subseteq Xci$. On the other hand $x \in Xici \subseteq Xic = Xi \cup Xib$ and $x \notin X$; hence $x \in Xib \subseteq Xbb$ and therefore $x \in Xci \cap Xbb$; thus $Xci \cap Xbb \neq \phi$. By the previous theorem $BC(X)$ consists of five distinct sets. Therefore every 14-set is an example of a set X that $|BC(X)| = 5$.□

Of course, we can simplify some of the theorems or obtain new results under special conditions on the set X or the underlying space S. For example ([4]) if S is a connected topological space, then subset X of S is a 14-set iff $Xbi \neq \phi$, $X - Xcic \neq \phi$ and $Xici - X \neq \phi$.

Also it is easily proved that if S is a connected topological space then $BK(X)$ consists of precisely six sets iff $Xbib \neq \phi$.

Finally, we conjecture that in a connected topological space, if X is a 14-set then $BI(X)$ consists of precisely eight distinct sets.

REFERENCES

1. J. Dugundji, "**Topology**", Allyn and Bacon. 1966.
2. H.H. Herda and R.C. Metzler, closure and interior in finite topological spaces, *Collquium Math.* **15**(1966) 211–216.
3. C.Kuratowski, Loperation $\bar{A}$ de lonalysis situs, *Fund. Math.*, **3**(1922) 182–199.
4. E. Langford, Characterization of Kuratowski 14-sets, *Amer. Math. Monthly*, **78**(1971) 362–367.
5. H. Nakano, "**Topology and linear topological space**". Tokyo 1951.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 10 - 21

THE STRUCTURE OF NEAR-RINGS

Raja Mohammad Latif
Department of Mathematical Sciences,
King Fahd University of Petroleum & Minerals,
Dhahran 31261, Saudi Arabia.

1. GENERAL INTRODUCTION

In recent years considerable interest has arisen in algebraic systems with binary operations of addition and multiplication satisfying all the axioms of a ring except perhaps one of the distributive laws and commutativity of addition. Such systems are generally called near-rings. The properties of such systems have been implicitely used by mathematicians ever since the development of calculus. For instance, the systems $< C(R, R), +, o>$, $<D(R, R), +, o>$ and $<R[x], +, o>$ consisting of all real valued continuous functions in one variable x, all real valued differentiable functions in x, and all polynomials in an indeterminate x over the reals, respecitvely, in which the operation + is defined pointwise and o is the usual composiion of functions, satisfy all the axioms of a ring except left distributive law. Therefore, these systems are near-rings (but not rings). The first formal study of a near-ring like concept was however made in 1905 by Dickson, in connection with his studies about the independence of the axioms of a field. He showed by constructing examples that the commutativity of multiplicaton and one of the distributive laws were not a consequence of the other field axioms. In terms of the present day terminology, Dickson's example was that of a "near-field", which is a very important type of a near-ring. Shortly afterwards, in 1907, Veblen and Wedderburn used Dickson's near-fields to give examples of non-desarguesian planes. In the mid 1930's, H. Zassenhaus determined all finite near-fields, and used them in describing sharply transitive permutation groups. In the late 1930's, H. Wielandt initiated the study of near-rings which were not near-fields. In 1950, D. Blackett published some important papers on simple and semisimple near-rings. In the late 1950's and early 1960's. A. Fröhlich published a series of important papers dealing with distributively generated near-rings. In his papers, Fröhlich developed a non-abelian homological algebra of modules over near-rings. Since the publication of the papers of Blackett, Fröhlich, and others, there has been a steady flow of papers on near-rings and related topics.

In terms of applications, near-rings have proven to be useful in other areas of Mathematics. For example, in addition to their applications in geometry and group theory as mentioned above, near-rings have been applied to combinatorics, design of statistical experiments, coding theory and cryptography. A near-ring is exactly what is needed to describe the structure of the endomorphisms of various mathematical structure in an adequate manner.

For basic notions in near-ring and their structure theory, we refer to [Pilz; 1983], and for their links with group theory, we refer to [Meldrum; 1985]. We also refer to [Clay; 1992] for geometrical aspects of near-ring theory. For a complete and up to date bibliography of the literature on near-rings and related topics we refer to a recent issue of the Near-ring News Letter, No. 16, 1995.

2. FUNDAMENTAL CONCEPTS

2.1 Basic Definitions and Examples

We begin with the definition of a near-ring.

Definition 2.1.1

A *(right) near-ring* R is a triple $<R, +, .>$ consisting of a set R with two binary operations "+" and ".", called addition and multiplication, respectively, such that

(1) $<R, +>$ is a (not necessarily commutative) group;

(2) $<R, .>$ is a semigroup;

(3) For all $a, b, c \in R$: $(a + b)\,c = ac + bc$ (Right distributivity).

Left near-rings can be defined similarly.

Let us consider a natural example of a right near-ring.

Example 2.1.2

Let $G = <G, +>$ be a (not necessarily commutative) group G $(\neq (0))$. Let $T(G)$ denote the set of all mappings from G into itself. On T(G) we define + to be the usual pointwise addition of mappings and multiplication is the composition of maps: for $f, g \in T(G)$ and $x \in G$:

$$(f + g)(x) = f(x) + g(x), \quad (f.g)(x) = f(g(x)).$$

Then $<T(G), +, .>$ is a right near-ring which is not a left near-ring (and so not a ring).

A ring is a right and left near-ring but not conversely. A near-ring with a commutative multiplication is both a right and a left near-ring. Yet such a near-ring need not be a ring, since, for example, any non abelian group $G = <G, +>$ with trivial multiplication, $xy = 0$ for all $x, y \in G$ is a near-ring $<G, +, .>$ with commutative multiplication but non commutative addition. One may, however, prove the following.

Proposition 2.1.3 *[Pilz; 1983]*

A near-ring R satisfying both distributive laws and $R = R^2$ is a ring.

One may note the following two properties which follow immediately from the basic definitions.

Fact 1: $0.x = 0$ for all $x \in R$ and 0 is the neutral element of $<R, +>$.

Fact 2: $-(ab) = (-a)b$ for all $a, b \in R$.

Remark: It is possible to have $x.0 \neq 0$ in a (right) near-ring.

Example 2.1.4

Let $G = <G, +>$ be a (nontrivial) group, and let f_a be the constant map:

$f_a(x) = a, x \in G$. Then $C(G) = \{f_a : a \in G\}$ is a near-ring in which f_0 is the neutral element.

but $\quad f_a \cdot f_0 = f_a \neq f_0$ for $a \neq 0$.

The transfer of many interesting results from righs to near-rings is made possible by the following axiom:

(4) $\quad x \,.\, 0 = 0$ for all $x \in R$.

Definition 2.1.5

Near-rings with the additional axiom (4) are called *zero symmetric*. Thus all rings are zero symmetric near-rings.

Let us consider an example of zero symmetric near-ring which is not a ring.

Example 2.1.6

Let $T_0(G)$ be the subset of $T(G)$ in Example 2.2 which consists of these mappings which leave 0 fixed is a (right) zero symmetric near-ring.

In the sequal, all near-rings are assumed to be zero symmetric.

If R contains an identity element 1 that is $x.1 = 1.x = x$ for all $x \in R$, then it is unique and is called the *identity* of R; R is, in this case a *near-ring with identity*. A right near-ring $R = <R, +, .>$ is called an *abelian near-ring* if $<R, +>$ is abelian. If the nonzero elements of R form a multiplicative gorup, then R is called a *near-field*. In [Pilz; 1983], it has been shown by Neumann that a near-field is an abelian near-ring.

Definition 2.1.7

An element r of a right near-ring R is called *(left) distributive* if for all $s, t \in R$: $r(s + t) = rs + rt$ (for example, 0 is always a distributive element). A near-ring R is called *distributively generated (d.g.)* if R contains a multiplicative subsemigroup D of (left) distributive elements which is a generating set for $<R, +>$.

Example 2.1.8

Let $G = <G, +>$ be a (not necessarily commutative) group and let E denote a multiplicative semigroup of endomorphism of G, and let

$$E(G) = \left\{ \sum_{i=1}^{n} \pm f_i : f_i \in E \right\}.$$

Then $E(G)$ is a d.g. near-ring whose generating semigroup is E.

Remark: Distributively generated near-rings are zerosymmetric.

Definition 2.1.9

A subset A of a (right) near-ring R is called a *(left) R-subgroup* of R if A is a subgroup of $(R, +)$ and $RA \subseteq A$ (Here $BC = \{bc : b \in B, c \in C\}$ for any subsets B and C of R). Right R-subgroups of R are defined analogously. *A (right) ideal* of R is a normal subgroup A of $<R, +>$ such that $AR \subseteq A$. A *(left) ideal* of R is a normal subgroup A of $<R, +>$ such that $r_1(r_2 + a) - r_1r_2 \in A$ for all $r_1, r_2 \in R$ and for all $a \in A$. The word *ideal* will always mean a subset of R which is both a right and a left ideal of R. If S is any subset of R, then $<S>$ will denote the *ideal generated by S.*

Definition 2.1.10

For near-rings R and R^*, a mapping $f : R \rightarrow R^*$ is a *near-ring homomorphism* if for all $x, y \in R$:

$$f(x + y) = f(x) + f(y), \; f(x\,y) = f(x) \,.\, f(y).$$

In [Pilz; 1983], it has been shown that A is an ideal of a near-ring R if and only if A is the kernel of a near-ring homomorphism.

Remark: The sums and products of ideals are defined as in rings. However, *unlike ring*, the product IJ, for ideals I and J of a near-ring need not be an ideal.

2.2 GEOMETRICAL ASPECTS OF NEAR-RINGS AND SOME OF THEIR APPLICATIONS

Planar near-rings were defined by J.R. Clay in 1967 in hope of extending the construction of Veblen and Wedderburn [Weblen & Wedderburn; 1907] of a plane having unusual properties using near-fields. It turned out that this class of near-rings has a close connection with incidence geometry, combinatorics and experimental designs (see [Ferrero; 1970] and [Chen; 1991]. Following the example of circles in the Euclidean plane, Clay singled out a subclass of planar near-rings in 1988, which has "circular" property. This class of near-ring has been proved to have nice applications in coding theory and cryptography (see [Modisett; 1988], [Fuchs, Hofer & Pilz; 1990] and [Clay & Kiechle; 1993]).

We review here some basic definitions and ideas of planar near-rings and it's connection with Ferrero pairs. For any given near-ring N, define a relation $^{=}m$ on N, and define $a \; {}^{=}m \; b$ if and only if $ax = bx$ for all $x \in N$. One can readily check that $^{=}m$ is an equivalence relation. Following J.R. Clay, a near-ring N is called planar if $N/{}^{=}m$ has at least three equivalence classes, and for all $a, b, c \in N$ with $a \; {}^{\neq}m b$ there exists a unique $x \in N$ such that $ax = bx + c$ (see [Clay; 1992]).

To any given planar near-ring N there corresponds a uniquely determined Ferrero pair, i.e., a pair (N, Φ), where N is a group and $\Phi \subseteq \mathrm{Aut}(N)$ such that if $\phi \in \Phi$ is not the identity mapping, then $-\phi + {}^{id}N$ is bijective. Actually, numerous planar near-rings can be constructed from any Ferrero (N, Φ) (see [Clay; 1992] for more details).

Let N be a finite planar near-ring with corresponding Ferrero pair (N, Φ), and set

$$N^* = N \setminus \{x \in N \mid x \,{}^{\neq}m\, 0\}$$

and $$B^* = \{N^* a + b = a^{\Phi} + b \mid a, b \in N, a \neq 0\},$$

where a^{Φ} is the orbit of a, then the incident structure $(N, B^*, \in)$ is a balanced incomplete block design, which is to say that there are positive integers κ, τ and λ and so that (i) each block $B \in B^*$ contains exactly κ elements, (ii) any given element $x \in N$ belongs to exactly τ blocks from B^* and (iii) any given pair $y, z \in X$ of distinct elements belongs to exactly λ subsets from B^*. The term "balanced incomplete block design" is usually abbreviated as "BIBD". (see [Clay; 1992]).

Next, a finite planar near-ring N (and the corresponding Ferrero pair (N, Φ) and BIBD $(N, B^*, \in)$ as well) is called circular if any three distinct points x, y, z ∈ N belong to at most one block $B \in B^*$. In this case, a block $N^*a + b$ is referred to as a circle with center b and radius a. The class of circular planar near-rings is a proper subclass of the class of all planar near-rings. The circularity property provides some important and useful additional properties of geometries, combinatorial objects (for example, BIBD's), codes and cryptography systems arising from planar near-rings as well as related Frobenius groups.

Ke and Wang [Ke & Wang; 1991] investigated the Frobenius groups having kernels of order 64 since 64 is the smallest possible order for a nonabelian kernel to exist (cf. adams [Adams; 1976]). Among the 267 nonisomorphic groups of order 64 (cf. Hall and Senior [Hall & Senior; 1964]), only three nonabelian ones can be the kernels of a Frobenius group. In these three cases, only two of them yield circular Ferrero pairs. However, both resulting circular BIBD's of these two Ferrero pairs have block size three, i.e., they are automatically circular.

In order to get a closer investigation of circular planar near-rings, Clay used computer to calculate circular pairs (Zp, Φ) where p is a prime, $4 \leq p < 1000$, and Φ is a multiplicative subgroup of Z^* with $|\Phi| > 3$ (see [Clay; 1988] or pp. 61–64 of [Clay; 1992]). These examples are proved to be very useful for exploring the structure of circular planar near-rings in the following few years.

Modisett characterized circular Ferrero pairs constructed from finite fields, i.e., those Ferrero pairs (F, Φ) with F being a field while Φ being a subgroup of the multiplicative group of F such that $|\Phi| \geq 3$. For more details we refer to [Clay; 1988, 1992 & 1993] and [Modisett; 1988].

2.3 MODULES OVER NEAR-RINGS

Analogous to the notion of modules over a ring, we have the notion of 'modules over a near-ring' defined in the following way.

Definition 2.3.1

A near-ring module R^M (briefly M) over a ring R is a pair (M, θ) where $M = <M, +>$ is a (not necessarily commutative) group and $\theta: R \times M \to M$ is a mapping such that if $\theta(r, m)$ is denoted by rm, then the following conditions hold:

(i) $(r_1 + r_2)\, m = r_1 m + r_2 m$, (ii) $(r_1 r_2)\, m = r_1\, (r_2 m)$

for all $r_1, r_2 \in R$ and $m \in M$.

If R is a near-ring with 1 and $1x = x$ for all $x \in M$, we say that R^M is a *unitary near-ring module*.

Example 2.3.2

(i) A near-ring R is a near-ring module over itself, denoted by R^{R}.

(ii) For any group $G = <G, +>$, G is a unitary near-ring module over $T_0(G)$.

Definition 2.3.3

A mapping $f: M \rightarrow M^*$ between left R-modules M and M^* is called an R-*module homomorphism* (or for short, R-*homomorphism*) if for all, $x, y \in M$, $r \in R$:

$$f(x + y) = f(x) + f(y), \ f(rx) = rf(x)$$

Definition 2.3.4

(i) A subset A of an R-module (that is, a near-ring left R-module) M is an R-*subset* of M if $RA = \{ra : r \in R, a \in A\} \subseteq A$.

(ii) An R-subset A is an R-*subgroup of* M if $<A, +>$ is a subgroup of $<M, +>$.

(iii) An R-subset A is an R-*submodule* of M if

(a) $<A, +>$ is a normal R-subgroup of $<M, +>$;

(b) $r(m + a) - rm \in A$, for all $r \in R$, $a \in A$, $m \in M$.

Remark: The intersection of an arbitrary collection of R-subsets (R-subgroups; R-submodules) of an R-module M is an R-subset (R-subgroup; R-submodule). The intersection of all R-subsets (R-subgroups; R-submodules) containing a subset B of M is called the R-subset (R-subgroup; R-submodule) generated by B. If $a \in M$, then Ra is an R-subgroup of M.

Definition 2.3.5

Let M be an R-module over a near-ring R. M is called *simple* if M has no proper non-zero R-submodules. In particular, if $_RM = {}_RR$, simple R-submodules of $_RR$ are called *simple ideals* of R. M is called *irreducible* if M contians no proper non-zero R-subgroups.

Remark: Every R-submodule is an R-subgroup. The converse, however, need not be true. Hence every irreducible R-module is simple, but every simple R-module is not necessarily irreducible.

Definition 2.3.6

Let A be an R-submodule of an R-module M (respectively ideal of R). The factor group M/A (respectively R/A) can be regarded as an R-module (respecitvely near-ring) called the *factor module* (respectively *factor near-ring*) by defining $r(m + A) = rm + A$ (respectively $(r_1 + A)(r_2 + A) = r_1r_2 + A$). The natural group epimorphism $\theta : M \to M/A$ (respectively $\theta : R \to R/A$) becomes an R-epimorphism (respectively near-ring epimorphism).

Definition 2.3.7

Let $\{M_i : i \in I\}$ be a family of R-submodules of an R-module M. Then M is the *direct sum* of the family $\{M_i : i \in I\}$ if the additive group $<M, +>$ is the direct sum of the normal subgroups $\{<M_i, +>: i \in I\}$. In this case, we write $M = \sum_{i \in I} \oplus M_i$.

Using the standard arguments, the following result can be proved.

Proposition 2.3.8

(i) Let $\{M_i : i \in I\}$ be a collection of R-submodules of an R-module M. The subgroup H of the additive group $(M, +)$ generated by

$$M^* = \bigcup_{i \in I} M_i$$

is an R-submodule of M and $H = \sum_{i \in I} M_i$ where $\sum_{i \in I} M_i$ denotes the collection of all finite sums of the elements from M^*.

(ii) $M = \sum_{i \in I} \oplus M_i \Leftrightarrow M = \sum_{i \in I} M_i$

and $M_{i0} \cap M = \sum_{i \neq i_0} M_i = (0)$ for each $i_0 \in I$.

(iii) $M = \sum_{i \in I} \oplus M_i \Leftrightarrow M = \sum_{i \in I} M_i$, every $m \in M$ has a unique representation $m = \sum m_i$, $m_i \in M_i$, and every element of M_i commutes with every element of M_j, for all $i, j, i \neq j$.

A submodule A of M is a *direct summand* if there exists a submodule B such that $A \oplus B = M$. If (0) and M are the only direct summands, M is *indecomposable.*

Using properties of submodules and direct sums, the following important near-ring analogue of the classical Wedderburn structure theorem for ring modules can be proved.

Theorem 2.3.9 [Pilz; 1983]

For a near-ring R-module M the following assertions are equivalent:

(1) Every R-submodule of M is a sum of simple submodules;

(2) M is a sum of simple submodules;

(3) M is a direct sum of simple submodules;

(4) Every submodule of M is a direct summand.

Definition 2.3.10

A near-ring module M is called *semisimple* if M satisfies one of the conditions of the above theorem. A near-ring R is called *semisimple* if ${}_R R$ is semisimple. Therefore, a near-ring R is semisimple if and only if R is the direct sum of simple left ideals. If R has an identity 1, the number of summands is finite.

Concluding Remarks: One of the most important questions in the structure theory of near-rings is to determine conditions which characterize semisimple and "strictly semisimple" near-rings (that is, near-rings which are direct sum of irreducible ideals). Such investigations have led to the discovery of many unring like properties of near-rings. For earlier developments in this direction we refer to [Pilz; 1983].

ACKNOWLEDGEMENT

The author is highly indebted to the King Fahd University of Petroleum and Minerals for providing all necessary research facilities during the preparation of this paper. The author is also greatly thankful to Professor Javed Ahsan for the encouragement and the valuable comments for the preparation of this paper.

REFERENCES

1. W.B. Adams, Near integral domains on nonabelian groups. *Monatsh. Math.* 81(1976), 178–183.

2. S.A. Amitsur, Finite subgroups of division rings. *Trans. Amer. Math. Soc.* 80(1955), 361–386.

3. I.H. Chen, Some combinatorial structures arising from finite planar near-rings. Ph. D. thesis, *Nat'l Chiao Tung Univ., Hsinchu*, 1991.

4. James R. Clay, Circular block designs from planar near-rings. *Ann. Discrete Math.* **37**(1988), 95–106.

5. James R. Clay, Near-rings: Geneses and Applications, Oxford Univ. Press, Oxford, 1992.

6. James R. Clay, Imbedding an arbitrary ring in a non-trivial near-ring, *Amer. Math. Monthly* **74**(1967), 406–407. MR 35-5476.

7. James R. Clay, and H. Kiechle, Linear codes from planar near-rings and möbius planes. *Algebras, Groups and Geometries* **10**(1993), 333–344.

8. G. Ferrero, Stems planari e bib-disegni. *Riv. Mat. Univ. Parma* (2) 11(1970), 79–96.

9. Y. Fong, G. Pilz, A. Oswald, and K.C. Smith, *Near-ring News Letter*, No. 16, January 1995.

10. P. Fuchs, G. Hofer, and G. Pilz, Codes from planar near-rings. IEEE *Trans. Inform. Theory* **36**(1990), 647–651.

11. W.-F. Ke and K.-S. Wang. On the Frobenius groups with kernel of order 64. *Contributions to General Algebra* **7**(1991), 221–233.

12. W.-F. Ke., Structures of circular planar near-rings. Ph.D. dissertation, Univ. Arizona, Tucson, 1992.

13. J.M. Hall and J.K. Senior. Groups of order 2^n, ($n \leq 6$). The Macmillan Company, New York, 1964.

14. J.D.P. Meldrum. Near-rings and their links with groups. Pitman, London, (1985).

15. M.C. Modisett. A characterization of the circularity of certain balanced incomplete block designs. Ph.D. dissertation, University of Arizona, 1988.

16. M. Modisett. A characterization of the circularity of balanced incomplete block designs, 83–94.

17. Gunter Pilz. Near-rings. North Holland Pub. Co. (1983).

18. O. Veblen and J.H. MacLagan-Wedderburn. Non-desarguesian and non-pascalian geometries. *Trans. Amer. Math. Soc.* **26**(1907), 379–388.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 22 – 29

ON SEMI-WEAKLY SEMI-CONTINUOUS MAPPINGS

Raja Mohammad Latif

Department of Mathematical Sciences,
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia.

ABSTRACT

We introduce semi-weakly semi-continuous mappings and investigate some of their properties.

1. INTRODUCTION

In 1985, T. Noiri and B. Ahmad [Noiri & Ahmad; 1985] introduced the concept of semi-weakly continuous mappings and studied their several properties. The purpose of the present note is to introudce a new class of mappings called semi-weakly semi-continuous mappings and investigate some properties analogous to those given in [Noiri & Ahmad; 1985] and [Noiri; 1974] concerning semi-weakly continuous and weakly continuous mappings respectively.

2. PRELIMINARIES

Let X be a topological space and let S be a subset of X. The closure and the interior of S are denoted by $Cl(S)$ and $\mathrm{Int}(S)$ respectively. A subset S is said to be semi-open [Levine; 1963] if there exists an open set U such that $U \subseteq S \subseteq Cl(U)$. $SO(X)$ will denote the class of all semi-open sets in a topological space X. Te cmplement of a semi-open set is called semi-closed. The union of all semi-open subsets of X contained in S is called the semi-interior of S and denoted by $s\mathrm{Int}(S)$. The intersection of all semi-closed subsets of X containing S is called the semi-closure of S and denoted by $sCl(S)$.

1991 Mathematics Subject Classification: Primary 54C; Secondary 54H.

Keywords and Phrases: Topological space semi-open set, semi-closed set, semi-weakly semi-continuous mapping, irresolute, semi-

weakly continuous mapping, S-connected space, T_2-space, semi-T_2 space, Urysohn space, s-Urysohn space.

Throughout this note, X and Y denote topological spaces and by $f : X \to Y$ we denote a mapping f of a space X into a space X.

3. SEMI-WEAKLY SEMI-CONTINUOUS MAPPINGS

Definition 1

A mapping $f : X \to Y$ is called semi-weakly semi-continuous (briefly s.w.s.c.) if for each point $x \in X$ and each semi-oepn set $V \subseteq Y$ containing $f(x)$, there exists a semi-open set $U \subseteq X$ containing x such that $f(U) \subseteq sCl(V)$.

Definition 2

A mapping $f : X \to Y$ is called an irresolute if and only if the inverse image of each semi-open set in Y, is a semi-open set in X.

Theorem 3

[Latif; 1993]. A mapping $f : X \to Y$ is called an irresolute if and only if for each point $x \in X$ and each semi-open set V containing $f(x)$ there exists a semi-open set U containing x such that $f(U) \subseteq V$.

Theorem 4

Let $f : X \to Y$ be an irresolute. Then f is semi-weakly semi continuous.

Proof

By using theorem 3, the result follows immediately.

The following example shows that the converse of theorem 4 may not be true in general.

Example 5

Let $X = \{1, 2, 3\}$. Let $T^* = \{\phi, \{1\}, X\}$ and $T = \{\phi, \{2\}, \{1, 2\}, X\}$ be topologies on X. Let $Id_X : (X, T^*) \to (X, T)$ be the identity map. Then Id_X is not an irresolute. We note that Id_X is semi-weakly semi-continuous because $sCl\{2\} = X$ in (X, T).

Definition 6

[Noiri, Ahmad; 1985]. A mapping $f : X \to Y$ is called semi-weakly continuous (briefly s.w.c) if for each point $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, there exists a semi-open set $U \subseteq X$ containing x such that $f(U) \subseteq sCl(V)$.

Theorem 7

Let $f : X \to Y$ be semi-weakly semi-continuous. Then f is semi-weakly continuous.

Proof

Note that every open set is a semi-open set.

The next example reveals that the converse of theorem 7 may not be true in general.

Example 8

Let $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3\}$. Let $T = \{\phi, \{1\}, X\}$ and $T^* = \{\phi, \{1\}, \{2\}, \{1, 2\}, Y\}$ be topologies on X and Y respectively. Define $f : X \to Y$ by $f(1) = f(2) = 1$, $f(3) = 3$ be the identity map. Then clearly f is semi-weakly continuous. Note that $f(3) = 3 \in \{2, 3\} \in SO(Y)$. The semi-open sets in X containing 3 are only $\{1, 3\}$ and X. Now $f(\{1, 3\}) = \{1, 3\}$ and $sCl(\{2, 3\}) = \{2, 3\}$. Thus $f(\{1, 3\}) \not\subset sCl(\{2, 3\})$. Hence f is not semi-weakly semi-continuous.

Theorem 9

Let $f : X \to Y$ be semi-weakly continuous. Then

$$sCl[f^{-1}(V)] \subseteq f^{-1}[Cl(V)]$$

for each semi-open set $V \subseteq Y$.

Proof

Suppose there exists a point $x \in sCl[f^{-1}(V)] - f^{-1}[Cl(V)]$. Then $f(x) \notin Cl(V)$. Hence there exists an open set W containing $f(x)$ such that $W \cap V = \phi$. Since V is semi-open, we have $V \cap sCl(W) = \phi$. Since f is semi-weekly continuous, there exists a semi-open set $U \subseteq X$ contianing x such that $f(U) \subseteq sCl(W)$. Thus we obtain $f(U) \cap V = \phi$. On the other hand, since $x \in sCl[f^{-1}(V)]$, we have $U \cap f^{-1}(V) \neq \phi$ and hence $f(U) \cap V \neq \phi$. We have a contradiction. Therefore we have $sCl[f^{-1}(V)] \subseteq f^{-1}[Cl(V)]$.

Theorem 10

Let $f: X \to Y$ be semi-weakly semi-continuous. Then

$$sCl[f^{-1}(V)] \subseteq f^{-1}[sCl(V)].$$

for each semi-open set $V \subseteq Y$.

Proof

Suppose there exists a point $x \in sCl[f^{-1}(V)] - f^{-1}[sCl(V)]$. Then $f(x) \notin sCl(V)$. Hence there exists a semi-open set W containing $f(x)$ such that $W \cap V = \phi$. Since V is semi-open, we have $V \cap sCl(W) = \phi$. Since f is semi-weakly semi-continuous, there exists a semi-open set $U \subseteq X$ containing x such that $f(U) \subseteq sCl(W)$. Thus we obtain $f(U) \cap V = \phi$. On the other hand sinec $x \in sCl[f^{-1}(V)]$, we have $U \cap f^{-1}(V) \neq \phi$ and hence $f(U) \cap V \neq \phi$. We have a contradiction. Thus we have $sCl[f^{-1}(V)] \subseteq f^{-1}[sCl(V)]$.

Theorem 11

Prove that a mapping $f : X \to Y$ is semi-weakly semi-continuous if and only if for every semi-open set T in Y $f^{-1}(T) \subseteq \text{sInt}[f^{-1}(\text{sCl}(T))]$.

Proof

Let $x \in X$ and T a semi-open set containing $f(x)$. Then $f(x) \in f^{-1}(T) \subseteq \text{sInt}[f^{-1}(sCl(T))]$. Put $S = \text{sInt}[f^{-1}(sCl(T))]$. Then S is semi-open and $f(S) \subseteq sCl(T)$. This shows that f is semi-weakly semi-continuous.

Conversely let T be a semi-open set of Y and $x \in f^{-1}(T)$. Then there exists a semi-open and $f(S) \subseteq sCl(T)$. Therefore we have $x \in S \subseteq f^{-1}[sCl(T)]$ and hence $x \in \text{sInt}[f^{-1}(sCl(T))]$. This proves that $f^{-1}(T) \subseteq \text{sInt}[f^{-1}(sCl(T))]$.

Theorem 12

Let $f: X \to Y$ be a mapping and $g: X \to X \times Y$ be the graph mapping of f, given by $g(x) = (x, f(x))$ for every point $x \in X$. If g is semi-weakly semi-continuous, then f is semi-weakly semi-continuous.

Proof

Let $x \in X$ and T be any semi-open set contianing $f(x)$. Then by theorem 11 of [Levine; 1963], $X \times T$ is a semi-open set in $X \times Y$ containing $g(x)$. Since g is semi-weakly semi-continuous, there exists a semi-open set S containing x such that $g(S) \subseteq sCl(X \times T)$. It follows from lemma 4 of [Noiri; 1978] that $sCl(X \times T) \subseteq X \times sCl(T)$. Since g is the graph mapping of f, we have $f(S) \subseteq sCl(T)$. This shows that f is semi-weakly semi-continuous.

Theorem 13

A space (X, T) is semi-T_2 if and only if for every $x, y \in X$ such that $x \neq y$, there exist disjoint semi-open sets U and V such that $x \in U$ and $y \in V$.

Theorem 14

If $f : X \rightarrow Y$ is a semi-weakly semi-continuous mapping and Y is semi-T_2, then the graph $G(f)$ is a semi-closed of $X \times Y$.

Proof

Let $(x, y) \notin G(f)$. Then, we have $y \notin f(x)$. Since Y is semi-T_2, there exist disjoint semi-open sets S and T such that $f(x) \in S$ and $y \in T$. Since f is semi-weakly semi-continuous, there exists a semi-open set R containing x such that $f(R) \subseteq \mathrm{sCl(S)}$. Since S and T are disjoint, we have $T \cap sCl(S) = \phi$ and hence $T \cap f(R) = \phi$. This shows that $(R \times T) \cap G(f) = \phi$. It follows from theorem 2 and 11 in [Levine; 1963] that $G(f)$ is semi-closed.

Definition 15

By a semi-weakly semi-continous retraction, we mean a semi-weakly semi-continuous mapping $f : X \rightarrow A$, where $A \subseteq X$ and f/A is the identity mapping on A.

Theorem 16

Let $A \subseteq X$ and $f : X \rightarrow A$ be a semi-weakly semi-continuous retraction of X onto A. If X is a Hausdorff space then A is a semi-closed set in X.

Proof

Note that f is semi-weakly continuous by theorem 7. Now the result follows from theorem 4 of [Noiri & Ahmad; 1985].

4. S-CONNECTED SPACES

Theorem 17

[Thompson; 1981]. A space X is said to be S-connected if X can not be written as the disjoint union of two non-empty semi-open sets.

It is already known that S-connectedness is invariant under semi-continuous surjections. Our next result shows that S-connectedness is invariant under semi-weakly semi-continuous surjections.

Theorem 18

If X is an S-conected space and $f : X \to Y$ is a semi-weakly semi-continuous surjection then Y is S-connected.

Proof

Suppose Y is not S-connected. Then there exist non-empty semi-open sets V_1 and V_2 of Y such that $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = Y$. Hence we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since f is surjective, $f^{-1}(V_i) \neq \phi$ for $i = 1, 2$. By theorem 11, we have $f^{-1}(V_i) \subseteq \mathrm{sInt}[f^{-1}(sCl(V_i))]$ because f is semi-weakly semi-continuous. Since V_i is semi-open and also semi-closed, we have $f^{-1}(V_i) \subseteq \mathrm{sInt}[f^{-1}(V_i)]$. Hence $f^{-1}(V_i)$ is semi-open for $i = 1, 2$. This implies that X is not S-connected. This is contrary to the hypothesis that X is S-connected. Therefore Y is S-connected.

Definition 19

A space X is called a Urysohn space if for every pair of distinct points x and y in X, there exist open sets U and V in X such that $x \in U, y \in V$ and $Cl(U) \cap Cl(V) = \phi$.

Theorem 20

If Y is Urysohn space and $f : X \to Y$ is a semi-weakly continuous injection, then X is semi-T_2 space.

Proof

For any distinct points x_1, $x_2 \in X$, we have $f(x_1) \neq f(x_2)$ because f is injective. Since Y is Urysohn, there exist open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $Cl(V_1) \cap Cl(V_2) = \phi$. Then $sCl(V_1) \cap sCl(V_2) = \phi$ since $sCl(V_j) \subseteq Cl(V_j)$ for $j = 1, 2$. Hence we have $\text{sInt}[f^{-1}(sCl(V_1))] \cap \text{sInt}[f^{-1}(sCl(V_2))] = \phi$. Since f is semi-weakly coontinuous, so by theorem 1 of [Noiri & Ahmad; 1985], we have $x_j \in f^{-1}(V_j) \subseteq \text{sInt}[f^{-1}(sCl(V_j))]$ for $j = 1, 2$. This implies that X is semi-T_2.

Definition 21

A topological space (X, T) is said to be s-Urysohn if for each pair x, y of distinct points in X, there exist U, $V \in S(X)$ such that $x \in U$, $y \in V$ and $Cl(U) \cap Cl(V) = \phi$.

Theorem 22

If Y is a s-Urysohn space and $f : X \rightarrow$ is a semi-weakly semi-continuous injection then X is semi-T_2.

Proof

For any distinct points $x, y \in X$ we have $f(x) \neq f(y)$ because f is injective. Since Y is s-Urysohn there exist semi-open sets U and V in Y such that $f(x) \in U$, $f(y) \in V$ and $Cl(U) \cap Cl(V) = \phi$. Hence we have $\text{sInt}[f^{-1}(sCl(U))] \cap \text{sInt}[f^{-1}(sCl(V))] = \phi$. Since f is semi-weakly semi-continuous, so by theorem 11 we have $x \in f^{-1}(U) \subseteq \text{sInt}[f^{-1}(sCl(U))]$ and $y \in f^{-1}(V) \subseteq \text{sInt}[f^{-1}(sCl(V))]$. This implies that X is semi-T_2.

Theorem 23

If X is an S-connected sapce and $f : X \rightarrow Y$ is an irresolute mapping with the semi-closed graph, then f is constant.

Proof

Suppose that f is not constant. Then tere exist distinct points x, y in X such that $f(x) \neq f(y)$. Since the graph $G(f)$ is semi-closed and $(x, f(y))$ is not in $G(f)$, there exist semi-open sets U and V containing x and $f(y)$, respecitvely, such that $f(U) \cap V = \phi$. Since f is irresolute, U and $f^{-1}(V)$ are disjoint non-empty semi-open sets. It follows from theorem 17 of [Thompson; 1981] that X is not S-connected. Therefore, f is constant.

Corollary 24

Let X be irreducible. If $f : X \to Y$ is an irresolute mapping with the semi-closed graph, then f is constant.

ACKNOWLEDGEMENT

The author is highly indebted to the University of Alberta and the King Fahd University of Petroleum and Minerals for providing financial aid and all necessary research facilities during the preparation of this paper.

REFERENCES

1. C. Dorsett, **Semi-T_2**, semi-R_1, and semi-R_0 topological sapces, *Annales de la Societe Scientifique de Bruxelles*, T. 92, III(1978), 143-150.
2. R.M. Latif, On characterizations of Mappings, *Soochow Journal of Mathematics*, Volume 19, No. 4 (1993), 475-495.
3. N. Levine, Semi-open sets and semi-coontinuity in topological spaces, *Amer. Math. Monthly*, 70(1963), 36-41.
4. P.E. Long and D.A. Carnahan, Comparing almost continuous functions, *Proc. Amer. Math. Soc.* 38(1973), 413-418.
5. S. Maheshwari and R. Prasad, Some new separation axioms, Annales Soc. Scient. Bruxelles, 89, III (1975), 395-402.
6. T. Noiri, A note on s-regular spaces *Glansik Mat.*, 13(33) (1978), 107-110.
7. T. Noiri and B. Ahmad, On semi-weakly continuous mappings, *Kyungpook Math. J.*, 25 (1985), 123-126.
8. T. Noiri, On weakly continuous mappings, *Proc. Amer. Math. Soc.*, 46(1974), 120-124.
9. M.K. Singal and A.R. Singal, Almost continuous mappings, *Yokahama Math. J.*, 16(1968), 63-73.
10. T. Thompson, Characterizations of irreducible spaces, *Kyungpook Math. J.*, 21(1981), 191-194.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 30 - 38

ON A CHARACTERIZATON OF $\hat{F}_4(2)$ THE REE-EXTENSION OF THE CHEVALLEY GROUP $F_4(2)$ ... II

Syed Muhammad Husnine

Department of Mathematics,
University of The Punjab, Lahore.

ABSTRACT

In this paper we obtain further results on the structure of a finite group G having a subgroup isomorphic to the centralizer of an involution in $F_4(2)$ such that the image of this involution in G is not central.

1. INTRODUCTION

The centre of a Syllow 2-subgroup, S, of the Chevalley Group $F_4(2)$ is the four-group. Following [3], we denote the three involutions of this centre by $x_{21}, x_{24}, x_{21}x_{24}$. According to section-2 of [3], we have, $S = \Pi S_i$, $i = 1, 2, ..., 24$; $M = D_{10} = \Pi S_i$, $i \neq 10$; $D_5 = \Pi S_i$, $i \neq 5$, where, S_i is a subgroup of order 2 generated by x_i. The centralizer of x_{21} in $F_4(2)$ is generated by the x_i's, w_1, w_2 and w_5 subject to the action of w_i's on the x_j's given in Table 1, the commutator relations between x_i and x_j given in Table 2, and the relations $w_1^2 = w_2^2 = w_5^2 = (w_1w_2)^4 = (w_2w_5)^3 = (w_1w_5)^2 = 1$.

We are investigating a finite group G in which, the centralizer of a noncentral involution y_1 is isomorphic to $C(x_{21})$ in $F_4(2)$. In Husnine [3], the following results have been proved.

Theorem A ([3])

Let G be a finite group with a noncentral involution y_1 such that $C = C_G(y_1)$ is isomorphic to $C_{F_4(2)}(x_{21}) = C_1$. Identify C with C_1. Then the following hold:

(i) $N_G(S)$ is a Sylow$_2$-subgroup of G and $[N_G(S) : S] = 2$.

(ii) $N_G(M)/M \cong \Sigma_3$ and $N_G(M) \subseteq C_G(x_{24})$.

(iii) S is its own normalizer in $N_G(M)$. In particular no two of the involutions x_{21}, x_{24} and $x_{21}x_{24}$ are conjugate in $N_G(M)$.

For necessary details about the group $F_4(2)$, we refer the reader to [3]. We, however, mention the following results from [1] and [2], which are repeatedly used in this paper.

(1.1) [1]. For any three elements x, y, z of a group G, we have

$$[x, yz] = [x, y]^z [x, z]$$

(1.2) [2]. Let x be an involution of S. Then x is conjugate in S to an involution of one of the forms $\prod_{i\in I} x_i$ or $\Phi\ (\prod_{i\in I} x_i)$ where I is one of the sets listed in Table 3 and Φ is the graph automorphism of $F_4(2)$.

The integers c(I) listed in Table 3 is such that the number of conjugate elements of $\prod_{i\in I} x_i$ in S is $2^{c(I)}$.

Table - 3

I	*c(I)*	*I*	*c(I)*	*I*	*c(I)*	*I*	*c(I)*	*I*	*c(I)*
24	0	9;17	4	7,8	7	4,18,21	7	5,7,17	9
21,24	0	9,17,21	4	7,8,21	7	4,16	8	5,7,17,21	9
23	1	9,16	5	7,8,16	7	4,16,18	8	5,7,16	10
21,23	1	9,16,21	5	7,8,16,21	7	4,15	9	5,7,16,21	10
17,23	2	9,15	7	7,8,14	9	4,15,18	9	5,11	11
22	2	9,14	7	6	6	4,12	10	5,10	12
21,22	2	9,13	8	6,21	6	4,11	11	2	7
17,22	3	18,	5	6,17	7	4,7	10	2,24	7
16,20	4	18,21	5	6,16	6	3,4	8	2,21	7
20	3	17,18	6	6,16,21	6	3,4,21	8	2,21,24	7
20,21	3	16,18	7	6,16,17	7	3,4,21,24	8	2,18	8
17,20	4	15,18	8	6,15	9	3,4,7	10	2,18,24	8
16,20	4	14,18	9	6,14	7	5	7	2,17	8
16,17,20	4	13,18	6	6,15,21	7	5,21	7	2,17,24	8
15,20	6	13,18,24	6	6,14,17	8	5,17	8	2,17,18	9
19	4	13,18,20	7	6,14,15	9	5,16	9	2,14	10
19,21	4	13,18,19	8	6,12	10	5,15	7	2,14,18	10
17,19	5	8	5	6,7	8	5,15,21	7	2,13	10
16,19	6	8,21	5	6,7,21	8	5,15,17	8	2,13,18	10
15,19	5	8,17	5	6,7,17	8	5,15,16	9	2,10	12
15,17,19	5	8,17,21	5	6,7,17,21	8	5,14	8	2,7	11
15,19,20	5	8,16	7	6,7,15	10	5,14,21	8	2,3	8
14,19	6	8,15	6	4	6	5,14,17	9	2,3,24	9
14,19,22	6	8,15,21	6	4,24	6	5,14,16	9	2,3,21	9
14,15,19,20	6	8,15,16	7	4,21	6	5,14,15,16	9	2,3,21,24	9
9	4	8,14	9	4,21,24	6	5,7	9	2,3,18	11
9,21	4	8,13	9	4,18	7	5,7,21	9	2,3,7	11

(1.3) [2]. Let $x \in S$ be an involution. Then there is an involution y of one of the forms listed in Table 4, such that x is conjugate in C_3, te centralizer of $x_{21}x_{24}$ in $F_4(2)$, to y or to $\Phi(y)$, where Φ is the graph automorphism of $F_4(2)$ defied in (2.14) of [3]. The sets $F_3(y)$, listed in Table 4 are complete sets of representatives of the distinct conjugacy classes of involutions in S, which are used in C_3 to give $S \cap \mathrm{ccl}_{C_3}(y)$. The integers $d(y)$ in Table 4 satisfy $d(y) = |S \cap \mathrm{ccl}_{C_3}(y)|$.

Table 4

y	$F_3(y)$	$d(y)$
x_{21}	$\{x_{21}\}$	1
$x_{21}x_{24}$	$\{x_{21}x_{24}\}$	1
x_{23}	$\{x_i \mid i = 19, 20, 22, 23\}$	$2(2^4-1)$
$x_{21}x_{23}$	$\{x_{21}u \mid u \in F_3(x_{23})$	$2(2^4-1)$
$x_{17}x_{23}$	$\{\Pi_{i\in I}x_i \mid I = \{17, 23\}, \{17, 22\}, \{16, 23\}, \{16, 20\}, \{15, 22\}, \{15, 19\}, \{14, 20\}, \{14, 19\}\}$	$2^2(2^2-1)(2^4-1)$
$x_{66}x_{22}$	$\{\Pi_{i\in I}x_i \mid I = \{17, 20\}, \{17, 19\}, \{16, 22\}, \{16, 19\}, \{16, 17, 20\}, \{15, 20\}, \{15, 23\}, \{15, 17, 19\}, \{15,22,23\}, \{15, 10, 20\}, \{14,22\}, \{14,23\}, \{14,16,19\}, \{14, 15, 20\}, \{14, 20, 23\}, \{14, 19, 22\}, \{14, 15, 19, 20\}\}$	$2^4(2^2-1)(2^4-1)$
x_9	$\{x_i \mid i = 5, 6, 8, 9\}$	$2^4(2^4-1)$
x_9x_{21}	$\{ux_{21} \mid u \in F_3(x_9)\}$	$2^4(2^4-1)$
x_9x_{17}	$\{\Pi_{i\in I}x_i \mid I = \{9, 17\}, \{9, 16\}, \{8, 17\}, \{8, 15\}, \{6, 16\}, \{6, 14\}, \{5, 15\}, \{5, 14\}\}$	$2^4(2^4-1)(2^4-1)$
$x_9x_{17}x_{21}$	$\{ux_{21} \mid u \in F_3(x_9x_{17})\}$	$2^4(2^2-1)(2^4-1)$
x_9x_{15}	$\{\Pi_{i\in I}x_i \mid I = \{9, 15\}, \{9, 14\}, \{8, 16\}, \{8, 14\}, \{8, 15, 16\} \{6, 17\}, \{6, 15\}, \{6, 14, 17\}, \{6, 16, 17\}, \{6, 14, 15\}, \{5, 17\}, \{5, 16\}, \{5, 14, 17\}, \{5, 15, 16\}, \{5, 14, 15, 16\}, \{5, 15, 17\}, \{5, 14, 16\}\}$	$2^7(2^2-1)(2^4-1)$

y	$F_3(y)$	$d(y)$
x_9x_{13}	$\{\Pi_{i\in I}\, x_i \mid I = \{9, 13\}, \{9, 12\},$ $2^8(22-1)\ (24-1)\ \{8, 11\}, \{8, 13\},$ $\{6,10\}, \{6,12\}, \{5, 16\}, \{5, 11\}\ \}$	$2^8(2^2-1)\ (2^4-1)$
x_{18}	$\{x_{18}\}$	2^5
$x_{18}x_{21}$	$\{x_{18}x_{21}\}$	2^5
$x_{14}x_{18}$	$\{x_ix_{18} \mid i = 14, 15, 16, 17\}$	$2^6(2^4-1)$
$x_{13}x_{18}$	$\{x_ix_{18} \mid i = 10, 11, 12, 13\}$	$2^6(2^4-1)$
$x_{13}x_{18}x_{24}$	$\{ux_{24} \mid u \in F_3(x_{13}x_{18})\}$	$2^6(2^4-1)$
$x_{13}x_{20}x_{18}$	$\{(\Pi_{i\in I}\, x_i)\, x_{18} \mid I = \{13, 20\},$ $\{13, 19\}, \{12, 22\}, \{12, 19\},$ $\{12, 22, 24\}, \{11, 23\}, \{11, 20\},$ $\{11, 23, 24\}, \{10, 23\}, \{10, 22\},$ $\{10, 23, 24\}, \{10, 22, 24\}\}$	$2^7(2^2-1)\ (2^4-1)$
x_4	$\{x_i \mid i = 2, 4\}$	$2^6(2^2-1)$
x_4x_{21}	$\{ux_{21} \mid u \in F_3(x_4)\}$	$2^6(2^2-1)$
x_4x_{24}	$\{ux_{24} \mid u \in F_3(x_4)\}$	$2^6(2^2-1)$
$x_4x_{21}x_{24}$	$\{ux_{21}x_{24} \mid u \in F_3(x_4)\}$	$2^6(2^2-1)$
x_4x_{16}	$\{\Pi_{i\in I}\, x_i \mid I = \{4, 16\}, \{4, 15\},$ $\{2, 17\}, \{2, 14\}, \{2, 17, 24\}\ \}$	$2^8(2^2-1)^2$
x_4x_7	$\{ux_7 \mid u \in F_3(x_4)\}$	$2^{10}(2^2-1)$
x_4x_{18}	$\{ux_{18} \mid u \in F_3(x_4)\}$	$2^7(2^2-1)$
$x_4x_{18}x_{21}$	$\{ux_{18}x_{21} \mid u \in F_3(x_4)\}$	$2^7(2^2-1)$
$x_4x_{16}x_{18}$	$\{(\Pi_{i\in I}\, x_i)\, x_{18} \mid I = \{4, 16\},$ $\{4, 15\}, \{2, 17\}, \{2, 14\}\ \}$	$2^8(2^2-1)^2$
x_4x_{12}	$\{(\Pi_{i\in I}\, x_i \mid x_{18} \mid I = \{4, 12\},$ $\{4, 11\}, \{2, 13\}, \{2, 10\},$ $\{2, 13, 18\}\ \}$	$2^{10}(2^2-1)^2$

y	$F_3(y)$	$d(y)$
x_3x_4	$\{(\Pi_{i\in I} x_i \mid x_{18} \mid I = \{3, 4\},$ $\{3, 2\}, \{1, 14\}\}$	$2^8(2-1)^2(1+2.2)$
$x_3x_4x_{24}$	$\{ux_{24} \mid u \in F_3(x_3x_4)\}$	$2^8(2-1)^2(1+2.2)$
$x_3x_4x_{21}x_{24}$	$\{ux_{21}x_{24} \mid u \in F_3(x_3x_4)\}$	$2^8(2-1)^2(1+2.2)$
$x_3x_4x_{18}$	$\{ux_{18} \mid u \in F_3(x_3x_4)\}$	$2^{10}(2-1)^2(1+2.2)$

We investigate the action of $N_G(S)$ on S in the next two sections. For this we denote by $Z_1(S)$ the centre of S and by $Z_i(S)$ the inverse image of the centre of $S/Z_{i-1}(S)$ in S for $i > 1$. Thus the action of $N_G(S)$ on $Z_1(S)$ is already given in Lemma (3.2) of [3].

2. THE ACTION OF $N_G(S)$ ON $Z_2(S)$:

Here we prove the following result:

Theorem B

Let G be a finite group with a noncentral involution y_1 such that $C = C_G(y_1)$ is isomorphic to $C(x_{21})$ in $F_4(2)$. Identify C with this centralizer. Then there exists an element u in $N_G(S)\backslash S$ such that $x_{21}^u = x_{24}$, $x_{24}^u = x_{21}$, $x_{23}^u = x_{17}$ and $x_{17}^u = x_{23}$; $u^2 \in M \cap D_5$.

Remark

The action of the involution to on $Z_4(S)$ turns out to be the same as that of the automorphism Φ of $F_4(2)$ mentioned in (2.14) section 2 of [3] as can be verified directly from Table I. Hence the importance of Theorem B.

Lemma (2.1)

$Z_1(S) = S_{21}S_{24}$; $Z_2(S) = S_{17}S_{23}Z_1(S)$; $Z_3(S) = S_{16}S_{22}Z_2(S)$;
$Z_4(S) = S_{15}S_{20}Z_3(S)$; $Z_1(M) = S_{17}S_{21}Z_{24}$; $Z_2(M) = S_{16}S_{23}Z_1(M)$;
$Z_3(S) = S_{15}S_{22}Z_2(M)$; $Z_1(D_5) = S_{23}S_{21}S_{24}$; $Z_2(D_5) = S_{17}S_{22}Z_1(D_5)$;
$Z_3(D_5) = S_{16}S_{20}Z_2(D_5)$;

Proof

This is directly verified from Table 2.

From now onwards, we write T for $N_G(S)$. So we have:

Lemma (2.2)

There is an element u in $T\backslash S$, such that $x_{17}^{u}=x_{23}$ and $x_{23}^{u}=x_{17}$ or $x_{17}^{u}=x_{23}x_{21}$ and $x_{23}^{u}=x_{17}x_{24}$.

Proof

Let $u \in T\backslash S$. Then by Lemma (3.2) of [3], we have, $x_{21}^{u} = x_{24}$, $x_{24}^{u} = x_{21}$, $M^u = D_5$ and $(D_5)^u = M$. Thus $Z_1(M)^u = Z_1(D_5)$ and $Z_1(D_5)^u = Z_1(M)$. Since u normalizes $Z_1(S)$, Lemma 1 gives us, $x_{23}^{u} = x_{17}z$, z $Z_1(S)$. If $x_{23}^{u} = x_{17}x_{21}$ or $x_{17}x_{21}x_{24}$, then $x_{23}^{ux_{10}} = x_{17}$ or $x_{17}x_{24}$ from Table 2 and ux_{10} acts upon x_{21} and x_{24} in the same manner as u. We write u for ux_{10} in such a case. Now $x_{23}^{u} = x_{17}$ implies x_{17}^{u} is a conjugate of x_{23} by u^2. Since $u^2 \in S$, we must have $x_{17}^{u} = x_{23}$ or $x_{23}x_{24}$ from Tables 2 and 3. If $x_{17}^{u} = x_{23}x_{24}$, then ux_5 conjugates x_{17} to x_{23}. Again $x_{23}^{u} = x_{17}x_{24}$ implies $(x_{17}x_{24})^u = x_{23}$ or $x_{23}x_{24}$ i.e. $x_{17}^{u} = x_{23}x_{21}$ or $x_{23}x_{21}x_{24}$. If $x_{17}^{u} = x_{23}x_{21}x_{24}$ then ux_5 conjugates x_{17} to $x_{23}x_{21}$. In both the cases ux_5 acts upon x_{21}, x_{24}, x_{23} in the same manner as u. We write u for ux_5 in such a case. This proves the lemma.

Lemma (2.3)

There is an element u in $T\backslash S$, such that $x_{23}^{u} = x_{17}$, $x_{17}^{u} =$ x23 and $u^2 \in M \cap D_5$.

Proof

Let u be an element of $T\backslash S$, satisfying Lemma (2.2). Then $(Z_2(M))^u = Z_2(D_5)$ and u normalizes $Z_2(S) = Z_1(M)\ Z_1(D_5)$. Thus $x_{16}^{u} = x_{22}z$, $z \in Z_2(S)$. Now from Table 2, $x_{17}^{u} = [x_1, x_{16}]^u = [x_1^u, x_{22}z]$ by (1.1). But $x_1^u \in M \cap D_5$ by Lemma 3.2 of [3]. Hence by Table 2, $[x_1^u, z] = 1$ and $[x_1^u, x_{22}] = 1$, x_{23} or x_{24}. This forces $[x_1^u, x_{22}]^z = [x_1^u, x_{22}]$, since $z \in Z_2(S)$. Thus $x_{17}^{u} = [x_1^u, x_{22}]$. This means $x_{17}^{u} = x_{22}^{x_1^u}x_{22}$.

Now let $x_{17}^{u} = x_{23}x_{21}$. Then $x_{22}^{x_1^u} = x_{23}x_{22}x_{21} = (x_{22}x_{21})^{x_2}$ by Table 2. This implies x_{22} is conjugate to $x_{22}x_{21}$ in S. But according to Table 3, these two elements belong to different conjugacy classes in S. Thus $x_{17}^{u} = x_{23}$.

Finally, we note that $u_2 \in S$ and centralizes x_{17} and x_{23}. But $C_S(x_{17}) = M$ and $C_S(x_{23}) = D_5$. Hence $u^2 \in M \cap D_5$. This proves the Lemma and thereby Theorem B is established.

3. ACTION OF $N_G(S)/S$ ON $Z_4(S)$

We prove the following theorem in this section:

Theorem C

Let G be a finite group with a noncentral involution y_1 such that, $C = C_G(y_1)$ is isomorphic to the centralizer of x_{21} in $F_4(2)$. We identify C with this centralizer. Then the following hold:

(i) There is an element u in $N_G(S)\backslash S$, such that u acts on $Z_4(S)$ as the graph automorphism of $F_4(2)$, and u^2 does not involve $x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_{10}, x_{11}, x_{12}$.

(ii) There is an element w in $N_G(M)$, such that w acts upon $Z_4(S)$ as w_{10} in $F_4(2)$.

Lemma (3.1)

There is an element u in $N_G(S)\backslash S$ such that u acts on $Z_2(S)$ as in Theorem B and $x_{22}^{u} = x_{16}$.

Proof

Let u be as in Theorem B. Then u permutes M and D_5. Thus u permutes $Z_2(M)$ and $Z_2(D_5)$. Since u normalizes $Z_2(S)$, we have, due to Lemma (2.1), $x_{22}^{u} = x_{16}^{z}$, $z \in Z_2(S)$. Now, if x_{23} occurs in z, we have $[x_5,x_{22}^{u}] = [x_5,x_{23}] = x_{24}$ by (1.1) and Table 2. Thus $[x_5^{u},x_{22}^{u^2}] = x_{21}$ which means $x_{21} = [x_5^{u}, x_{22}x_{23}(\alpha)\, x_{24}(\beta)]$, $\alpha, \beta \in \{0, 1\}$. But $x_5^{u} \in D_5$. Thus, $x_{21} = [x_5^{u}, x_{22}^{u^2}]$ which implies x_{22} is conjugate to $x_{21}x_{22}$ in S. Since this contradicts Table 3, x_{23} cannot occur in z.

If x_{17} occurs in z, $x_{22}^u = x_{16}x_{17}w$, $w \in Z_1(S)$. Thus ux_1 conjugates x_{22} to $x_{16}w$ and acts on $Z_2(S)$ as in Theorem B according to Table 2. We write u for ux_1. Thus, $x_{22}^u = x_{16}w$. If x_{21} occurs in w, then ux_{11}; takes x_{22} to x_{16} or $x_{16}x_{24}$ and acts on $Z_2(S)$ as in Theorem B. We write u for ux_{11}. Thus, $x_{22}^u \in \{x_{16}, x_{16}x_{24}\}$. Let $x_{22}^u = x_{16}x_{24}$. Then uw_1u^{-1} takes x_{22} to $x_{23}x_{21}$ and centralizes x_{21} and x_{24}. This contradicts Table 4. Thus $x_{22}^u = x_{16}$ and the Lemma is established.

Lemma (3.2)

There is an element u in $N_G(S)\backslash S$ such that u acts on $Z_2(S)$ as in Theorem B, and permutes x_{16} and x_{22}.

Proof

Let u be as in Lemma (3.1). Then $x_{16}^u = x_{22}^{u^2}$, where $u^2 \in M \cap D_5$. Thus, from Table 2, $x_{16}^u = x_{22}x_{23}(\alpha)\, x_{24}(\beta)$, α, $\beta \in \{0, 1\}$. Now from Table 2, $ux_2(\alpha)x_6(\beta)$ takes x_{16} to x_{22} and x_{22} to x_{16} and acts on $Z_2(S)$ as u. We write u for $ux_2(\alpha)\, x_6(\beta)$ and the lemma is proved.

Lemma (3.3)

There is an element u in $N_G(S)\backslash S$, such that u satisfies Lemma (3.2) and $x_{20}^u = x_{15}$.

Proof

Let u be as in Lemma (3.2). Then u normalizes $Z_3(S)$ and $Z_4(S)$ and permutes $Z_3(M)$ and $Z_3(D_5)$. Thus by Lemma (2.1), $x_{20}^u = x_{15}z$, $z \in Z_3(S)$. If x_{23} occurs in z, then, $[x_5, x_{20}^u] = x_{24}$ by (1.1) and Table 2. Thus $x_5^{u^{-1}}$ takes x_{20} to $x_{20}x_{21}$. This contradicts Table 3. If x_{22} occurs in z, then $x_6^{u^{-1}}$ conjugates x_{20} to $x_{20}x_{21}$, again contradicting Table 3.

If x_{17} occurs in z, then $x_{20}^u = x_{15}x_{17}w$, $w \in S_{16}S_{21}S_{24}$. Thus ux_3 conjugates x_{20} to $x_{15}w$ and satisfies Lemma (3.2). We write u for ux_3. If x_{21} occurs in w, we take ux_{12} for us and call it u. This u satisfies Lemma (3.2) and $x_{20}^u = x_{15}s$, $s \in S_{16}S_{24}$.

Let $x_{20}^{u} = x_{15}x_{24}$. Then uw_2u^{-1} conjugates x_{20} to $x_{21}x_{22}$. But uw_2u^{-1} centralizes x_{21} and x_{24}. This contradicts Table 4 since x_{20} is conjugate to x_{23} and $x_{21}x_{22}$ is conjugate to $x_{21}x_{23}$ in $C_3 = C(x_{21}) \cap C(x_{24})$ in $F_4(2)$.

Let $x_{20}^{u} = x_{15}x_{16}$. Then $ux_2w_2u^{-1}$ is in C_3 and conjugates x_{20} to $x_{21}x_{22}$. This also contradicts Table 4.

Finally let $x_{20}^{u} = x_{15}x_{16}x_{24}$. Then uw_2u^{-1} conjugates $x_{17}x_{20}$ to $x_{16}x_{20}$. But $x_{17}x_{20}$ is conjugate to $x_{16}x_{22}$ and $x_{16}x_{22}$ is conjugate to $x_{17}x_{23}$ as shown in Tanle 4. This again is a contradiction since uw_2u^{-1} is in C_3. Hence the Lemma.

Lemma (3.4)

There is an element u in $N_G(S)\backslash S$ such that u satisfies Lemma (2.2) and permutes x_{15} and x_{20}.

Proof

Let u be an element as in Lemma (3.3). Then $x_{15}^{u} = x_{20}^{u^2}$, $u^2 \in M \cap D_5$ and u^2 centralizes x_{16} and x_{22}. Thus, from Table 2, and (2.8) of [3], we find that x_1, x_2, x_5, x_6, x_{10}, x_{11} cannot occur in u^2. Thus, $x_{15}^{u} = x_{20}x_{23}(\alpha)\, x_{24}(\beta)$ where $\alpha, \beta \in \{0, 1\}$. Now $ux_4(\alpha)\, x_8(\beta)$ conjugates x_{15} to x_{20} and satisfies Lemma (3.3). Thus renaming $ux_4(\alpha)\, x_8(\beta)$ as u, we have proved the Lemma.

Proof of Theorem C

Let u be an element in $NG(S)\backslash S$ such that u satisfies Lemma (3.4). Then u^2 centralizes x_i for i = 15, 16, 17, 20, 21, 22, 23, 24. Thus, from Table 2, u^2 cannot involve x_i for i = 1, 2, 3, 4, 5, 6, 8, 10, 11, 12.

Finally, we write $w = u^{-1}w_5u$. Then w acts on $Z_5(S)$ as w_{10} due to Lemma (3.4) and Table 1. This proofs all parts of the Theorem.

REFERENCES

1. Gorenstein, D. Finite Groups, Harper & Row, New York, 1968.
2. Gutermann, M.A., A Charaterizaion of $F_4(2^n)$, J. Algebra, 20(1972), 1–23.
3. Husnine, S.M. On a Characterization of $\widehat{F_4(2)}$, The Ree-Extension of the Chevalley Group $F_4(2)$... I. PUJM, Vol. XXVI. (1993), 101–110.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 39 - 46

ON A CHARACTERIZATON OF $\widehat{F_4(2)}$ THE REE-EXTENSION OF THE CHEVALLEY GROUP $F_4(2)$ ... III

Syed Muhammad Husnine

Department of Mathematics,
University of The Punjab,
Lahore.

ABSTRACT

In this paper we improve the result of Husnine [4] on a group G with a noncentral involution y_1 such that the centralizer of y_1 in G is isomorphic to the centralizer of a central involution in the Chevalley Group $F_4(2)$.

ABSTRACT

The centre of a Syllow 2-subgroup, S, of the Chevalley Group $F_4(2)$ is the four-group. Following [3], we denote the three involutions of this centre by $x_{21}, x_{24}, x_{21}x_{24}$.

According to secton 2 of [3], we have $S = \Pi S_i$, $i = 1, ..., 24$; $M=D_{10}=\Pi S_i$, $i \neq 10$; $D_5=\Pi S_i$, $i \neq 5$, where, $S_i = <x_i>$ is a subgroup of order 2. The centralizer of x_{21} in $F_4(2)$ is generated by the x_i's, w_1, w_2, and w_5 subject to the action of w_i's on the x_j's given in Table-I, the commutator relations between x_i and x_j given in Table-2, and the relations $w_1^2 = w_2^2 = w_5^2 = (w_1w_2)^4 = (w_2w_5)^3 = (w_1w_5)^2 = 1$ as stated in [3].

In Husnine [4], the following result has been proved:

Theorem C

Let G be a finite group with a noncentral involution y_1 such that, $C = C_G(y_1)$ is isomorphic to the centralizer of x_{21} in $F_4(2)$. We identify C with this centralizer. Then the following hold:

(i) There is an element u in $N_G(S)\backslash S$, such that u acts on $Z_4(S)$ as the graph automorphism of $F_4(2)$, and u^2 does not involve $x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_{10}, x_{11}, x_{12}$.

(ii) There is an element w in $N_G(M)$, such that w acts upon $Z_4(S)$ as w_{10} in $F_4(2)$.

We refer the reader to [3], for the description of the group $F_4(2)$. All notations are standard and follow [1] and [2]. The only information we are quoting from [2] on the conjugacy of involutions in $C(x_{21})$ in $F_4(2)$ is as follows:

(1.1) [2]. Let x be an involution in S. Then x is conjugate in C_1, $C_1 = C_{F_4(2)}(x_{21})$ to an involution y of one of the forms listed in Table-5. The sets $F_1(y)$ satisfy $S \cap \mathrm{ccl}_{C_1}(y) = \bigcup_{v \in F_1(y)} (S \cap \mathrm{ccl}_{C_3}(v))$.

Throughout the remainder of this paper, we shall refer to the Tables 1 & 2 of [3], Table-3 and Table-4 of [4] and Table-5 of this paper by their numbers without refering to the papers.

Table - 5

y	$F_1(y)$
x_{21}	$\{x_{21}\}$
x_{24}	$\{x_i \mid i = 18, 23, 24\}$
$x_{21}x_{24}$	$\{ux_{21} \mid u \in F_1(x_{24})\}$
x_{17}	$\{x_i \mid i = 13, 17\}$
$x_{17}x_{24}$	$\{\Pi_{i \in I} x_i \mid I = \{17,24\}, \{17,23\}, \{13,18\}, \{13,23\}\}$
$x_{16}x_{22}$	$\{\Pi_{i \in I} x_i \mid I = \{16,22\}, \{13,24\}, \{13,20\}, \{17,18\}, \{13, 18, 24\}, \{13, 23, 24\}, \{13, 18, 20\}\}$
x_9	$\{x_i \mid i = 4, 9\}$
x_9x_{21}	$\{ux_{21} \mid u \in F_1(x_9)\}$
x_9x_{17}	$\{\Pi_{i \in I} x_i \mid I = \{9,17\}, \{4,18\}, \{4,24\}\}$
$x_9x_{17}x_{21}$	$\{ux_{21} \mid u = F_1(x_9x_{17})\}$
x_9x_{15}	$\{\Pi_{i \in I} x_i \mid I = \{9,15\}, \{9,13\}, \{4,12\}, \{4, 16\}, \{4, 16, 18\}\}$
x_7	$\{x_i \mid i = 3, 7\}$
x_7x_{24}	$\{x_3x_{21}, x_7 x_{24}\}$

y	$F_1(y)$
x_7x_{23}	$\{\Pi_{i\in I} x_i \mid I = \{7,23\}, \{3,18\}, \{3,22\}, \{3,24\}, \{3, 21, 24\}\}$
x_7x_9	$\{\Pi_{i\in I} x_i \mid I = \{7,9\}, \{3,4\}, \{3,7\}\}$
$x_7x_9x_{21}$	$\{x_7x_9x_{21}, x_3x_4x_{21}, x_3x_7x_{24}\}$
$x_7x_9x_{15}$	$\{\Pi_{i\in I} x_i \mid I = \{7,9,15\}, \{3,4,18\}, \{3,7,22\}, \{3,4,24\}, \{3,4,21,24\}\}$
x_4x_7	$\{\Pi_{i\in I} x_i \mid I = \{4, 7\}, \{3, 8\}, \{3, 4, 7\}\}$

act of $N_G(S)/S$ on $Z_6(S)$.

We prove the following result in this section.

Theorem D

Let G be a finite group with a noncentral involution y_1, such that $C = C_G(y_1)$ is isomorphic to the centralizer of x_{21} in $F_4(2)$. We identify C with this centralizer. Then the following hold:

(i) There is an element u in normalizer of S in G such that u acts on $Z_5(S)$ as the graph automorphism of $F_4(2)$, permutes x_8 and x_{12}, $x_7^u = x_{18}x_{21}(\alpha)\, x_{24}(\beta)$ and $x_{18}^u = x_7x_{21}(\beta)x_{24}(\alpha)$; $\alpha, \beta \in \{0, 1\}$.

(ii) There is an element w in $N_G(M)$, such that w acts upon $Z_5(S)$ as w_{10} in $F_4(2)$.

Lemma (2.1)

$$Z_5(S) = S_9S_{13}S_{14}S_{19}Z_4(S);\ Z_6(S) = S_7S_8S_{12}S_{18}Z_5(S)$$
$$Z_4(D_5) = S_9S_{13}S_{18}S_{19}Z_4(S);\ Z_5(D_5) = S_4S_8S_{12}S_{18}Z_5(S)$$
$$Z_4(M) = S_7S_9S_{13}S_{14}Z_4(S);\ Z_5(M) = S_3S_7S_8S_{12}Z_5(S).$$

Proof

This is verified by direct calculation from Table-2, keeping in view (2.8) of [3] and (2.1) of [4].

Lemma (2.2)

There is an element u in $N_G(S)$ such that u satisfies Theorem C and $x_{19}^u = x_{14}$.

Proof

Let u be an element of $N_G(S)$ such that u satisfies theorem C. Then u normalizes $Z_5(S)$ and maps $Z_4(D_5)$ onto $Z_4(M)$. Thus, due to (2.1), we must have $x_{19}^u = x_9(\alpha)\, x_{13}(\beta)\, x_{14}(\gamma)\, z$, $z \in Z_4(S)$; $\alpha, \beta, \gamma \in \{0, 1\}$. If $\alpha \neq 0$, we have, by Table-2, $[x_{18}, x_{19}^u] = x_{23}$. This implies $[x_{18}^u, x_{19}] = x_{17}$ which means x_{19} is conjugate to $x_{17}x_{19}$ in S. This contradicts Table-3. Hence $\alpha = 0$. Thus we have $x_{19}^u = x_{13}(\beta)\, x_{14}(\gamma)\, z$, $z \in Z_4(S)$. If $\beta \neq 0$, we have, $[x_7, x_{19}^u] = x_{17}$ which means $[x_7^{u^{-1}}, x_{19}] = x_{23}$. This implies, x_{19} is conjugate to $x_{19}x_{23}$ in S, a contradiction to Table-3. Thus $\beta = 0$. Now γ cannot be 0. So, $x_{19}^u = x_{14}z$, $z \in Z_4(S)$. If x_{15} appears if z, then $[x_3, x_{19}^u] = x_{16}x_{17}$. This implies $x_3^{u^{-1}}$ conjugates x_{19} to $x_{19}x_{22}x_{23}$. But $(x_{19}x_{22}x_{23})^{x_4x_3} = x_{19}x_{21}$. Thus x_{19} becomes a conjugate of $x_{19}x_{21}$, a contradicton to Table-3. Hence x_{15} does not appear in z. Let x_{16} appear in z. Then $x_{19}^u = x_{14}x_{16}x_{20}(\alpha)z'$, $\alpha \in \{0, 1\}$ $z' \in S_4(S) \cap C_S(x_1)$. So, ux_1u^{-1} conjugates x_{19} to $x_{19}x_{20}x_{23}$ $x_{16}(\alpha)x_{24}(\alpha)$, which is conjugate to $x_{16}(\alpha)$ $x_{19}x_{21}$ in S through $x_2x_3x_9(\alpha)$. This shows x_{19} is conjugate in S to $x_{19}x_{21}$ if $\alpha = 0$, and to $x_{16}x_{19}x_{21} = (x_{16}x_{19})^{x_{11}}$ if $\alpha = 1$. Thus x_{16} cannot appear in x_{19}^u, due to Table-3.

Now $x_{19}^u = x_{14}x_{20}z'$ implies ux_1u^{-1} conjugates x_{19} to $x_{16}x_{19}x_{20}x_{24}$ which is conjugate to $x_{16}x_{19}$ via x_2x_9. This contradicts Table-3. Thus $x_{19}^u = x_{14}z'$. If x_{22} appears in z', ux_2u^{-1} conjugates x_{19} to $x_{17}x_{19}$ and Table-3 is violated. Thus $x_{19}^u \in x_{14}S_{17}S_{21}S_{23}S_{24}$. If x_{23} appears in x_{19}^u, then $[x_5, x_{19}^u] = x_{24}$. Applying u^{-1} on both sides, we find that x_{19} becomes conjugate to $x_{19}x_{21}$ in S, a contradiction to Table-3. Thus $x_{19}^u \in x_{14}S_{17}S_{21}S_{24}$. Let $x_{19}^u = x_{14}x_{21}z$, $z \in S_{17}S_{24}$, then ux_{13} conjugates x_{19} to $x_{14}z$, and acts on $Z_4(S)$ in the same way as u. Writing u for ux_{13}, we have $x_{19}^u = x_{14}x_{17}(\alpha)\, x_{24}(\beta)$, $\alpha,\beta \in \{0,1\}$. Now $uw_1u^{-1} \in C_3 = C(x_{21}) \cap C(x_{24})$ in G, and it conjugates x_{19} to $x_{14}z$, and acts on α, $\beta \in \{0, 1\}$. Now $uw_1u^{-1} \in C_3 = C(x_{21}) \cap C(x_{24})$ in G, and it conjugates x_{10} to $x_{20}x_{21}(\beta)\, x_{22}(\alpha)$. If $\alpha \neq \beta$, x_{19} becomes conjugate to $x_{20}x_{21}$ or to $x_{20}x_{22}$ in C_3. But w_1w_2 conjugates $x_{20}x_{21}$ to $x_{21}x_{23}$ and x_1 conjugates $x_{20}x_{22}$ to $x_{20}x_{21}$ by Table 1 and 2. Thus x_{19} becomes conjugate to $x_{21}x_{23}$. This violates Table 4. Hence $x_{19}^u \in \{x_{14}, x_{14}x_{17}x_{24}\}$. Finally, if $x_{19}^u = x_{14}x_{17}x_{24}$, then uw_1u^{-1} is in C_3 and conjugates $x_{15}x_{19}$ to $x_{16}x_{20}x_{21}x_{22}$, which is conjugate to $x_{15}x_{17}x_{19}$ via x_1w_2. Since w_1w_2 is also in S_3, we find that $x_{15}x_{19}$ becomes conjugate to $x_{15}x_{17}x_{19}$ in C_3. This contradicts Table 4. Hence the lemma.

Lemma (2.3)

There is an element u in $N_G(S)$ such that u satisfies Theorem C

$$\text{and} \quad x_{19}^u = x_{14}\,; \quad x_{19}^u = x_{19}.$$

Proof

Let u be as in Lemma (2.2). Then $x_{14}^u = x_{19}^{u^2}$. Thus, from Theorem C and Table 2, we are left with $x_{14}^u = x_{19}$ or $x_{19}x_{24}$. If $x_{14}^u = x_{19}x_{14}$, then $x_{19}\, x_{14}$, then ux_9 conjugates x_{14} to x_{19}, x_{19} to x_{14} and acts as u on $Z_4(S)$. We replace u by ux_9 in this case and call it u. This proves the Lemma.

Lemma (2.4)

There is an element u in $N_G(S)$ such that u satisfies Lemma (2.3) and $x_9^u = x_{13}x_{24}(\alpha)$; $x_{13}^u = x_9x_{21}(\alpha)$, $\alpha \in \{0, 1\}$.

Proof

Let u be as in Lemma (2.3). Then u normalizes $Z_5(S)$ and permutes $\{Z_4(M), Z_4(D_5)\}$. Thus by Lemma (2.1), $x_9^u \in Z_4(M) \cap Z_4(D_5)$ $= S_9S_{13}Z_4(S)$. Let $x_9^u = x_9(\alpha)\, x_{13}(\beta)\, z$, $z \in Z_4(S)$, α, $\beta \in \{0,1\}$. Then $ux_{19}u^{-1}$ conjugates x_9 to $x_9x_{21}(\alpha)$. Thus by Table 3, we have $x_9^u = x_{13}z$, $z \in Z_4(S)$. If x_{23} appears in z, 5, $x_9^u] = [x_5, z]\, [x_5, x_{13}]^z =$

$x_{15}x_{22}x_{24}$. This implies x_9 is conjugate to $x_{16}x_{20}x_{21}$ via ux_5u^{-1}, which is in S. But x_{10} conjugates $x_9x_{16}x_{20}x_{21}$ to x_9x_{21}. Thus x_9 becomes a conjugate of x_9x_{21} in S, a contradiction to Table 3.

Let x_{22} appear in z. Then $[x_6, x_9^u] = [x_6, z]\,[x_6, x_{13}]$. Thus $[x_6^{u^{-1}}, x_9] = x_{17}x_{21}x_{22}$. This means $x_6^{u^{-1}}$ conjugates x_9 to $x_9x_{17}x_{21}x_{22} = (x_9x_{21})^{x_{11}}$. This violates Table 3.

Let x_{15} appear in z. Then $[x_2, x_9^u] = x_{16}x_{24}$. This implies x_9 is conjugate to $x_9x_{21}x_{22} = (x_9x_{17}x_{21})^{x_{11}}$ in S. This contradicts Table 3.

Let x_{20} appear in z. Then $[x_8, x_9^u] = x_{24}$. This forces x_9 to be a conjugate of x_9x_{21} in S, a violation of Table 3.

Now let $x_9^u = x_{13}x_{16}x_{17}\,(\alpha)\,x_{21}\,(\beta)\,x_{24}\,(\gamma)$. Then uw_2u^{-1} conjugates x_9 to $x_9x_{20}x_{22} = (x_9x_{17}x_{21})^{w_1x_1x_{11}x_{10}}$ and belongs to $C(x_{21})$ in G. This contradicts Table 5. Thus we have, $x_9^u = x_{13}x_{17}(\alpha)\,x_{21}\,(\beta)\,x_{24}\,(\gamma)$. But then, $ux_7(\alpha)\,x_{14}(\beta)$ conjugates x_9 to $x_9x_{24}(\gamma)$ and satisfies Lemma (2.3). We write u for $ux_7(\alpha)\,x_{14}(\beta)$. Thus $x_9^u = x_{13}x_{24}\,(\alpha)$. Now $x_{13}^u = x_9^{u^2}\,x_{21}(\alpha) = x_9x_{21}(\alpha)\,x_{23}(\beta)\,x_{24}(\gamma)$ by Theorem C and Table 2. Thus $ux_{18}(\beta)\,x_{19}(\gamma)$ conjugates x_9 to $x_{13}x_{24}(\alpha)$ x_{13} to $x_9x_{21}(\alpha)$ and satisfies Lemma (2.3). We write u for $ux_{18}(\beta)\,x_{19}(\gamma)$ and the Lemma is established.

Lemma (2.5)

There is an elemnt u in $N_G(S)$ such that u satisfies Lemma (2.4); $x_7^u = x_{18}x_{21}(\alpha)\,x_{24}(\beta)$ and $x_{18}^u = x_7x_{21}(\beta)\,x_{24}(\alpha)$.

Proof

Let u be an element of $N_G(S)$ satisfying Lemma (2.4). Then u normlizes $Z_4(S)$ and maps $Z_4(M)$ onto $Z_4(D_5)$. Thus from Lemma (2.1), $x_7^u = x_{18}x_{19}(\alpha)\,x_{13}(\beta)\,x_9(\gamma)\,z$, $z \in Z_5(S)$. Now $[x_{19}, x_7^u] = x_{24}(\gamma)$ by Table 2. Hence $x_{19}^{u^{-1}}$ conjugates x_7 to $x_7x_{21}\,(\gamma)$ which contradicts Table 3 if $\gamma = 1$. So $\gamma = 0$. Now $[x_{14}, x_7^u] = x_{21}(\beta)$ which implies x_7 is conjugate to $x_7x_{24}(\beta)$ in S. Table 3 again forces $\beta = 0$. Thus $x_7^u = x_{18}x_{19}(\alpha)\,z$, $z \in Z_4(S)$.

If x_{15} occurs in z, $[x_{12}, x_7^u] = x_{21}$ which makes x_7 a conjugate to x_7x_{21}, if x_{16} occurs in z, $[x_{12}, x_7^u] = x_{21}$ which makes x_7 a conjugate to x_7x_{24}, if x_{17} occurs in z, $[x_{10}, x_7^u] = x_{21}$ which makes x_7 a conjugate to x_7x_{24}, all contradicting Table 3. If x_{20} occurs in z, $[x_1, x_7^u] = x_{21}x_{22}$. This means x_7 is conjugate to $x_7x_{16}x_{24} = (x_7x_{24})^{x_{12}}$ in S, violating Table 3. Let x_{22} appear in z. Then $[x_6, x_7^u] = [x_6, x_{18}x_{22}] = x_{20}x_{24}$. This implies x_7 is conjugate to $x_7x_{15}x_{21} = (x_7x_{24})^{x_{11}x_{18}}$ by Table 2. This offends Table 3. If x_{23} occurs in z,$[x_5,x_7^u] = x_{24}$ making x_7 conjugate to x_7x_{21}. This contradiciton to Table 3 leaves us with $x_7^u = x_{18}x_{19}(\gamma)\, S_{21}S_{24}$. Finally, $[x_3, x_7^u] = x_{21}x_{23}$ if $\gamma = 1$. This shows x_7 is conjugate to $x_7x_{17}x_{24} = (x_7x_{24})^{x_{13}}$ in S, which violates Table 3. Thus γ must be 0. We have shown, $x_7^u = x_{18}x_{21}(\alpha)\, x_{24}(\beta)$. Now $x_{18}^u = x_7^{u^2} x_{21}(\beta)\, x_{24}(\alpha)$. But u^2 centralizes $Z_4(S)$ by Theorem C, x_{14} and x_{19} by Lemma (2.3), x_9 and x_{13} by Lemma (2.4). Thus, due to Table 2, u^2 cannot involve x_i for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 18, 19. So u^2 centralizes x_7 and x_{18}. Hence $x_{18}^u = x_7x_{21}(\beta)\, x_{24}(\alpha)$, and the Lemma is proved.

Lemma (2.6)

There is an element u in $N_G(S)$ such that u satisfies Lemma (2.5) and permutes x_8 and x_{12}.

Proof

Let u be an element of $N_G(S)$ satisfying Lemma (2.5). Then u normalizes $Z_6(S)$ and permutes $\{Z_5(M), Z_5(D_5)\}$. Thus by Lemma (2.1), we have, $x_8^u = x_8(\alpha)\, x_{12}(\beta) z$, $z \in Z_5(S)$. Now $x_8^{ux_{20}u^{-1}} = x_8x_{21}(\alpha)$ alongwith Table 3, forces α to be 0. Also, since x_8^u cannot belong to $Z_5(S)$, β cannot be 0. So we have $x_8^u = x_{12}z$, $z \in Z_5(S)$.

If x_{19} appears in z, then ux_9u^{-1} conjugates x_8 to x_8x_{21}, which contradicts Table 3. If x_9 appears in z, x_8 becomes conjugate to x_8x_{21} via $ux_{19}u^{-1}$, if x_{13} appears in z, x_8 becomes conjugate to x_8x_{17} via $ux_7u^{-1}x_{18}x_{12}$, if x_{20} appears in z, $ux_8u^{-1}x_{12}$ conjugates x_8 to x_8x_{21}, all contradicting Table 3. Now, if x_{23} appears in z, x_8 becomes conjugate to x_8x_{21} via $ux_5u^{-1}x_{10}$, if x_{22} appears in z, x_8 becomes conjugate to x_8x_{21} via ux_6u^{-1}, if x_{14} appears in z, then $ux_4u^{-1}x_{12}$ conjugates x_8 to $x_8x_{17}x_{21}$, all contradicting Table 3. If x_{21} appears in z, then ux_{15} satisfies Lemma (2.5) and x_8ux_{15} does not involve x_{21}. Hence by writing u for ux_{15}, if necessary, we can assume $x_8^u \in x_{12}S_{15}S_{16}S_{17}S_{24}$. Now if x_{15} appears in z, x_8 becomes conjugate to x_8x_{17} by $ux_2u^{-1}x_{18}$, if x_{16} appears in z, x_8 becomes conjugate to $x_8x_{17}x_{24}(\alpha)$, via $ux_1u^{-1}x_2x_{12}$ all contrading Table 3. Thus $x_8^u = x_{12}x_{17}(\alpha)x_{24}(\beta)$. But then, $uw_1u^{-1}x_{11}(\alpha)$ conjugates x_8 to $x_9x_{17}(\alpha)\,x_{21}(\beta)$, a contradiction to Table 4, if $\alpha \neq 0$, and $\beta \neq 0$, since, x_8 is conjugate to x_9 in C_3, but x_9 is not conjugate to any of x_9x_{17}, x_9x_{21}, $x_9x_{17}x_{21}$ in $C_3 = C_G(x_{21}) \cap C_G(x_{24})$. Hence $x_8^u = x_{12}$. Finally, $x_{12}^u = x_8^{u^2}$. Thus from the proof of Lemma (2.5), $x_{12}^u = x_8x_{24}(\alpha)$. Now $ux_{20}(\alpha)$, permutes $\{x_8, x_{12}\}$ and satisfies Lemma (2.5). We write u for $ux_{20}(\alpha)$ and the Lemma is proved.

Proof of Theorem D:

Let u be an element of $N_G(S)$ such that u satisfies Lemma (2.6). Then by Lemma (2.4), $x_9^u = x_{13}x_{24}(\alpha)$ and $x_{13}^u = x_9x_{21}(\alpha)$. Let $\alpha = 1$. Thus uw_1u^{-1} is in $C_G(x_{21})$ and conjugates x_9 to x_9x_{21}, a contradiction to Table 5. Thus u must permute x_9 and x_{13}.

Finally we put $w = uw_5u^{-1}$ and the theorem is established.

REFERENCES

1. Gorenstein, D. **Finite Groups, Harper** & Row, New York, 1968.
2. Gutermann, M.A. A Characterization of $F_4(2^n)$, *J. Algebra* **20**(1972), 1–23.
3. Husnine, S.M. On A Characterization of $\widehat{F_4(2)}$, The Ree-Extension of the Chevalley Group $F_4(2)$ ... I *PUJM*, Vol. XXVI (1993), 101-110.
4. Husnine, S.M. On A Characterization of $\widehat{F_4(2)}$, The Ree-Extension of the Chevalley Group $F_4(2)$ ... II *PUJM* Vol. XXVIII (1995), 23-32.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 47 - 68

MODULAR GROUP ACTION ON CERTAIN QUADRATIC FIELDS

M. Aslam, S.M. Husnine, A. Majeed

Mathematics Department,
University of the Punjab
Lahore.

ABSTRACT

In this paper we determine the ambiguous numbers of a subset $Q^*(\sqrt{n})$ of the real quadratic field where $Q^*(\sqrt{n})$ is invariant under the action of the Modular Group PSL(2, Z), for some non square positive rational integer n.

1. INTRODUCTION

For each non square positive rational integer n, let us define

$$Q^*(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} \;\middle|\; \frac{a^2-n}{c} \text{ is a rational integer and } \left(a, \frac{a^2-n}{c}, c\right) = 1 \right\}$$

A real quadratic irrational number $\alpha = a + b\sqrt{n} \in Q(\sqrt{n})$ is called an ambiguous number if α and its conjugate $\bar{\alpha}$, as real numbers, have different signs.

Mushtaq [2] has proved that $Q^*(\sqrt{n})$ is invariant under the group action of $G = \langle x, y : x^2 = y^3 = 1 \rangle$, where $x(\alpha) = -1/\alpha$, $y(\alpha) = (\alpha-1)/\alpha$.

He has further proved that $Q^*(\sqrt{n})$ contains a finite number of ambiguous numbers and those occurring in a particular orbit of $Q^*(\sqrt{n})$ under the action of G form a unique closed path in the coset diagram.

Thus it becomes interesting to know the actual number of ambiguous numbers in $Q^*(\sqrt{n})$ as a function of n on one hand and the number of distinct closed paths formed by these numbers on the other hand.

In this paper we attempt to attack the first part of the problem.

The notation is standard and we follow [1] and [2]. In particular, for any positive rational integer n, $\tau(n)$ denotes the number of positive divisors of n and for any real number n, the largest rational integer not greater than n is denoted by $[n]$.

We start with the following simple result whose proof is trivial.

Lemma 1

For any non-square positive rational integer n the number of elements of type $\frac{a+\sqrt{n}}{c}$, such that $\frac{a^2-n}{c}$ is a rational integer and $a^2 < n$, is

$$2\tau(n) + 4\sum_{a=1}^{[\sqrt{n}]} \tau(n-a^2)$$

For each non-square positive rational integer n, we shall denote the number

$$2\tau(n) + 4\sum_{a=1}^{[\sqrt{n}]} \tau(n-a^2) \quad \text{by } \tau^*(n)$$

In the following theorem we determine the number of ambiguous numbers of $Q^*(\sqrt{n})$, where n is a square free positive rational integer.

Theorem 2

Let n be a square-free positive rational integer. Then the number of ambiguous numbers in $Q^*(\sqrt{n})$ is $\tau^*(n)$.

Proof

Let n be a fixed square-free positive rational integer. Then $\alpha = \frac{a+\sqrt{n}}{c}$ is an ambiguous number of $Q^*(\sqrt{n})$ if and only if $\frac{a^2-n}{c}$ is a rational integer, $a^2 < n$ and $\left(a, \frac{a^2+n}{c}, c\right) = 1$.

By lemma 1, the number of ambiguous numbers of the form $\frac{a+\sqrt{n}}{c}$, such that $\frac{a^2+n}{c}$ is a rational integer, is $\tau^*(n) = 2\tau(n) + 4\sum_{a=1}^{[\sqrt{n}]} \tau(n-a^2)$.

Now we claim that for any c dividing $a^2 - n$, $\left(a, \frac{a^2-n}{c}, c\right) = 1$.

For if any rational prime p divides $\left(a, \frac{a^2-n}{c}, c\right)$, then $p \mid a, p \mid \left(\frac{a^2-n}{c}\right)$ and $p \mid c$.

Now $p \mid c \Rightarrow c = c'p$ and $p \mid \left(\frac{a^2-n}{c}\right) \Rightarrow a^2-n = c''\,cp = c''c'p^2$

So $p^2 \mid (a^2 - n)$. But $p^2 \mid a^2$. So $p^2 \mid n$.

This contradicts our assumption that n is square-free.

Hence the number of ambiguous numbers of $Q^*(\sqrt{n})$ is $\tau^*(n)$.

Illustration

As an illustration we consider the following examples.

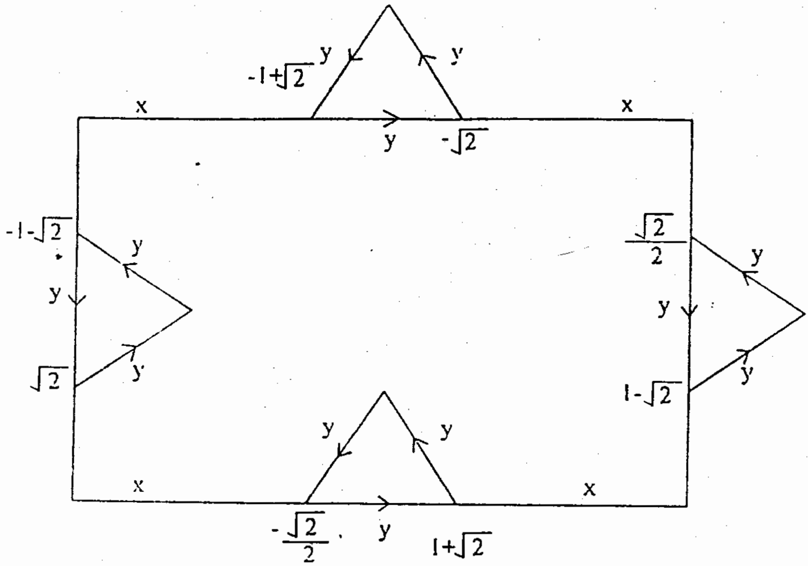
Example 1

Let $n = 2$. Then

$$\begin{aligned} \tau^*(n) &= 2\,\tau(2) + 4\,\tau(2-1) \\ &= 2(2) + 4(1) \\ &= 8 \end{aligned}$$

These 8 numbers are $\frac{\pm\sqrt{2}}{1}, \frac{\pm\sqrt{2}}{2}, \frac{1+\sqrt{2}}{\pm 1}, \frac{1-\sqrt{2}}{\pm 1}$, and only these are all the ambiguous numbers of $Q^*(\sqrt{2})$.

The distribution of the ambiguous numbers in $Q^*(\sqrt{2})$ on the coset diagram under the action of G on $Q^*(\sqrt{2})$ is shown in figure 1



The only closed path in the coset diagram under the action of G on $Q^*(\sqrt{2})$

Figure 1

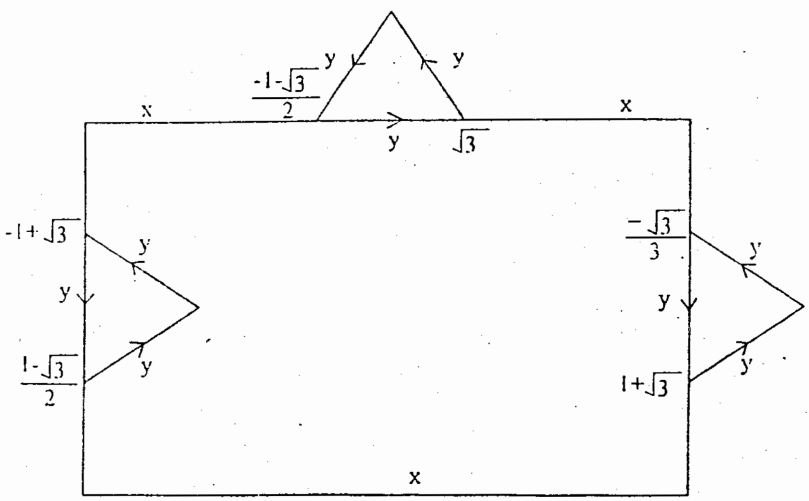
Example 2

Let $n = 3$. Then

$$\begin{aligned}\tau^*(n) &= 2\,\tau(3) + 4\,\tau(3-1)\\ &= 2(2) + 4(2)\\ &= 12\end{aligned}$$

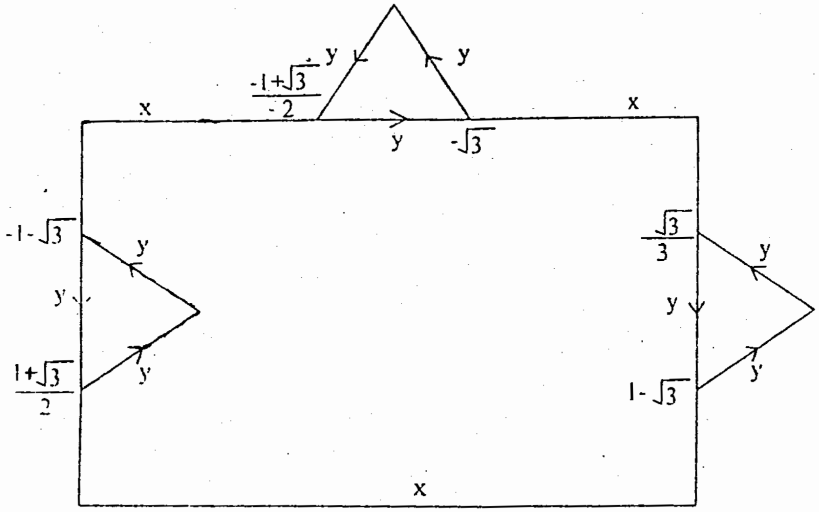
These 12 numbers are $\frac{\pm\sqrt{3}}{1}$, $\frac{\pm\sqrt{3}}{3}$, $\frac{1+\sqrt{3}}{\pm 1}$, $\frac{-1+\sqrt{3}}{\pm 1}$, $\frac{1+\sqrt{3}}{\pm 2}$, $\frac{-1+\sqrt{3}}{\pm 2}$, and only these are all the ambiguous numbers of $Q^*(\sqrt{3})$.

The distribution of the ambiguous numbers in $Q^*(\sqrt{3})$ on the coset diagram under the action of G on $Q^*(\sqrt{3})$ is shown in figures 2 and 3.



Closed path in the coset diagram for the orbit $(\sqrt{3})^G$

Figure 2



Closed path in the coset diagram for the orbit $(-\sqrt{3})^G$

Figure 3

Remarks:

1. According to theorem 6 of [2] the above diagrams show that there are exactly two orbits of G in $Q^*(\sqrt{3})$, namely $\sqrt{3}^G$ and $-\sqrt{3}^G$.

2. In the light of theorems 5 and 6 of [2], the above examples reveal that G acts transitively on $Q^*(\sqrt{2})$. But it does not do so on $Q^*(\sqrt{3})$.

The following theorem gives a generalization of theorem 3.2 for $n = p^{2u}n'$, where p is a rational prime, u is a positive rational integer and n' is a square free positive rational integer.

Theorem 3

Let $n = p^{2u}n'$ where p is a rational prime, u is a positive rational integer and n' is a square free positive rational integer. Then the number of ambiguous numbers in $Q^*(\sqrt{n})$ is $\tau^*(n) - \tau^*(p^{2u-2}n')$.

Proof

Suppose that $n = p^{2u}n'$, where p is a rational prime, u is a positive rational integer and n' is a square free positive rational integer.

Then $\alpha = \dfrac{a+\sqrt{n}}{c}$ is an ambiguous number of $Q^*(\sqrt{n})$ if and only if $a^2 < n$, $\dfrac{a^2-n}{c}$ is a rational integer and $\left(a, \dfrac{a^2-n}{c}, c\right) = 1$.

By lemma 1, the number of ambiguous numbers of the form $\dfrac{a+\sqrt{n}}{c}$, such that $\dfrac{a^2-n}{c}$ is a rational integer, is $\tau^*(n)$.

We claim that in the above $\tau^*(n)$ numbers exactly $\tau^*(p^{2u-2}n')$ numbers are not in $Q^*(\sqrt{n})$ and all the rest are in $Q^*(\sqrt{n})$.

Now consider the following two cases.

Case I p does not divide c

Here $\left(a, \frac{a^2-n}{c}, c\right)$ is not divisible by p. Now if any other rational prime q divides $\left(a, \frac{a^2-n}{c}, c\right)$, then $q \mid a$, $q \mid c$ and $\frac{a^2-n}{c}$ is a rational integer.

Thus $q^2 \mid (a^2 - n)$ and $q^2 \mid a^2$ forces q^2 divides $n = p^{2u}n'$ i.e. $q^2 \mid n'$, a contradiction to the choice of n'.

Hence $\left(a, \frac{a^2-n}{c}, c\right) = 1.$

Which shows that $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$.

Case II p divides c

Here $c = pc'$

Further $\frac{a^2-n}{c} = \frac{a^2-p^{2u}n'}{pc'}$ being a rational integer implies $p \mid (a^2 - p^{2u}n')$.

So $p \mid a^2$ i.e. $p \mid a$ and hence

$a = a'p$, where $a' \in \{0, \pm 1, \ldots.., \pm [\sqrt{p^{2u-2}\,n'}]\}$

Now $c \mid (a^2 - n) \Rightarrow pc' \mid (a'^2p^2 - p^{2u}n')$

$\Rightarrow \quad c' \mid p\,(a'^2 - p^{2u-2}n')$

We consider the cases

(i) $c' \dagger (a'^2 - p^{2u-2}n')$

(ii) $c' \mid (a'^2 - p^{2u-2}n')$

(i) $c' \dagger (a'^2 - p^{2u-2}n')$

Since $\frac{a^2-n}{c} = \frac{p(a^2 - p^{2u-2}n')}{c'}$ is a rational integer, so $c' = pc''$.

Also we claim that $\frac{(a'^2 - p^{2u-2}n')}{c''}$ is not divisible by p. For if $\frac{(a'^2 - p^{2u-2}n')}{c''}$ is divisible by p, then $\frac{(a'^2 - p^{2u-2}n')}{pc''}$ must be a rational integer.

i.e. $c' = pc''$ divides $(a'^2 - p^{2u-2}n')$, a contradiction to the assumption that $c' \dagger (a'^2 - p^{2u-2}n')$.

$$\text{Thus} \quad \left(a, \frac{a^2 - n}{c}, c\right) = \left(a'p, \frac{p(a'^2 - p^{2u-2}n')}{c'}, pc'\right)$$

$$= \left(a'p, \frac{p(a'^2 - p^{2u-2}n')}{pc''}, pc'\right)$$

$$= \left(a'p, \frac{(a'^2 - p^{2u-2}n')}{c''}, pc'\right) \text{ is not divisible by } p.$$

Also if any other rational prime q divides $\left(a, \frac{a^2 - n}{c}, c\right)$, then $q \mid a$, $q \mid c$ and $\frac{a^2 - n}{c}$ is a rational integer.

Thus $q^2 \mid (a^2 - n)$ and $q^2 \mid a^2$ forces q^2 divides $n = p^{2u}n'$ i.e. a contradiction to the choice of n'.

$$\text{Therefore} \quad \left(a, \frac{a^2 - n}{c}, c\right) = \left(a'p, \frac{(a'^2 - p^{2u-2}n')}{c''}, pc'\right) = 1$$

$$\text{So} \qquad \alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n}).$$

(ii) $c' \mid (a'^2 - p^{2u-2}n')$

$$\text{Here} \quad \left(a, \frac{a^2 - n}{c}, c\right) = \left(a'p, \frac{p^2\,(a'^2 - p^{2u-2}n')}{pc'}, pc'\right)$$

$$= \left(a'p, \frac{p\,(a'^2 - p^{2u-2}n')}{c'}, pc'\right)$$

is divisble by p.

Therefore $\alpha = \dfrac{a+\sqrt{n}}{c} = \dfrac{a'p+\sqrt{p^{2u}n'}}{pc'}$

$= \dfrac{a'+\sqrt{p^{2u-2}n'}}{c'}$ does not belong to $Q^*(\sqrt{n})$.

So $\tau^*(p^{2u-2}n') = 2\tau(p^{2u-2}n') + 4 \sum_{a'=1}^{[\sqrt{p^{2u-2}n'}]} \tau(p^{2u-2}n'-a'^2)$

numbers are not in $Q^*(\sqrt{n})$.

Thus the number of ambiguous numbers in $Q^*(\sqrt{n})$ is $\tau^*(n) - \tau^*(p^{2u-2}n')$.

Illustration

Example 3

Let $n = 2^2 . 3 = 12$. Then

$$\tau^*(12) - \tau^*(3)$$

$$= 2\,\tau(12) + 4 \sum_{a=l}^{3} \tau(12 - a^2) - \{2\,\tau(3) + 4\,\tau(3-1)\}$$

$$= 12 + 4(2 + 4 + 2) - \{2(2) + 4(2)\}$$

$$= 12 + 32 - 12$$

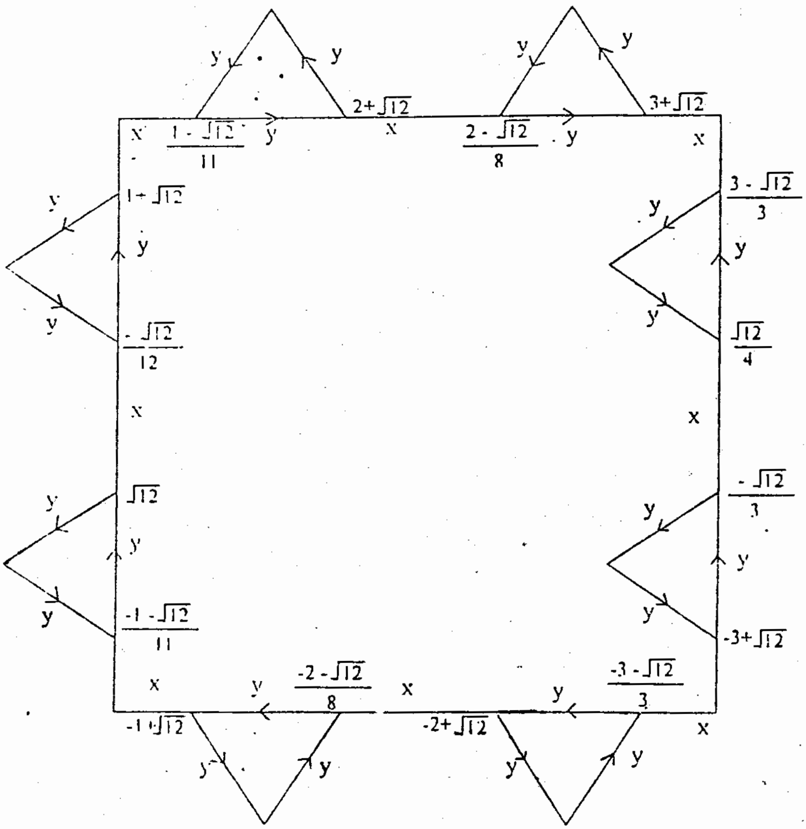
$$= 32$$

These 32 numbers are $\dfrac{\pm\sqrt{12}}{1}, \dfrac{\pm\sqrt{12}}{3}, \dfrac{\pm\sqrt{12}}{4}, \dfrac{\pm\sqrt{12}}{12}, \dfrac{\pm1+\sqrt{12}}{\pm1}, \dfrac{\pm1+\sqrt{12}}{\pm11}, \dfrac{\pm2+\sqrt{12}}{\pm1}, \dfrac{\pm2+\sqrt{12}}{\pm8}, \dfrac{\pm3+\sqrt{12}}{\pm1}, \dfrac{\pm3+\sqrt{12}}{\pm3}$, and only these are all the ambiguous numbers of $Q^*(\sqrt{12})$.

But the numbers $\dfrac{\pm\sqrt{12}}{2} = \dfrac{\pm\sqrt{3}}{1}, \dfrac{\pm\sqrt{12}}{6} = \dfrac{\pm\sqrt{3}}{3}, \dfrac{\pm2+\sqrt{12}}{\pm2}, \dfrac{\pm1+\sqrt{3}}{\pm4}, \dfrac{\pm2+\sqrt{12}}{\pm4} = \dfrac{\pm1+\sqrt{3}}{\pm2}$ are all the ambiguous numbers of $Q^*(\sqrt{3})$.

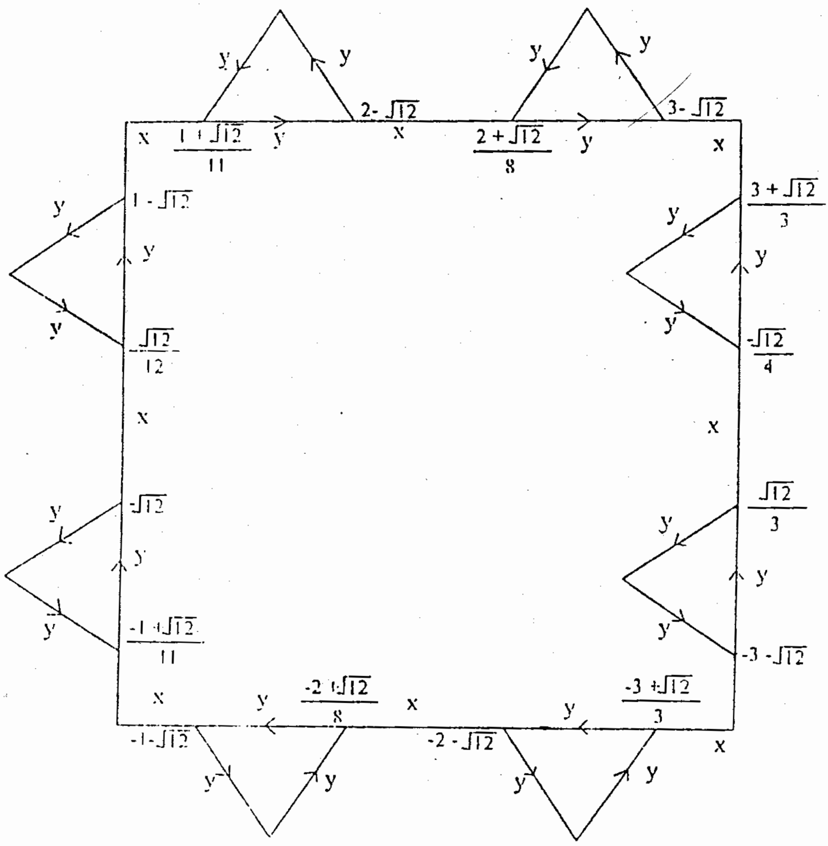
Thus the number of ambiguous numbers of $Q^*(\sqrt{12})$ is $\tau^*(12)-\tau^*(3)$.

The distribution of the ambiguous numbers in $Q^*(\sqrt{12})$ on the coset diagram under the action of G on $Q^*(\sqrt{12})$ is shown in figures 4 and 5.



Closed path in the coset diagram for the orbit $(\sqrt{12})^G$

Figure 4



Closed path in the coset diagram for the orbit $(-\sqrt{12})^G$

Figure 5

Example 4

Let $n = 2^4, 3 = 48$. Then

$\tau^*(2^4 . 3) - \tau^*(2^2 . 3)$

$= 124 - 44 = 80$

These 80 numbers are $\frac{\sqrt{48}}{\pm 1}, \frac{\sqrt{48}}{\pm 3}, \frac{\sqrt{48}}{\pm 16}, \frac{\pm 1 + \sqrt{48}}{\pm 1}, \frac{\pm 1 + \sqrt{48}}{\pm 47},$
$\frac{\pm 2 + \sqrt{48}}{\pm 1}, \frac{\pm 2 + \sqrt{48}}{\pm 4}, \frac{\pm 2 + \sqrt{48}}{\pm 11}, \frac{\pm 2 + \sqrt{48}}{\pm 44}, \frac{\pm 3 + \sqrt{48}}{\pm 1}, \frac{\pm 3 + \sqrt{48}}{\pm 3},$
$\frac{\pm 3 + \sqrt{48}}{\pm 13}, \frac{\pm 3 + \sqrt{48}}{\pm 39}, \frac{\pm 4 + \sqrt{48}}{\pm 1}, \frac{\pm 4 + \sqrt{48}}{\pm 32}, \frac{\pm 5 + \sqrt{48}}{\pm 1}, \frac{\pm 5 + \sqrt{48}}{\pm 23},$
$\frac{\pm 6 + \sqrt{48}}{\pm 1}, \frac{\pm 6 + \sqrt{48}}{\pm 3}, \frac{\pm 6 + \sqrt{48}}{\pm 4}$, and only these are all the ambiguous of $Q^*(\sqrt{48})$.

But $\frac{\pm\sqrt{48}}{\pm 2} = \frac{\pm\sqrt{12}}{\pm 1}, \frac{\pm\sqrt{48}}{\pm 8} = \frac{\pm\sqrt{12}}{\pm 4}, \frac{\pm\sqrt{48}}{\pm 6} = \frac{\pm\sqrt{12}}{\pm 3}, \frac{\pm\sqrt{48}}{\pm 24} = \frac{\pm\sqrt{12}}{\pm 12},$

$\frac{\pm 2+\sqrt{48}}{\pm 2} = \frac{\pm 1+\sqrt{12}}{\pm 1}, \frac{\pm 2+\sqrt{48}}{\pm 22} = \frac{\pm 1+\sqrt{12}}{\pm 11}, \frac{\pm 4+\sqrt{48}}{\pm 2}, \frac{\pm 2+\sqrt{12}}{\pm 1},$

$\frac{\pm 4+\sqrt{48}}{\pm 16}, \frac{\pm 2+\sqrt{12}}{\pm 8}, \frac{\pm 6+\sqrt{48}}{\pm 2} = \frac{\pm 3+\sqrt{12}}{\pm 1}, \frac{\pm 6+\sqrt{48}}{\pm 6} = \frac{\pm 3+\sqrt{12}}{\pm 3},$

are all ambiguous numbers of $Q^*(\sqrt{12})$.

And $\frac{\pm\sqrt{48}}{\pm 4} = \frac{\pm\sqrt{12}}{\pm 2}, \frac{\pm\sqrt{3}}{\pm 1}, \frac{\pm\sqrt{48}}{\pm 12} = \frac{\pm\sqrt{12}}{\pm 6} = \frac{\pm\sqrt{3}}{\pm 3}, \frac{\pm 4 + \sqrt{48}}{\pm 4}$

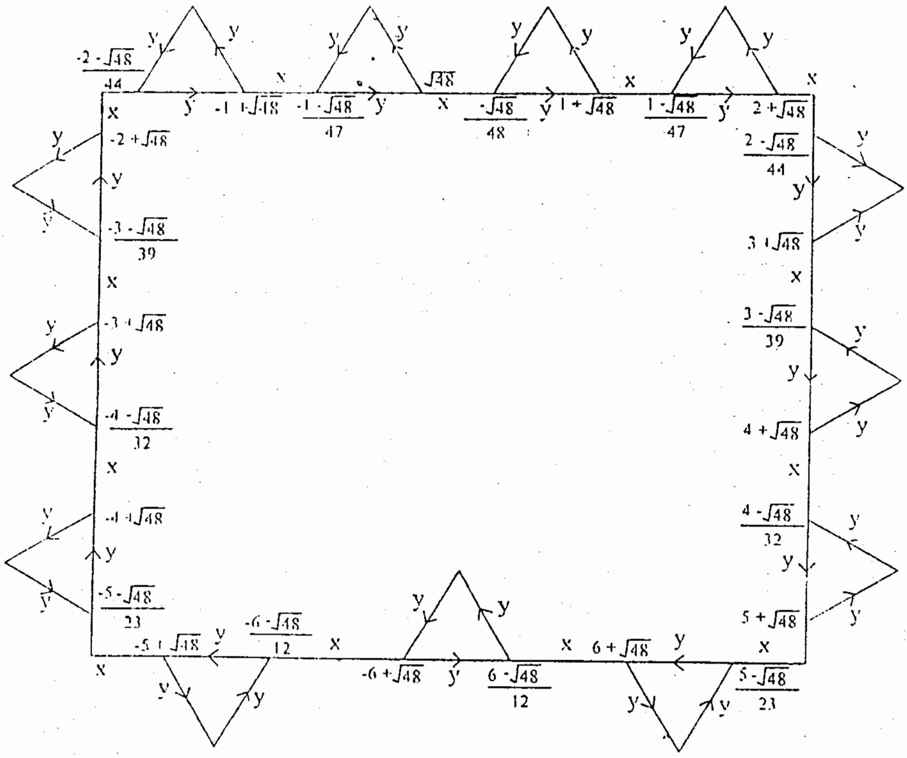
$= \frac{\pm 2 + \sqrt{12}}{\pm 2} = \frac{\pm 1 + \sqrt{3}}{\pm 1} = \frac{\pm 4 + \sqrt{48}}{\pm 8} = \frac{\pm 2 + \sqrt{12}}{\pm 4} = \frac{\pm 1 + \sqrt{3}}{\pm 2}$

are all ambiguous number of $Q^*(\sqrt{3})$.

Thus the number of ambiguous numbers of $Q^*(\sqrt{2^4 . 3})$ is

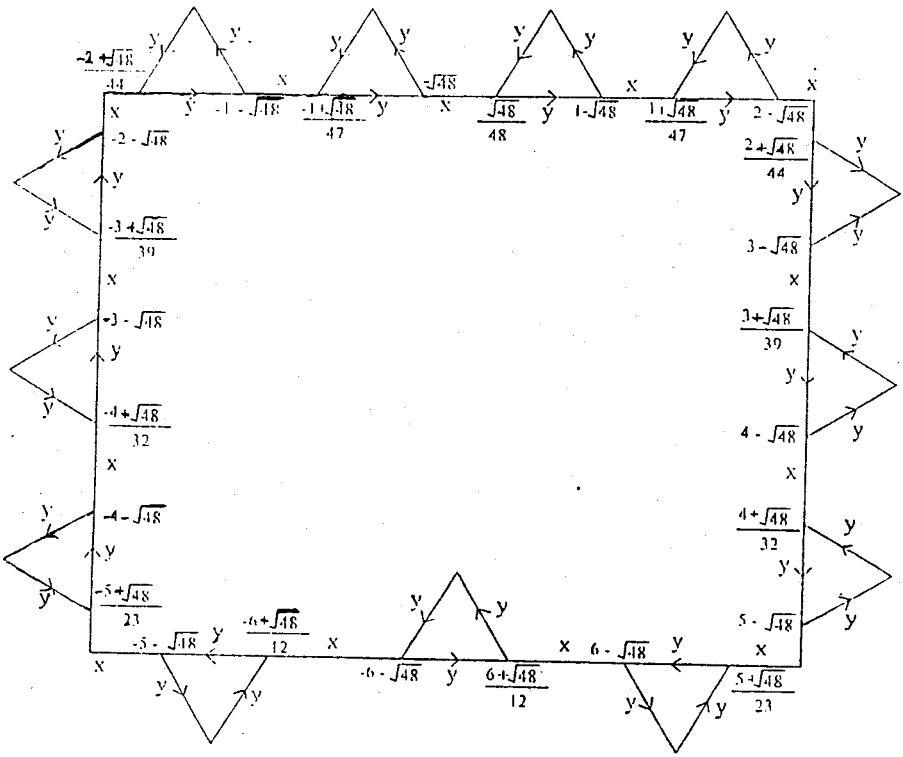
$\tau^*(2^4 . 3) - \tau^*(2^4 . 3)$

The distribution of the ambiguous numbers in $Q^*(\sqrt{48})$ on the coset diagram under the action of G on $Q^*(\sqrt{48})$ is shown in figures 6, 7, 8 and 9.



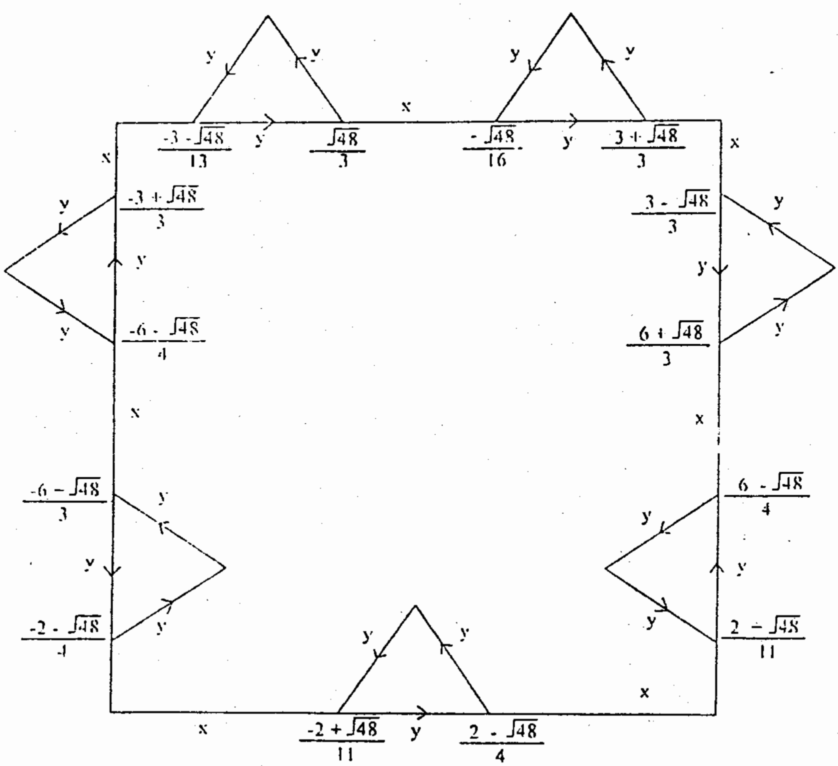
Closed path in the coset diagram for the orbit $(\sqrt{48})^G$

Figure 6



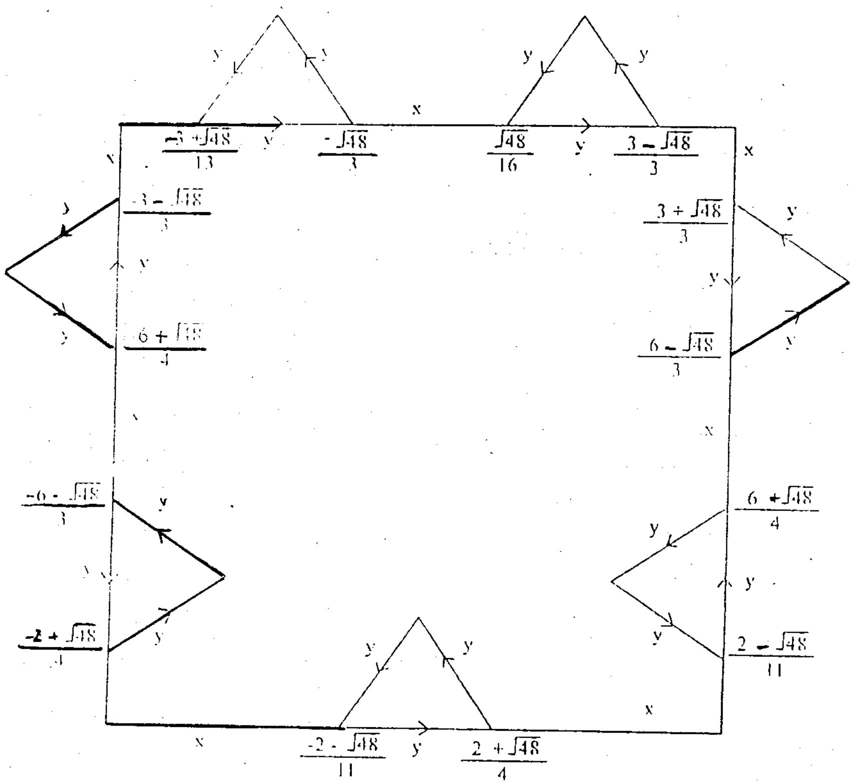
Closed path in the coset diagram for the orbit $(-\sqrt{48})^G$

Figure 7



Closed path in the coset diagram for the orbit $\left(\frac{\sqrt{48}}{3}\right)^G$

Figure 8



Closed path in the coset diagram for the orbit $\left(\frac{-\sqrt{48}}{3}\right)^{G}$

Figure 9

In the following theorem we generalize theorem 3. For $n = p_1^{2u} p_2^{2u_2} n'$, where p_1, p_2 are rational integers and n' is a square-free positive rational integer.

Theorem 4

Let $n = p_1^{2u_1} p_2^{2u_2} n'$, where p_1, p_2 are rational primes, u_1, u_2 are positive rational integers and n' is a square-free positive rational integer. Then the number of ambiguous numbers in $Q^*(\sqrt{n})$ is $\tau^*(n) - \tau^*(p_1^{-2} n) - \tau^*(p_2^{-2} n) + \tau^*(p_1^{-2} p_2^{-2} n)$.

Proof

Suppose that $n = p_1^{2u_1} p_2^{2u_2} n'$, where p_1, p_2 are rational primes, u_1, u_2 are positive rational integers and n' is a square-free positive rational integer.

Then $\alpha = \dfrac{a + \sqrt{n}}{c}$ is an ambiguous number of $Q^*(\sqrt{n})$ if and only if $a^2 < n$, $\dfrac{a^2 - n}{c}$ is a rational integer and $\left(a, \dfrac{a^2 - n}{c}, c\right) = 1$.

By lemma 1, the number of ambiguous numbers of the form $\dfrac{a + \sqrt{n}}{c}$, such that $\dfrac{a^2 - n}{c}$ is a rational integer, is $\tau^*(n)$.

We claim that in the above $\tau^*(n)$ numbers exactly

$\{\tau^*(p_1^{-2} n) - \tau^*(p_2^{-2} n) + \tau^*(p_1^{-2} p_2^{-2} n)\}$ numbers are not in $Q^*(\sqrt{n})$ and all the rest are in $Q^*(\sqrt{n})$.

Now consider the following two cases.

Case I. p_j does not divide c, where $j = 1, 2$

Here $\left(a, \dfrac{a^2 - n}{c}, c\right)$ is not divisible by p_j, where $j = 1, 2$

Also if any other rational prime q divides $\left(a, \frac{a^2 - n}{c}, c\right)$, then $q \mid a$, $q \mid c$ and $q \mid \left(\frac{a^2 - n}{c}\right)$ implies $q^2 \mid (a^2 - n)$ and $q^2 \mid a^2$. Thus $q^2 \mid n$ i.e. $q^2 \mid n'$, a contradiction to the choice of n'. So $\left(a, \frac{a^2 - n}{c}, c\right) = 1$.

This shows that $\alpha = \frac{a + \sqrt{n}}{c}$ belong to $Q^*(\sqrt{n})$.

Case II. p_j divides c, where $j = 1, 2$

Here $c = c_j p_j$

We know that if $p_j \mid c$, then $p_j \mid a$.

So $a = a_j p_j$, where $a_j \in \left\{0, \pm1, \pm2, \ldots, \pm[\sqrt{p_j^{-2} n}\right\}$.

Now since $\frac{a^2 - n}{c}$ is a rational integer, so $c \mid (a^2 - n)$ which implies that $p_j c_j \mid (a_j^2 p_j^2 - n) \Rightarrow c_j \mid p_j (a_j^2 - p_j^{-2} n)$.

We consider the cases

(i) c_j does not divide $(a_j^2 - p_j^{-2} n)$

(ii) c_j divides $(a_j^2 - p_j^{-2} n)$

(i) If c_j does not divide $(a_j^2 - p_j^{-2} n)$

Then as $\frac{a^2 - n}{c} = \frac{p_j (a_j^2 - p_j^{-2} n)}{c_j}$ is a rational integer, so $c_j = p_j c_j'$.

Also we claim that $\frac{(a_j^2 - p_j^{-2} n)}{c_j'}$ is not divisible by p_j.

For if $\frac{(a_j^2 p_j^{-2} n)}{c_j'}$ is divisible by p_j. Then $\frac{(a_j^2 - p_j^{-2} n)}{p_j c_j'}$ must be a rational integer i.e. $c_j = p_j c_j'$ divides $(a_j^2 - p_j^{-2} n)$, a contradiction.

$$\text{Therefore} \quad \left(a, \frac{a^2 - n}{c}, c\right) = \left(a_j p_j, \frac{(a_j^2 p_j^2 - n)}{p_j c_j'}, p_j c_j\right)$$

$$= \left(a_j p_j, \frac{p_j (a_j^2 - p_j^{-2} n)}{c_j}, p_j c_j\right)$$

$$= \left(a_j p_j, \frac{p_j (a_j^2 - p_j^{-2} n)}{p_j c_j'}, p_j c_j\right)$$

$$= \left(a_j p_j, \frac{(a_j^2 - p_j^{-2} n)}{c_j'}, p_j c_j\right)$$ is not divisible by p_j

Also if any other rational prime q divides $\left(a, \frac{a^2 - n}{c}, c\right)$, then $q | a$, $q | c$ and $q^2 | (a^2 - n)$ and $q^2 | a^2$. Thus $q^2 | n$ i.e. $q^2 | n'$, a contradiction to the choice of n'. Hence $\left(a, \frac{a^2 - n}{c}, c\right) = 1$.

Thus $\alpha = \frac{a + \sqrt{n}}{c}$ belongs to $Q^*(\sqrt{n})$.

(ii) On the other hand if c_j divides $(a_j^2 - p_j^{-2} n)$, then

$\left(a, \frac{a^2 - n}{c}, c\right) = \left(a_j p_j, \frac{p_j (a_j^2 - p_j^{-2} n)}{c_j}, p_j c_j\right)$ is divisible by p_j.

Therefore $\alpha = \frac{a + \sqrt{n}}{c} = \frac{a_j p_j + \sqrt{n}}{p_j c_j} = \frac{a_j + \sqrt{p_j^{-2} n}}{c_j}$ does not belong to $Q^*(\sqrt{n})$. However, in $\tau^*(p_1^{-2} n)$ and $\tau^*(p_2^{-2} n)$ numbers,

$\tau^*(p_1^{-2} p_2^{-2} n)$ numbers are repeated. Thus $\{\tau^*(p_1^{-2} n) + \tau^*(p_2^{-2} n) - \tau^*(p_1^{-2} p_2^{-2} n)\}$ numbers are not in $Q^*(\sqrt{n})$.

Therefore the number of ambiguous numbers in $Q^*(\sqrt{n})$ is:

$$\tau^*(n) - \tau^*(p_1^{-2} n) - \tau^*(p_2^{-2} n) + \tau^*(p_1^{-2} p_2^{-2} n)$$

Example 5

Let $n = 2^4 \times 3^4 \times 2 = 2592$. Then the number of ambiguous numbers in $Q^*(\sqrt{n})$ is 952.

These ambiguous numbers form eight closed paths in the coset diagram under the action of G on $Q^*(\sqrt{n})$. Hence these numbers belong to 8 distinct orbits with representatives $\sqrt{2592}$, $-\sqrt{2592}$, $\frac{\sqrt{2592}}{32}$, $-\frac{\sqrt{2592}}{32}$, $\frac{3+\sqrt{2592}}{7}$, $\frac{3-\sqrt{2592}}{7}$, $\frac{-3+\sqrt{2592}}{7}$, and $\frac{-3-\sqrt{2592}}{7}$.

The number of ambiguous numbers in the orbits of each of $\frac{3+\sqrt{2592}}{7}$, $\frac{3-\sqrt{2592}}{7}$, and $\frac{-3-\sqrt{2592}}{7}$ is 70, in the orbits of each of $\frac{\sqrt{2592}}{32}$, $-\frac{\sqrt{2592}}{32}$ is 86, and in the orbits of each of $\sqrt{2592}$, $-\sqrt{2592}$ is 250.

REFERENCES

1. Ivan Niven, Herbert S. Zuckerman and Hugh L. Montgomery, An Introduction to the theory of numbers: John Wiley and sons, Inc. 1991.

2. Q. Mushtaq, Modular Group acting on real quadratic fields, *Bull Austral, Math. Soc.* Vol. **37**(1988), 303–309.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 69 - 82

A NEW METHOD FOR APPROXIMATING SUMS BY MEANS OF COMPLEX INTEGRATION

Hasan M. Ymeri

Higher Technical School,
University of Prishtina,
38220 Mitrovice e Titos,
Yugoslavaia

ABSTRACT

In this paper we derive a new method for evaluating or approximating sums by means of complex integration. Our result is sufficiently general that it is applicable to a wide variety of functions. We consider examples that illustrate the power of the technique; our first example is an alternative derivation of the Euler-Maclaurin sum formula for the case in which the remainder term vanishes, and our other two examples show how our technique can be applied when the Euler-Maclaurin formula is not useful.

1. INTRODUCTION

There exists a class of techniques by which sums may be converted into integrals and vice versa by selecting appropriate integration contours in the complex plane. One such technique, the Watson transformation [7], has proved valuable in such diverse areas of electrical engineering as microwave theory and techniques, electromagnetic theory and propagation, and has yielded considerable insight into physical processes in these and other fields. Related techniques can be found scattered throughout the standard texts on applied mathematics an analysis. For example, Whittaker and Watson [8], consider a transformation that uses a contour integral around an elongated rectangle of a function multiplied by $(e^{2\pi iz} - 1)^{-1}$, which has poles at $z = n$, where n is an integer. Carrier *et al.* [3], perform some simple by $\pi\cot(\pi z)$ and integrating over a square centered at the origin. Numerous other examples of techniques of this type abound in the mathematics and electro-technical engineering literature.

The purpose of this paper is to present a related new technique for evaluating or approximating sums. The generality of our main result, Equation (12), more than compensates the complexity introduced by the use of two contour integrals with two different integrands.

In our first example, we offer an alternative (valid under certain conditions) to the derivation usually used [8], [3], to obtain the Euler-Maclaurin sum formula. Under our assumptions, the remainder term in the sum formula vanishes. Our derivation is simpler in many respects: Few steps are needed, the Bernoulli numbers appear naturally, and the convergence question is answered in a straightforward manner. There is a scientific utility in that we can compare this method with the conventional one and thereby gain some insight into the power of complex analysis. The remaining examples arise from many problems of a type frequently encountered in the applied mathematics and theoretical electrotechniques. They are given to show that the technqiues of this paper are practical and can be applied when the Euler-Maclaurin series is not useful.

2. DERIVATON OF THE SUM FORMULA

We examine a finite sum of the form

$$S = \sum_{n=M}^{N} f(n) \tag{1}$$

We assume that there exists a function $F(z)$ in the complex plane such that $F(n) = f(n)$ for n integer, with $M \leq n \leq N$. Further, we assume (this assumption is not necessary but simplifies the mathematics that follows) that $F(z)$ has no singularities on the real axis for $M \leq z \leq N$. We introduce two auxiliary functions

$$F_+(z) = F(z) \,/\, (1 - e^{-2\pi i z}), \tag{2a}$$

$$F_-(z) = F(z) \,/\, (e^{-2\pi i z} - 1), \tag{2b}$$

These auxiliary functions have two useful properties. First, their poles on the real axis select out the desired terms for the sum, i.e.,

$$\lim_{\varepsilon \to 0} \in F_+(n + \varepsilon) = \lim_{\varepsilon \to 0} \in F_-(n + \varepsilon) = \frac{1}{2\pi i} f(n), \tag{3}$$

Second, their difference is equal to the original function $F(z)$

$$F_+(z) - F_-(z) = F(z) \left\{ \frac{e^{\pi i z}}{e^{\pi i z} - e^{-\pi i z}} - \frac{e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} \right\} = F(z) \tag{4}$$

Consider a rectangular region R_+ in the complex z plane (see Fig. 1) bounded by the straight lines $y = 0$, $x = M$, $x = N$ and $y = R$, where $z = x + iy$. We define a contour C_+ as the perimeter of R_+, traversed in a counterclockwise direction, with circular identation of radius r (as shown in Fig. 1) to avoid the poles of F_+ on the real axis. Similarly, R_- is the mirror image of R_+ reflected through the real axis, and the contour C_- is the perimeter of R_-, also traversed in a counterclockwise direction, and also indented to avoid the poles of F_- on the real axis. We assume for simplicity that $F(z)$ has no singularities except poles in R_+ and R_-.

Let us integrate $F_+(z)$ around the closed contour C_+. This integral is given by

$$\int_{C_+} F_+(z)\, dz = 2\pi i \sum_{\text{res}}$$

$$= \sum_{n=M}^{N} \int_{n+r}^{n+1-1} R_+(x)\, dx - i \int_0^{\pi/2} re^{i\theta} F_+(M + re^{i\theta})\, d\theta$$

$$- \sum_{n=M+n}^{N-1} \int_0^{\pi/2} re^{i\theta} F_+(n + re^{i\theta})\, d\theta - i \int_{\pi/2}^{\pi} i$$

$$- e^{i\theta} F_+(N+re^{i\theta})\, d\theta + i \int_r^R F_+(N+iy)\, dy$$

$$- i \int_r^R F_+(M+iy)\, dy - \int_M^N F_+(x + iR)\, dx \qquad (5)$$

In Eq. (5), the first sum is over the residues arising from poles of $F(z)$ in the region R_+. On the right-hand side of Eq. (5), referring to Fig. 1, the first term is the sum of integrals over the segments between poles on the real axis; the next three terms are the integrals over the circular segments around these poles; the next two terms are the integrals along paths parallel to the y axis; and the final term is the integral along the line $y = R$. Integrating $F_-(z)$ around the closed contour C_-, we arrive at a similar expression:

$$\int_{C_-} F_-(z)\, dz = 2\pi i \sum_{\text{res}} F_-(z)$$

$$= \sum_{n=M}^{N-1} \int_{n+r}^{n+1-r} F_-(x)\, dx - i \int_{\pi/2}^{0} re^{i\theta} F_-(M + re^{i\theta})\, d\theta$$

$$- \sum_{n=M+1-\pi}^{N-1} \int^{0} re^{i\theta} F_-(n + re^{i0})\, d\theta$$

$$- i \int_{-\pi/2}^{-\pi} r\, e^{\,i\theta} F_-(N+re^{i\theta})\, d\theta + i \int_{r}^{R} F_-(N+iy)\, dy$$

$$- i \int_{r}^{R} F_-(M-iy)\, dy + \int_{M}^{N} F_-(x + iR)\, dx \qquad (6)$$

where the first sum is over the residues arising from of $F(z)$ in the region R_- and where the terms on the right-hand side of Eq. (6) are interpreted similarly to those of Eq. (5).

We are interested in adding Eqs. (5) and (6) and taking the limits $r \rightarrow 0$, $R \rightarrow \infty$. Note that the former limit cannot be taken in Eq. (5) or Eq. (6) alone, since they both contain divergent integrals. By virtue of Eq. (4), we have

$$\lim_{r\to 0} \sum_{n=M}^{N} \int_{n+r}^{n+1-r} \left\{F_+(x) - F_-(x)\right\} dx$$

$$= \lim_{r\to 0} \sum_{n=M}^{N} \int_{n+r}^{n+1-r} F(x)\, dx = \int_{M}^{N} F(x)\, dx, \qquad (7)$$

since, when we subtract $F_-(x)$ from $F_+(x)$, the singularities at integer values of x are removed. Furthemore, Eq. (3) can be used to obtain the integrals around the circular segments in the limit as $r \rightarrow 0$:

$$\int_{0}^{\pi/2} re^{i0} F_+(M + re^{i\theta})\, d\theta$$

$$= \int_{-\pi/2}^{0} re^{i\theta} F_-(M + re^{i\theta})\, d\theta = \frac{1}{4i} f(M), \qquad (8a)$$

$$\int_0^{\pi/2} re^{i\theta} F_+(M + re^{i\theta})\, d\theta$$

$$= \int_{-\pi}^{0} re^{i\theta} F_-(n + re^{i\theta})\, d\theta = \frac{1}{2i} f(n), \tag{8b}$$

$$\int_{\pi/2}^{\pi} re^{i\theta} F_+(N + re^{i\theta})\, d\theta$$

$$= \int_{-\pi}^{-\pi/2} re^{i\theta} F_-(N + re^{i\theta})\, d\theta = \frac{1}{4i} f(N), \tag{8c}$$

The terms corresponding to integrals along paths parallel to the y axis combine as follows. The sum of the two integrals in Eqs. (5) and (6) with $x = N$ is

$$i \int_R^r \{F(N + iy) - F(N - iy)\} \frac{dy}{e^{2\pi y} - 1} \tag{9}$$

The integrand of Eq. (9) has a well-defined limit as $y \to 0$:

$$\lim_{y\to 0} \{F(N + iy) - F(N - iy)\} \frac{dy}{e^{2\pi y} - 1} = \frac{i}{\pi} F'(N), \tag{10}$$

according to 1'Hospital's rule; therefore the limit $r \to 0$ in Eq. (9) causes no difficulty. If $F(N \pm iy)\, e^{-2\pi y} \to 0$ faster than $1/y$ as $y \to \infty$, we may let $R \to \infty$ in Eq. (9). Similar considerations apply to the two integrals with $x = M$. Finally, if

$$\lim_{y\to\infty} e^{-2\pi y} F(X \pm iy) = 0 \tag{11}$$

for $M \leq x \leq N$, the two integrals along $y = \pm R$ vanish in the limit $R \to \infty$. When Eqs. (5) and (6) are added, the results of Eqs. (7) through (11) lead us to the identity

$$S = \int_M^N F(X)\, dX + \frac{1}{2}\{f(N) + f(M)\}$$

$$- 2\pi i \;\{\sum_{\substack{\text{res} \\ R+}} F_+(Z) + \sum_{\substack{\text{res} \\ R-}} F_-(Z)\}$$

$$i \int_{\infty}^{0} dy \, \{F(N + iy) - F(N - iy)$$

$$- F(M + iy) + F(M - iy)\} \frac{1}{e^{2\pi y} - 1} \tag{12}$$

which is the desired result. To our knowledge, this result has not been published previously.

3. APPLICATION OF THE SUM FORMULA

In many practical cases it is convenient to treat potential or fields problems by using a new method proposed in our paper. In the examples that follows, the analyticity properties of $F(z)$ will determine the manner in which Eq. (12) is carried further to obtain a useable result. For example, if $F(z)$ is analytic everywhere, a special case of the Euler-Maclaurin series is obtained in which the remainder term vanishes. In our other examples, we are interested in the case where the number of terms in the sum is large, and we wish to obtain expansions of Eq. (12) in which few terms need be taken.

3.1 *An alternative derivation of the Euler-Maclaurin series*

The Euler-Maclaurin series may be written in the following form [4],

$$\sum_{n=M}^{N} f(n) = \int_{M}^{N} F(x) \, dx + \frac{1}{2} \{f(N) + f(M)\}$$

$$+ \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!} \{F^{2k-1}(N) - F^{2k-1}(M)\}$$

$$+ \frac{1}{(2K + 1)!} \int_{M}^{N} B_{2k+1} (x - [x]) \, F_{2k+1} (x) \, dx) \tag{13}$$

where $B_m = B_m(0)$ and where the $B_m(x)$ are Bernoulli polynomials, defined by the generating function [4]:

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m(x)}{m!} t^m \tag{14}$$

In Eq. (13), $[x]$ denotes the integer in the interval $(x - 1, x]$, and F^{2k-1} is the $(2k - 1)$th derivative of $F(z)$. In numerical computation, Eq. (13) is implemented by choosing the number of terms, K, in the expansion such that the remainder [the last term in Eq. (13) is small and can be neglected. Equation (13) is, in general, an asymptotic expansion which is easily derived by the method of real analysis [8], [3], [5].

We may use our sum formula, Eq. (12), to formulate an alternative derivation of the Euler-Maclaurin series for the case when $F(z)$ is entire. If the latter is true, then $F(z)$ has $F(z)$ possesses a Taylor series about any point in the complex plane. In particular, we have

$$F(x \pm iy) = \sum_{p=0}^{\infty} \frac{1}{p!} F^{p}(x)\,(\pm iy)^{p}, \tag{15}$$

Since $F(z)$ is entire, these expansions converge uniformly for all x and y. We may therefore substitute Eq. (15) with $x = M$ and N into the last term of Eq. (12) to obtain

$$\begin{aligned} & i \int_0^{\infty} dy\{F(N + iy) - F(N - iy)\} - F(M + iy) \\ & + F(M - iy)\}/(e^{2\pi y}-1) \\ & + \sum_{k=1}^{K} \frac{(-1)^{k-1}}{(2k-1)!} \{F^{2k-1}(N) - F^{2k-1}(M)\} \\ & \times \int_0^{\infty} y^{2k-1} \frac{dy}{(e^{2\pi y}-1)} \end{aligned} \tag{16}$$

Note that the even-derivative terms in Eq. (5) cancel. The uniform convergence of Eq. (15) allows us to interchange the sum and the integral in Eq. (16). Now the integral representation for Bernoulli numbers [6],

$$B_{2k} = 4k\,(-1)^{k-1} \int_0^{\infty} \frac{y^{2k-1}\,dy}{e^{2\pi y}-1} \tag{17}$$

can be used in Eq. (16) to obtain the desired result from Eq.(12):

$$\sum_{n=M}^{N} f(n) = \int_{M}^{N} F(x)\, dx + \frac{1}{2}\{f(N) + f(M)\}$$

$$+ \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\{F^{2k-1}(N) - F^{2k-1}(M)\} \tag{18}$$

This result is an important special case of the general Euler-maclaurin series, Eq. (13), in which the remainder term vanishes and the number of terms, K, is infinite.

It is useful to repeat here the conditions under which Eq. (18) holds:

(a) $F(z)$ is entire;

(b) $\lim_{y\to\infty} e^{-\pi y} F(x \pm iy) = 0,\ M \le x \le N$;

(c) $\lim_{y\to\infty} e^{-2\pi y} \begin{Bmatrix} F(N\pm iy) \\ F(M\pm iy) \end{Bmatrix} = 0.$

We also point out that Eq. (18) may not converge in the limit $N \to \infty$ or $M \to \infty$, although it is clear from our derivation that Eq.(18) converges for finite M and N.

3.2 *Functons with simple poles in R_+ or R_-*

To illustrate the use of Eq. (12) for a function with simple poles, we consider an elementary problem in two dimensions. Let $2N+1$ infinite line charges of density λ per unit length be positioned parallel to the z axis at positions $(x, y) = (n, 0)$, for $-N \le n \le N$. We wish to compute the electric field at the point $(0, a)$ in the $x-y$ plane due to this charge distribution. The electric field is

$$E = 2\lambda a \sum_{n=-N}^{N} (n^2 + a^2)^{-1}$$

$$= 2i\lambda \sum_{n=-N}^{N} (n - ia)^{-1} \tag{19}$$

The application of Eq. (12) (with $M=-N$) to this example is straightforward. The only pole of $F(z) = (z - ia)^{-1}$ is located at $z=ia$; the residue of $F_+(z)$ at this point is $-(e^{2\pi a}-1)^{-1}$. The last term in Eq. (12) is evaluated by expanding

$$(e^{2\pi y}-1)^{-1} = \sum_{m=1}^{\infty} e^{2\pi m y} \tag{20}$$

which is valid for all $y > 0$, and using the relation

$$\int_0^{\infty} dy\, e^{-\alpha y}\, (z + iy)^{-1} = ie^{-i\alpha z}\, Ei(i\alpha z) \tag{21}$$

which follows from the definition [1] of the exponential integral, $Ei(x)$:

$$Ei(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt \tag{22}$$

We obtain the following result from Eq. (12):

$$\begin{aligned} E &= 4\,\lambda \tan^{-1}(N/a) + 2\lambda a\,(N^2 + a^2)^{-1} - 4\pi\lambda\,(e^{2\pi a} - 1)^{-1} \\ &\quad - 2\lambda i \sum_{m=1}^{\infty} \{e^{-2\pi m a}\, Ei[2\pi m(a + iN)] \\ &\quad - e^{2\pi m a}\, Ei\,[2\pi m\,(-a + iN)] - \text{c.c}\} \end{aligned} \tag{23}$$

where c.c = complex conjugate. For large N, the last sum in Eq. (23) converges rapidly.

An interesting result is obtained from Eq. (23) by letting $N \to \infty$. In this limit, the second and fourth terms drop out, and the first and third terms combine to give the result [2]:

$$\begin{aligned} E &= 2\pi\lambda - 4\pi\lambda\,(e^{2\pi a} - 1)^{-1} \\ &= 2\pi\lambda \left\{\frac{e^{\pi a} - e^{-\pi a} + 2e^{-\pi a}}{e^{\pi a} - e^{-\pi a}}\right\} \\ &= 2\pi\lambda \cot h\,(\pi a) \end{aligned} \tag{24}$$

3.3 *Functions with branch points*

Our third example arises from the treatment of this sum relations, i.e.

$$\begin{aligned} W &= -\frac{C}{N} \sum_{n=0}^{N-1} \left\{1 - p\cos(2\pi n/N)\right\}^{1/2} \\ &= -\frac{C}{N} \left\{\sum_{n=0}^{N} \left[1 - p\cos(2\pi n/N)\right]^{1/2} - (1 - \text{p})^{1/2}\right\} \end{aligned} \tag{25}$$

where C is a constant and p is an arbitrary variable.

We may apply Eq. (12) (with $M=0$) to the sum in Eq. (25). The function $F(z) = [1 - p \cos(2\pi z/N)]^{1/2}$ has branch points for $p<1$ at $z = \pm iy_0$ and $z = N \pm iy_0$ (see Fig. 2), where $y_0 = \frac{N}{2\pi} \cos h^{-1} \frac{1}{p}$. These branch points lie on the boundaries of R_+ and R_-, and we deform C_+ and C_- slightly so that the branch points lie exterior to R_+ and R_-. The integral along the real axis in Eq. (12) is given by

$$\int_0^N \{1 - p \cos(2\pi z/N)\}^{1/2}\, dz$$

$$= \frac{2N}{\pi} \int_0^{\pi/2} \{1 + p - 2p \sin^2\theta\}^{1/2}\, d\theta$$

$$= \frac{2N}{\pi}(1 + p)^{1/2} E\{(\frac{2p}{1+p})^{1/2}\} \qquad (26)$$

where $E\{(\frac{2p}{1+p})^{1/2}\}$

is the complete elliptic integral of the second kind [1]. The term $\frac{1}{2}$ $[f(N) + f(0)]$ in Eq. (12) cancels the term $(1 - p)^{1/2}$ in Eq. (25). Furthermore, $F(z)$ has no poles in R_+ or R_-; thus the residue terms in Eq. (12) do not contribute. Finally, we consider the integrals along paths parallel to the imaginary axis in Fig. 2. Taking care to ensure that the integrands in Eq. (12) have the proper phases above and below the branch points, we have

$$F(N + iy) - F(N - iy) - F(iy) + F(-iy)$$

$$= \begin{cases} 0, & y \le y_0 \\ -4i\,[p \cos h\,(2\pi y/N) - 1]^{1/2}, & y > y_0 \end{cases} \qquad (27)$$

Therefore, substituting Eqs. (26) and (27) into Eq. (12), we obtain the singular part from Eq. (25):

$$W = -C\left\{ \frac{2}{\pi}(1 + p)^{1/2} E\,[(\frac{2p}{1+p})1/2] - \frac{4}{N}\int_{y_0}^{\infty} dy\, \frac{[p \cos h\,(2\pi y/N) - 1]^{1/2}}{e^{2\pi y} - 1} \right\} \qquad (28)$$

The first term in Eq. (28) is the singular part of this equation and gives rise to the well-known logarithmic singularity. Then, we expand the square root about $y = y_0$:

$$[p \cos h\ (2\pi y/N) - 1]^{1/2}$$

$$\simeq [p \cosh\ (2\pi y_0/N) + (2\pi/N)\, p(y - y_0) \times \sin h\ (2\pi y_0/N) - 1]^{1/2}$$

$$= (2\pi/N)^{1/2}\, (1 - p^2)^{1/4}\, (y - y_0)^{1/2} \tag{29}$$

In addition, we approximate $(e^{2\pi y} - 1)^{-1}$ by $e^{-2\pi y}$ in the denominator. (Both these approximations depend on the fact that y_0 1 for large N). We thus obtain

$$\int_{y_0}^{\infty} dy \frac{[p \cos h\ (2\pi y/N) - 1]^{1/2}}{e^{2\pi y} - 1}$$

$$\simeq \left(\frac{2\pi}{N}\right)^{1/2} (1 - p^2)^{1/4} \int_{y_0}^{\infty} (y - y_0)^{1/2}\, e^{-2\pi y}\, dy$$

$$= \frac{(1 - p^2)^{1/4}}{4(\pi N)^{1/2}} e^{-2\pi y_0} \tag{30}$$

Thus Eq. (28) becomes, for large N,

$$W = -C \left\{ \frac{2}{\pi} (1 + p)^{1/2}\, E\, \left[\left(\frac{2p}{1 + p}\right)^{1/2}\right] \right.$$

$$\left. - \frac{(1 - p^2)^{1/4}}{(\pi)^{1/2} N^{3/2}}\, e^{-N \cos h - (1/p)} \right\} \tag{31}$$

4. CONCLUSION

Based upon the complex integration, a new method for evaluating or approximating sums have been proposed. We consider examples that illustrate the power of the method: our first example is an alternative derivation of the Euler-Maclaurin sum formula for the case which the remainder term vanishes, and other two examples show how our method can be applied when the Euler-Maclaurin formula is not useful. The method proposed here have a more convenient for applications in applied mathematics and theoretical electromagnetics.

REFERENCES

1. Abramowitz, M., Stegum, I.A.: **Handbook of mathematical functions.** Washington, D.C. National Bureau of Standards 1964.

2. Bender, C.M., Orszag, S: **Advanced mathematical methods for scientists and engineers.** New York: McGraw-Hill 1978.

3. Carrier, G.F., Krook, M., Pearson, C.E.: **Functions of a complex variable.** New York: McGraw-Hill 1966.

4. **Henrici, P: Applied and computational complex analysis.** New York: Wiley 1977.

5. Knopp, K: **Infinite sequences and series.** New York: Dover 1956.

6. Pearson, C.E: **Handbook of applied mathematics.** New York: Van Nostrand-Reinhold 1974.

7. Sommerfeld, A: **Partial differential equations of Physics.** New York: Academic Press 1949.

8. Whittaker, E.T., Watson, G.N: **A course of modern analysis.** Cambridge: Cambridge University Press 1958.

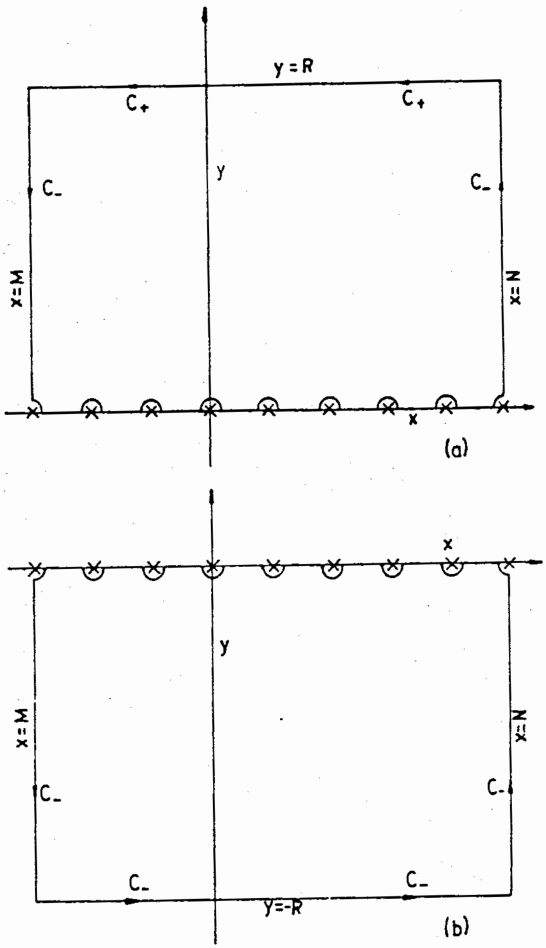


Fig.1. Complex plane with $z = x + iy$, showing the regions R_+ and R_- and their boundaries C_+ and C_-, as discussed in the text.

(a) The uppchalf-plane, showing R_+ bounded by C_+.

(b) The lower half-plane, showing R_- bounded by C_-.

The contous are indented near poles of F_+ and F_- on the real axis (at integer values of z).

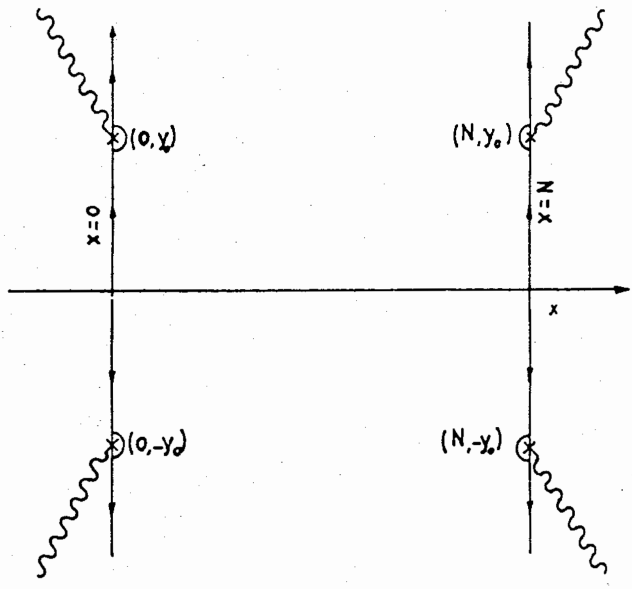


Fig. 1. Branch points and branch cuts of $F(z) = [1-p \cos(2\pi z/N)]$. The integration contours are indented so that the branch points are avoided.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 83 - 89

CONVEX ISOMETRIC FOLDING

E.M. El-Kholy

Department of Mathematics,
Faculty of Science,
Tanta University, Tanta,
Egypt.

ABSTRACT

In this paper we introduced a new type of isometric folding we called it "Convex Isometric Folding", then we have proved that the infimum of the ratio Vol. N/Vol $\phi(N)$, over all convex isoemtric foldings $\phi : N \to N$, where N is a compact 2-manifold (orientable or not), is $\frac{1}{4}$.

1. INTRODUCTION

A map $\phi : M \to N$, where M and N are C^∞ Riemannian manifolds of dimensions m and n respectively, is said to be an isometric folding of M into N if and only if for any piecewise geodesic path $\gamma : J \to M$, the induced path $\phi o \gamma : J \to N$ is a piecewise geodesic and of the same length [4]. The set of all isometric foldings $\phi : M \to N$ is denoted by $\Im(M, N)$.

Let $p : M \to N$ be a regular locally isometric covering, and let G be the group of convering transformations of p. An isometric folding $\phi \in \Im(M)$ is said to be p-invariant iff for all $g \in G$, and all $x \in M$, $p(\phi(x)) = p(\phi(g.x))$ [5]. The set of p-invariant isometric foldings is denoted by $\Im_i(M, p)$.

1.1 Definition

Let $\phi \in \Im(M, N)$, where M and N are C^∞ Riemannian manifolds of dimensions m and n respectively. We say that ϕ is a *convex isometric folding* if and only if $\phi(M)$ can be embedded as a convex set in R^n.

We will denote the set of all convex isometric foldings of M into N by $C(M, N)$, and if $C(M, N) \neq \phi$, then it forms a sub**semigroup** of $\Im(M, N)$.

1.2 Definition

We say that $\phi \in \Im_i(M, p)$ is a *p-invariant convex isometric folding* if and only if $\phi(M)$ can be embedded as a convex set in R^m.

We denote the set of p-invariant convex isometric foldings of M by $C_i(M, p)$. If $C_i(M, p) \neq \phi$, then for any convering map, $p : M \to N$, $C_i(M, p)$ is a subsemigroup of $C(M)$.

To solve our main problem we need the following results:

1.3 If N is an n-smooth Riemannian manifold, $p : M \to N$ is its universal convering and G is the group of convering transformations of p. Then $\Im(N)$ is isomorphic as a semigroup to $\Im_i(M, p)/G$ [5].

1.4 If N is an n-smooth Riemannian manifold, $p : M \to N$ is its universal covering and $\phi \in \Im(N)$ such that $\phi_* : \pi_1(N) \to \pi_1(N)$ is trivial, then the corresponding folding $\psi \in \Im_i(M, p)$ maps each fiber of p to a single point [1].

1.5 Under the same conditions of (1-4), if N is a compact 2-manifold, then Vol N/Vol $\phi(N)$ = Vol F/Vol $\psi(F)$, where F is a fundamental region of G in M[3].

2. CONVEX ISOMETRIC FOLDING AND COVERING SPACES

The next theorem establishes the relation between the set of convex isometric folding of a manifold, $C(N)$, and the set of p-invariant convex isometric folding of its universal covering space, $C_i(M, p)$.

2.1 Theorem

Let N be a manifold and $p : M \to N$ its universal covering. Let G be the group of covering transformations of p. If $C(N) \neq \phi$, then $C(N)$ is isomorphic as a semigroup to $C_i(M, p)/G$.

Proof

Let $C(N) \neq \phi$ then by using (1.3), there exists an isomorphism f from $\Im_i(M, p)/G$ into $\Im(N)$. Since $C_i(M, p)$ is a subsemigroup of $\Im_i(M, p)$, then $C_i(M, p)/G$ is a subsemigroup of $\Im_i(M,p)/G$.

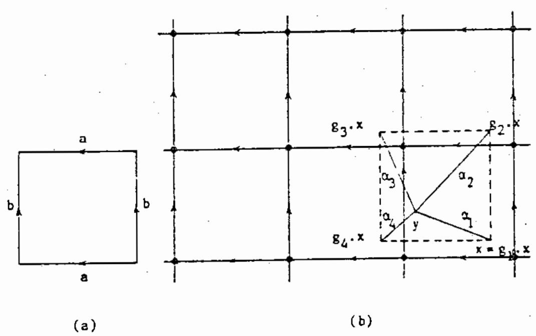
Let $h = f \mid (C_i(M, p)/G$, since $C_i(M, p)/G$ is a semigroup, thus h is a homomorphism and also is one-one. To show that h is onto we suppose that $\phi \in C(N)$, hence $\phi \in \Im(N)$ and consequently there will exists $\psi \in \Im_i(M, p)/G$. Since $\phi \in C(N)$, then ϕ_* is trivial and hence for all $x \in M$, $\psi(G.x) = \psi(x)$, and therefore $\psi \in C_i(M,p)/G$.

2.2 Theorem

Let N be a compact orientable 2-manifold and consider the universal covering space (R^2, p) of N. Let $\phi \in C(N)$ and $\psi \in C_i(R^2,p)$. Then for all $x, y \in R^2$, $d(\psi(x), \psi(y)) \leqq \Delta$, where Δ is the radius of a fundamental region for the convering space.

Proof

The theorem is true for $N = S^2$, [1], so we have to prove it for the connected sum of n-tori. First let $N = T$ be a torus homeomorphic to the quotient space obtained by identifying opposite sides of a square of length "a" as show in Fig. (1-a).

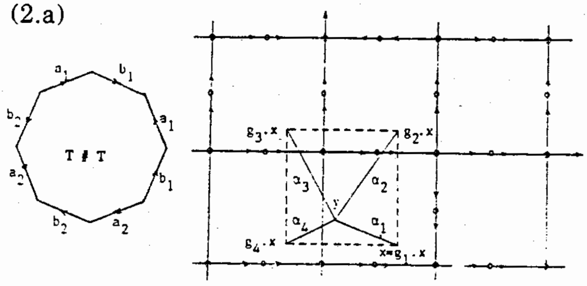


(Figure-1)

Suppose $\phi : T \to T$ is a convex isometric folding, then $\phi_*(\pi_1(T))$ is trivial. By theorem (2.1), there exists a convex isometric folding $\psi : R^2 \to R^2$ such that for all $x, y \in R^2$ and all $g \in G$, $p(\psi(x)) = p(\psi(g.x))$. Equivalently, for all $(P, Q) \in R^2$ and all $g \in Z \times Z$, there exists a unique $h \in Z \times Z$ such that $h o \psi(P, Q) = \psi(g.(P, Q))$, i.e., $\psi(P, Q) + (\sqrt{2}\,\Delta m', \sqrt{2}\,\Delta n') = \psi(P + \sqrt{2}\,\Delta m, Q + \sqrt{2}\,\Delta n)$, where $m, n, m', n' \in Z$.

Consider now any fundamental region F of the covering space (R^2, p) of T, i.e., a closed square of length a with sides identified as shown in Fig (1-b). Since ϕ_* is trivial then by (1.4) for all $x \in R^2$, $\psi(G.x) = \psi(x)$. Now let x, y be distinct points of R^2 such that $x \neq g.y$ for all $g \in G$ and let $d(x, y) = \alpha_1$. Then there is a point $x^* = g.x$ such that $d(y, x^*) = \min \alpha_i$, $\alpha_i = d(y, g_i.x)$, $i = 1, ..., 4$. Thus there are always four equivalent points $g_i.x$, $i = 1, ..., 4$ which form the vertices of a square of length a and such that $d(g_i.x, y) \leqq 2\Delta$. From Fig. (1-b) it is clear that $\max d(x^*, y) \leqq \Delta$, and since ψ is an isometric folding then $d(\psi(x), \psi(y) \leqq d(x, y)$ [4], i.e., $d(\psi(x), \psi(y)) = d(\psi(g_i.x), \psi(y)) \leq d(g_i.x, y) = d(x^*, y) \leqq \Delta$, and this proves the theorem of $N = T$.

Now consider the connected sum of two tori, $N = T \# T$, obtained as a quotient space of an octagon with sides identified as shown in Fig. (2.a)



(a) (Fig. 2) (b)

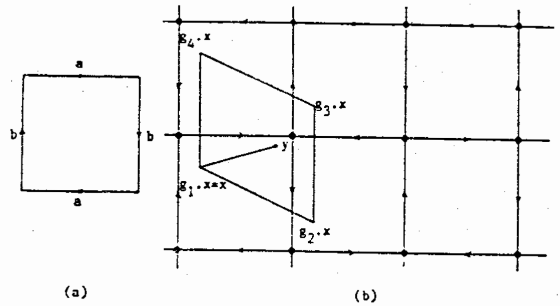
The group of convering transofrmations G will be isometric to $Z \times Z \times Z \times Z$. Using the same previous technique we can obtain four equivalent points as the vertices of a square of dimeter 2Δ, such that $\max d(y, x^*) \leqq \Delta$, and we have the result. This theorem, by using the above method, is true for the connected sum of n-tori.

2.3 Theorem

Let N be a compact non-orientable 2-manifold and consider the universal covering space (M, p) of N, let $\phi \in C(N)$ and $\psi \in C_i(M, p)$. Then for all $x, y \in M$, $d(\psi(x), \psi(y)) \leqq \Delta$, where Δ is the radius of a fundamental region for the covering space.

Proof

First let $N = P^2$ and $M = S^2$, then the theorem is true [2]. Now, consider the connected sum of two projective planes, the Klein bottle K, homeomorphic to the quotient space obtained by identifying the opposite sides of a square as shown in fig (3-a).



(Fig. 3)

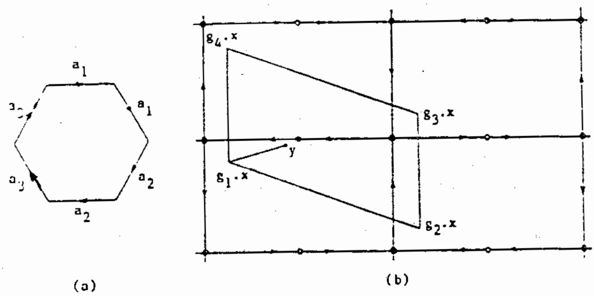
Suppose $\phi : K \rightarrow K$ is a convex isometric folding, then their exists a convex isometric folding $\psi : R^2 \rightarrow R^2$ such that for all $x \in R^2$ and $g \in G$, $p(\psi(x)) = p(\psi(g.x))$. Equivalently, for all $(P, Q) \in R^2$ and all $g \in Z \times Z_2$, there exists a unique $h \in Z \times Z_2$ such that $h o \psi(P, Q) = \psi(g.(P, Q))$, i.e., $\psi(P, Q) + (\sqrt{2}\ \Delta m', \sqrt{2}\ \Delta n') = \psi(P+\sqrt{2}\ \Delta m, \sqrt{2}\ \Delta n + (-)^m Q)$, where $m, n, m', n' \in Z$.

Any fundamental region F of the covering space (R^2, p) of K is a closed square of diameter 2Δ with boundary identified as shown in Fig. (3-b). Since ϕ_* is trivial, then for all $x \in R^2$, $\psi(G.x) = \psi(x)$. Now, let x, y be distinct points of R^2 such that $y \neq g.x$ for all $g \in G$, and let $d(x, y) = \alpha_1$. Thus there exist a point $x^* = g.x$ such that $d(y, x^*) = \min(\alpha_i)$, $\alpha_i = d(y, g_i.x)$, $i = 1, \ldots, 4$. Thus there are always four equivalent points $g_i.x$ which form the vertices of a parallelogram such that the shortest diameter is of length less than 2Δ.

Now, the point y is either inside or an the boundary of a triangle of vertices $g_1.x = x$, $g_2.x$, $g_3.x$. Let y' be a point equidistance from the verticies of this triangle. i.e., $d(y', x) = d(y', g_2.x) = d(y', g_3.x)$. From Fig. (3-b) it is clear that $d(y', x) < \Delta$, and hence $\max d(x^*, y) < \Delta$. Therefore

$$d(\psi(x), \psi(y)) = d(\psi(g_i.x), \psi(y)) \leq d(g.x_i, y) = d(x^*, y) < \Delta$$

Now, let N be the connected sum of three projective planes obtained as the quotient space of a hexagon with the sides identified in pairs as indicated in Fig. (4-a). In this case (R^2, P) is the universal



(Fig. 4)

cover of N and $G \cong Z \times Z \times Z_2$. Using the same method as that used above, we can have always equivalent points $g_i.x$, $i = 1, ..., 4$ which form the vertices of a parrellelogram its shortest diameter is of length less than 2Δ. From figure (4-b), we can see use that max $d(y,x^*) < \Delta$ and the theorem is proved. In general and by using the same technique the theorem is also true for the connected sum of n-projective planes.

3. VOLUME AND CONVEX FOLDING

The following theorem succeeded in estimating the maximum volume we may have if we convexly folded a compact 2-manifold into itself.

3.1 Theorem

The infimum of the ratio e_N = Vol N/Vol $\phi(N)$, where N is a compact 2-manifold over all convex isometric foldings $\phi \in C(N)$ of degree zero is 4.

Proof

Consider a compact 2-manifold N, and let $\phi : N \rightarrow N$ be a convex isometric folding, and since a convex isometric folding is an isometric folding, then the deg ϕ is $\pm$ 1 or 0, [4]. We will consider only the case for which deg ϕ is zero otherwise $\phi(N)$ can not be embeded as a convex subset of R^2 unless N is. In this cae the set of singularities of ϕ will decompose N into an even number of strata, say k, each of these stratum is homeomorphic to $\phi(N)$ and hence Vol N = k Vol $\phi(N)$, i.e., e_N should be an even number. To calculate the exact value of e_N consider first an orientable 2-compact manifold

N. By using (1-5), e_N = Vol F/Vol $\phi(F)$ and this means that e_N can be calculated by calculating volume of F and of its image $\phi(F)$, but F is a closed square of diameter 2Δ and $\phi(F)$ is a closed subset of F such that the distance $d(x, x')$ between any two points $x, x' \in \phi(F)$ is at most Δ. The supremum of 2-dimensional volume of such set is $\pi (\frac{\Delta}{2})^2$, and hence $2 < e_N$ but e_N is an even number, then $e_N = 4$.

Now, let N be a non-orientable 2-compact manifold, i.e., a connected sum of n-projective planes, the theorem is true for $n = 1$, [2]. The fundamental region in this case is a square or a rectangle of dimeter 2Δ according to n is even or odd. If n is an even number, then Vol $F = 2\Delta^2$ and we have the result.

Now, let n be an odd number, then F will be a rectangle of lengths $\frac{n-1}{2}a, \frac{n-1}{2}a$ and hence Vol $F = 4\Delta^2 \sin\theta\cos\theta =$

$$= 4\Delta^2 \frac{\frac{n^2+1}{2}a}{\sqrt{\frac{n^2+1}{2}}} \frac{\frac{n-1}{2}a}{\sqrt{\frac{n^2+1}{2}}} = \frac{n^2-1}{n^2+1} 2\Delta^2 .$$

Therefore $3 > e_N > 2$, for all $n > 1$. Since e_N is an even number then $e_N = 4$.

REFERNCES

1. E.M. Elkholy, Isometric and Topological Folding of Manifolds, Ph.D. thesis, Southampton University, England 1982.
2. E.M. Elkholy, Maximum Isometric Folding of the Real Projective Plane, Pakistanian *J. of Natural Science and Math.*, **Vol. 25**, No. 1, pp. 25–30, 1995.
3. E.M. Elkholy & A.E. Al-Ahmady, Maximum Volume of Isometric Foldings of Compact 2-Riemannian Manifolds, *Proc. Conf. Oper. Res. and Math. Methods Bull. Fac. Sci.* **Vol. 26A**(1), pp. 54–66, 1986.
4. S.A. Robertson, Isometric Folding of Riemannian manifolds, *Proc. Roy. Soc. Edinburgh*, (77), 275–284, 1977.
5. S.A. Robertson & E.M. Elkholy, Invariant and Equivariant Isometric Foldings, *Delta, J. Sci.* **Vol. 8(2)**, pp. 374–385, 1984.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 90 – 104

CONNECTEDNESS AND P-CONTINUITY IN BIFUZZY TOPOLOGICAL SPACES

A.S. Abu Safiya

Department of Mathematics,
University of Bahrain,
P.O. Box 32038, Isa Town,
Bahrain

A.A. Fora

Department of Mathematics,
City University Northampton Square,
London EC1V 0HB,
U.K.

ABSTRACT

The theory of bifuzzy topological spaces has recently introduced see [1]. If τ_1 and τ_2 are fuzzy topologies on X, then the tripel (X, τ_1, τ_2) is called a bifuzzy topological space. In this paper we introduce the concepts of connectedness and fixed point property in bifuzzy topological spaces. We show that these concepts are preserved under P-continuty and prove some related results in this area.

1. INTRODUCTION

Kelly [6] called the triple (X, T_1, T_2) where X is a non-empty set and T_1, T_2 are topologies on X a bitopological space. He initiated tue systematic study of such spaces. Since then several other authors have contributed to the subsequent developent of various bitopological properties. In particular connectedness in bitopological spaces was discussed by many authors see [3, 7, 8]. The purpose of this paper is to show that images and inverse images of different types of connectedness in bifuzzy topological spaces are preserved under P-continuity. We shall use P- to denote pairwise e.g. P-connected stands for pairwise connected. The τ_i-closure, τ_i-interior of a set λ will be denoted by $cl_i\lambda$, $int_i\lambda$ (or simply $cl\lambda$, $int\lambda$ in the case we have only one (fuzzy) topology on X), R denotes the real line with its usual order and χ_υ is the characteristic function of υ.

If (X, τ) is any topological space then there are two fuzzy topologies corresponding to τ, namely:

$$X/\tau = \{\chi_v : v \in \tau\} = \text{the set of all characteristic functions of open sets in } X$$

and $\omega(\tau)$ = the set of all lower semicontinuous functions from X into the colsed unit interval [0, 1].

It is clear that $X/\tau \subseteq \omega(\tau)$.

In this paper we shall follow (9) for the definitions of : fuzzy point, fuzzy topology, the direct and the inverse images of a fuzzy set and fuzzy continuous mapping. A set X on which are defined two fuzzy topologies τ_1, τ_2 is called a bifuzzy topological space (bfts for short).

2. BIFUZZY CONNECTEDNESS

Fatteh and Bassan [4] defined connectedness only for a crisp fuzzy set of a fuzzy topological space while Ajmal and Kohli [2] extended the notion of connectedness to an arbitrary fuzzy set. In [5] a fuzzy topological space (X, τ) is connected iff (X, τ) has no clopen fuzzy sets except 0 and 1.

We begin this section with the following definitions.

Definition 2.1

A bfts (X, τ_1, τ_2) is S-disconnected iff there exist non zero fuzzy sets $\lambda, \mu \in \tau_1 \cup \tau_2$ such that $\lambda + \mu = 1$ and $\lambda \cap \mu = 0$. A bfts (X, τ_1, τ_2) is called S-connected if it is not S-disconnected.

Definition 2.2

A bfts (X, τ_1, τ_2) is S_w-disconnected iff there exist non zero fuzzy sets $\lambda, \mu \in \tau_1 \cup \tau_2$ such that $\lambda + \mu = 1$. A bfts (X, τ_1, τ_2) is called S_w-connected if it is not S_w-disconnected.

Definition 2.3

A bfts (X, τ_1, τ_2) is P-disconnected iff there exist non zero fuzzy sets $\lambda \in \tau_1$ and $\mu \in \tau_2$ such that $\lambda + \mu = 1$ and $\lambda \cap \mu = 0$. A bfts (X, τ_1, τ_2) is called P-connected if it is not P-disconnected.

Definition 2.4

A bfts (X, τ_1, τ_2) is P_w-disconnected iff there exist non zero fuzzy sets $\lambda \in \tau_1$ and $\mu \in \tau_2$ such that $\lambda + \mu = 1$. A bfts (X, τ_1, τ_2) is called P_w-connected if it is not P_w-disconnected.

The implications of the above types of bifuzzy disconnectedness can be described by the following diagram.

$$\begin{array}{ccc} P & \Rightarrow & P_w \\ \Downarrow & & \Downarrow \\ S & \Rightarrow & S_w \end{array}$$

To show that all implications are not reversible we present the following examples.

Example 2.5

Let $X = [0, 1]$, $\tau_1 = \{0, 1, \lambda\}$ and $\tau_2 = \{0, 1, \mu\}$; where λ and μ are defined as follows:

$$\lambda(x) = \begin{cases} 2/3 \text{ if } 1/2 \leq x < 1 \\ 1 \text{ if } 0 \leq x < 1/2 \end{cases} \qquad \mu(x) = \begin{cases} 1/3 \text{ if } 1/2 \leq x < 1 \\ 0 \text{ if } 0 \leq x < 1/2 \end{cases}$$

It is clear that $\lambda + \mu = 1$ and $\lambda \cap \mu \neq 0$. Hence (X, τ_1, τ_2) is P_w-disconnected and S_w-disconnected but it is neither P-disconnected nor S-disconnected.

Example 2.6

Let $X = [0, 1]$, $\tau_1 = \{0, 1, \lambda, \mu\}$ and $\tau_2 = \{0, 1\}$; where λ and μ are defined as follows: $\lambda = \chi\,[0.5, 1]$, $\mu = \chi[0, 0.5)$, i.e.,

$$\lambda(x) = \begin{cases} 1 \text{ if } 1/2 \leq x < 1 \\ 0 \text{ if } 0 \leq x < 1/2 \end{cases} \qquad \mu(x) = \begin{cases} 0 \text{ if } 1/2 \leq x < 1 \\ 1 \text{ if } 0 \leq x < 1/2 \end{cases}$$

It is clear that $\lambda + \mu = 1$ and $\lambda \cap \mu = 0$. Hence (X, τ_1, τ_2) is S-disconnected and S_w-disconnected but it is neither P-disconnected nor P_w-disconnected.

Example 2.7

A fuzzy set σ in a bfts (X, τ_1, τ_2) is S-C_i-disconnected (P-C_i-disconnected) iff σ has S-C_i(P-C_i) disconnection ($i = 1, 2, 3, 4$). That is, there exist proper fuzzy sets $\lambda, \mu \in \tau_1 \cup \tau_2$ ($\lambda \in \tau_1, \mu \in \tau_2$) such that $\sigma \subseteq \lambda \cup \mu$ and

C_1: $\sigma \cap \lambda \cap \mu = 0, \lambda \not\subseteq 1-\sigma, \mu \not\subseteq 1-\sigma.$

C_2: $\lambda \cap \mu \cap 1-\sigma, \lambda \not\subseteq 1-\sigma, \mu \not\subseteq 1-\sigma.$

C_3: $\sigma \cap \lambda \cap \mu = 0, \sigma \cap \lambda \neq 0, \sigma \cap \mu \neq 0.$

C_4: $\lambda \cap \mu \subseteq 1-\sigma, \sigma \cap \lambda \neq 0, \sigma \cap \mu \neq 0.$

A fuzzy set σ in a bfts (X, τ_1, τ_2) is said to be S-C_i(P-C_i) connected if there does not exist an S-C_i(P-C_i) disconnected of σ in X ($i = 1,2,3,4$).

3 Connectedness and P-continuity

We start with the following definition.

Definition 3.1

Consider a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, then f is said to be:

1) continuous if $f:(X,\tau_1)\to(Y,\sigma_1)$ & $f:(X,\tau_2)\to(Y,\sigma_2)$ are continuous.

2) P-continuous iff for any $\mu \in \sigma_1 \cup \sigma_2, f^{-1}(\mu) \in \tau_1 \cup \tau_2$.

3) P-open iff for any $\mu \in \tau_1 \cup \tau_2, f^{-1}(\mu) \in \sigma_1 \cup \sigma_2$.

Clearly if f is continuous then it is P-continuous but the converse is not true in general.

Example 3.2

Let $X = I$ and $f : (X, X/\tau_{1.r.}, X/\tau_{r.r.}) \to (X, X/\tau_{r.r.}, X/\tau_{1.r})$ be defined by $f(x) = x$, then f is P-continuous but not continuous.

Theorem 3.3

If $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is a P-continuous surjection function and λ is an S-C_1-connected fuzzy set in X, then $f(\lambda)$ is an S-C_1-connected fuzzy set in Y.

Proof

Suppose that $f(\lambda)$ is not S-C_1-connected, then there exist non zero proper fuzzy sets $\mu, \nu \in \gamma_1 \cup \gamma_2$ such that $f(\lambda) \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap f(\lambda) \not\subset \mu^c$ and $f(\lambda) \not\subset \nu^c$. Using theorem 4.1($f$) of Chang (1968) page 185, since $\lambda \subseteq f^{-1}(f(\lambda))$ and $f^{-1}(f(\lambda)) \subseteq f^{-1}(\mu \cup \nu) = f^{-1}(\mu) \cup$

$f^{-1}(\nu)$, then $\lambda \subseteq f^{-1}(\mu) \cup f^{-1}(\nu) \in \tau_1 \cup \tau_2$. Also $f^{-1}(\mu) \cap f^{-1}(\nu) \cap \lambda = f^{-1}(0) = 0$. Since $f(\lambda) \not\subset \nu^c$, so there exist y_1 and y_2 such that

$$\mu(y_1) > 1 - f(\lambda)(y_1) \quad \text{..... (1)}$$

and $$\nu(y_2) > 1 - f(\lambda)(y_2) \quad \text{..... (2)}$$

As f is onto $f^{-1}(\{y_1\})$ and $f^{-1}(\{y_2\})$ are non-empty subsets of X. By inverse and direct image of f we have $f^{-1}(\mu)(x) = \mu(y_1)$ for every $x \in f^{-1}(\{y_1\})$ and $f(\lambda)(y_1) = \sup \{\lambda(x) : x \in f^{-1}(\{y_1\})\}$. We claim that $f^{-1}(\mu) \not\subset \lambda^c$ and $f^{-1}(\nu) \not\subset \lambda^c$. Suppose $f^{-1}(\mu) \subseteq \lambda^c$, then $f^{-1}(\mu)(x) \leq 1 - \lambda(x)$ for every $x \in f^{-1}(\{y_1\})$; i.e. $f(x) \in \{y_1\}$ and so we have $\mu(f(x)) \leq 1-\lambda(x)$ which implies $\lambda(x) \leq 1 - \mu(y_1)$; i.e. $\sup \{\lambda(x) : x \in f^{-1}(\{y_1\})\} \leq 1 - \mu(y_1)$ and so $f(\lambda)(y_1) \leq 1 - \mu(y_1)$ which contradicts (1). Similarly $f^{-1}(\nu) \subseteq \lambda^c$ contradicts (2). Hence $f^{-1}(\mu) \not\subset \lambda^c$ and $f^{-1}(\nu) \not\subset \lambda^c$ and so λ is S-C_1-disconnected which is a contradiction. Therefore $f(\lambda)$ is S-C_1-connected.

Theorem 3.4

If $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is a P-continuous surjection function and λ is an S-C_2-connected fuzzy set in X, then $f(\lambda)$ is S-C_2-connected.

Proof

Similar to the proof of Theorem 3.3.

Theorem 3.5

If $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is a bifuzzy P-continuous function and λ is an S-C_3-connected fuzzy set in X, then $f(\lambda)$ is S-C_3-connected.

Proof

Suppose that $f(\lambda)$ is S-C_3-disconnected, then there exist fuzzy sets $\mu, \nu \subseteq \gamma_1 \cup \gamma_2$ such that $f(\lambda) \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap f(\lambda) = 0$, $f(\lambda) \cap \mu \neq 0$ and $f(\lambda) \cap \nu \neq 0$. Since $\lambda \subseteq f^{-1}(f(\lambda))$ and $f^{-1}(f(\lambda)) \subseteq f^{-1}(\mu \cup \nu) = f^{-1}(\mu) \cup f^{-1}(\nu)$ then $\lambda \subseteq f^{-1}(\mu) \cup f^{-1}(\nu) \in \tau_1 \cup \tau_2$. Also $f^{-1}(\mu) \cap f^{-1}(\nu) \cap \lambda = f^{-1}(0) = 0$. Since $f(\lambda) \cap \mu \neq 0$, so there exists $y_0 \in Y$ such that $f(\lambda)(y_0) \cap \mu(y_0) \neq 0$ which implies that $f(\lambda)(y_0) > 0$ where $f(\lambda)(y_0) = \sup \{\lambda(x): x \in f^{-1}(\{y_0\})\}$ and gives that

$f^{-1}(\{y_0\}) \neq \phi$. So there exists $x_0 \in X$ such that $x_0 \in f^{-1}(\{y_0\})$; i.e., $f(x_0) = y_0$. Since $f(\lambda)(y_0) > 0$, then there exists $x_1 \in X$ such that $x_1 \in f^{-1}(\{y_0\})$) and so $0 < \lambda(x_1) \leq f(\lambda)(y_0)$. Now $f^{-1}(\mu)(x_1) = \mu(f(x_1)) = \mu(y_0) \neq 0$ and $\lambda(x_1) \neq 0$. Hence $f^{-1}(\mu) \cap \lambda \neq 0$. Similarly we can show that $f^{-1}(v) \cap \lambda \neq 0$. This shows that λ is S-C_3-disconnected which is a contradiction. Hence $f(\lambda)$ is an S-C_3-connected fuzzy set in Y.

Theorem 3.6

If $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is a P-continuous function and λ is an S-C_4-connected fuzzy set in X, then $f(\lambda)$ is an S-C_4-connected.

Proof

Similar to the proof of Theorem 3.5.

Theorem 3.7

A fuzzy P-continuous image of a fuzzy S-connected space is fuzzy S-connected.

Proof

Let $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ be a P-continuous function and suppose on the contrary that Y is not S-connected. Then there exist non-zero fuzzy sets $\lambda, \mu \in \sigma_1 \cup \sigma_2$ such that $\lambda + \mu = 1$ and $\lambda \cap \mu = 0$. Since f is P-continuous then $f^{-1}(\lambda), f^{-1}(\mu) \in \tau_1 \cup \tau_2$. We claim that $f^{-1}(\lambda) + f^{-1}(\mu) = 1$. To prove our claim, suppose not. Then there exists $x \in X$ such that $f^{-1}(\lambda)(x) + f^{-1}(\mu)(x) \neq 1$ which implies that $\lambda(f(x)) + \mu(f(x)) \neq 1$ which contradicts $\lambda + \mu = 1$. Hence $f^{-1}(\lambda) + f^{-1}(\mu) = 1$. We claim that $f^{-1}(\lambda) \cap f^{-1}(\mu) = 0$. To prove our claim, suppose not. Then there exists $x \in X$ such that $f^{-1}(\lambda)(x) \cap f^{-1}(\mu)(x) > 0$ which implies that $\lambda(f(x)) \cap \mu(f(x)) > 0$ which contradicts $\lambda \cap \mu = 0$. Hence (X, τ_1, τ_2) is S-disconnected which is again a contradiction. Hence (Y, τ_1, τ_2) is S-connected.

Theorem 3.8

A fuzzy P-continuous image of a fuzzy P-connected space is fuzzy P-connected.

Proof

Similar to the proof of theorem 3.7.

Definition 3.9

A function $f : X \to X$ has a fixed point if there exist $t \in X$ such that $f(t) = t$. The point t is called a fixed point of f.

Definition 3.10

A fts (X, τ) has the fixed point property (f.p.p) if every continuous function from X into itself has a fixed point.

Definition 3.11

Consider a bfts (X, τ_1, τ_2),

(i) if every continuous function from (X, τ_1, τ_2) into itself has a fixed point we say that X has f.p.p.

(ii) if every P-continuous function from (X, τ_1, τ_2) into itself has a fixed point we say that X has P-f.p.p.

Proposition 3.12

If f is a P-continuous function from (X, τ_1, τ_2) into (Y, σ_1, σ_2), then f is continuous as a function from $(X, <\tau_1, \tau_2>)$ into $(Y, <\tau_1, \tau_2>)$.

Proof

Let μ be a subbasic open fuzzy set in $(Y, <\sigma_1, \sigma_2>)$ then $\mu \in \sigma_1 \cup \sigma_2$ and so $f-1(\mu) \in \tau_1 \cup \tau_2$ but $\tau_1 \cup \tau_2 \subseteq <\tau_1, \tau_2>$, therefore f is a continuous function from $(X, <\tau_1, \tau_2>)$ into $(Y, <\sigma_1, \sigma_2>)$.

Theorem 3.13

If (X, τ_1, τ_2) is a bfts such that $(X, <\tau_1, \tau_2>)$ uas the f.p.p, then (X, τ_1, τ_2) has the P-f.p.p.

Proof

Let $f : (X, \tau_1, \tau_2) \to (X, \tau_1, \tau_2)$ be a P-continuous function, then from proposition 2.14 $f : (X, <\tau_1, \tau_2>) \to (X, <\tau_1, \tau_2>)$ is continuous and so f has a fixed point. Hence (X, τ_1, τ_2) has the P-f.p.p.

Proposition 3.14

Let $f : X \to X$ be a function. Then the following are equivalent:

(i) $f : (X, T_1, T_2) \to (X, T_1, T_2)$ is P-continuous.

(ii) $f : (X, \omega(T_1), \omega(T_2)) \to (X, \omega(T_1), \omega(T_2))$ is P-continuous.

(iii) $f : (X, X/T_1, X/T_2) \to (X, X/T_1, X/T_2)$ is P-continuous.

Proof

(i) $\Rightarrow$ (ii) Let $\mu \in \omega(T_1) \cup \omega(T_2)$. We are going to show that $f^{-1}(\mu) \in \omega(T_1) \cup \omega(T_2)$. Now $\mu : (X, T_i) \to (I, T_{r.r})$ is continuous and so $\mu^{-1}(0, 1] \in T_1 \cup T_2$ and so we have by (i) f−1(μ−1(0, 1]) $\in$ $T_1 \cup T_2$ but $f^{-1}(\mu^{-1}(0, 1]) = f^{-1}(\mu)]^{-1}$ (0, 1]) $\in T_1 \cup T_2$. Hence $f^{-1}(\mu) : (X, T_i) \to (I, T_{r.r})$ is continuous and so $f^{-1}(\mu) \in \omega(T_1) \cup \omega(T_2)$ which completes the proof.

(ii) $\Rightarrow$ (iii) Let $\chi_u \in X/T_1 \cup X/T_2$. Then $\chi_u \in \omega(T_1) \cup \omega(T_2)$ and so by (ii) $f^{-1}(\chi_u) \in \omega(T_1) \cup \omega(T_2)$. That is $f^{-1}(\chi_u) : (X, T_i) \to (I, T_{r.r})$ is continuous. Therefore $[f^{-1}(\chi_u)]^{-1}$ (0, 1] $\in T_1 \cup T_2$ but $[f^{-1}(\chi_u)]^{-1}$ (0, 1] $= [\chi f^{-1}(u)]^{-1}$ (0,1] $\in T_1 \cup T_2$ which implies that $\chi f^{-1}(u) : (X, T_i) \to (I, T_{r.r})$ is continuous. Hence $\chi f^{-1}(u) \in X/T_1 \cup X/T_2$. That is $f^{-1}(\chi_u) \in X/T_1 \cup X/T_2$.

(iii) $\Rightarrow$ (i) Let $u \in T_1 \cup T_2$. Then $\chi_u \in X/T_1 \cup X/T_2$ and so $f^{-1}(\chi_u) \in X/T_1 \cup X/T_2$ but $f^{-1}(\chi_u) = \chi f^{-1}(u) \in X/T_1 \cup X/T_2$ which implies that $f^{-1}(u) \in T_1 \cup T_2$. Hence f is P-continuous.

Theorem 3.15

Let (X, T_1, T_2) be a bts. Then

(1) (X, T_1, T_2) has P-f.p.p iff $(X, \omega(T_1), \omega(T_2))$ has the P-f.p.p.

(2) (X, T_1, T_2) has P-f.p.p iff $(X, \omega(T_1), \omega(T_2))$ has the P-f.p.p.

Proof

(1) Let $f : (X, \omega(T_1), \omega(T_2)) \to (X, \omega(T_1), \omega(T_2))$ be P-continuous. We are going to prove that f has a fixed point. Using proposition 3.14, $f : (X, T_1, T_2) \to (X, T_1, T_2)$ is

P-continuous and hence f has a fixed point. Similarly we treat the other implication.

(2) The proof is similar to (1).

4 More results on connectedness

We start this section by extending some of the results obtained by Ajmal and Kohle [2] to bifuzzy topological spaces.

Definition 4.1

Let (X, τ_1, τ_2) be a bfts and λ_1, λ_2 be two fuzzy sets in X. Then

1) λ_1 and λ_2 are said to be disjoint iff $\lambda_1 \cap \lambda_2 = 0$.

2) λ_1 and λ_2 are said to be interesting iff $\lambda_1 \cap \lambda_2 = 0$.

3) λ_1 and λ_2 are said to be overlapping if there exists $x \in X$ such that $\lambda_1(x) > 1 - \lambda_2(x)$. In this case λ_1 and λ_2 are said to be overlap at x.

Theorem 4.2

If λ_1 and λ_2 are intersecting S-C_3-connected fuzzy sets in a bfts (X, τ_1, τ_2) then $\lambda_1 \cup \lambda_2$ is S-C_3-connected.

Proof

Suppose that $\lambda_1 \cup \lambda_2$ is S-C_3-disconnected, then there exist $\mu, \nu \in \tau_1 \cup \tau_2$ such that $\lambda_1 \cup \lambda_2 \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap (\lambda_1 \cup \lambda_2) = 0$, $\mu \cap (\lambda_1 \cup \lambda_2) \neq 0$ and $\nu \cap (\lambda_1 \cup \lambda_2) \neq 0$. Since $\lambda_1 \cup \lambda_2 \subseteq \mu \cup \nu$, then it is clear that $\lambda_1 \subseteq \mu \cup \nu$ and $\lambda_2 \subseteq \mu \cup \nu$. Since $(\mu \cap \nu) \cap (\lambda_1 \cup \lambda_2) = 0$, then we have $[(\mu \cap \nu) \cap \lambda_1] \cup [(\mu \cap \nu) \cap \lambda_2] = 0$ which implies that $(\mu \cap \nu) \cap \lambda_1 = 0$ and $(\mu \cap \nu) \cap \lambda_2 = 0$. Since λ_1 and λ_2 are S-C_3-connected, then ($\mu \cap \lambda_1 = 0$ or $\nu \cap \lambda_1 = 0$) and ($\mu \cap \lambda_2 = 0$ or $\nu \cap \lambda_2 = 0$). Suppose $\mu \cap \lambda_1 = 0$. Since λ_1 and λ_2 are intersecting, then there exists $x \in X$ such that $(\lambda_1 \cap \lambda_2)(x) \neq 0$ which implies that $\lambda_1(x) \neq 0$ and $\lambda_2(x) \neq 0$. We claim that $\nu \cap \lambda_2 \neq 2$. To prove claim, suppose, the contrary $\nu \cap \lambda_2 = 0$. Then $(\nu \cap \lambda_2)(x)$ gives that $\nu(x) = 0$ and so $(\mu \cup \nu)(x) = 0$ which contradicts that $(\lambda_1 \cup \lambda_2)(x) \subseteq (\mu \cup \nu)(x)$ because $(\lambda_1 \cup \lambda_2)(x) \neq 0$. Therefore $\nu \cap \lambda_2 \neq 0$ and so $\mu \cap \lambda_2 = 0$. Hence $\mu \cap (\lambda_1 \cup \lambda_2) = 0$ which contradicts that $\mu \cap (\lambda_1 \cup \lambda_2) \neq 0$. Similarly if $\nu \cap \lambda_1 = 0$ we can show that $\mu \cap \lambda_2 = 0$ is not possible. Hence $\nu \cap \lambda_2 = 0$. Therefore $\nu \cap (\lambda_1 \cup \lambda_2) = 0$ which contradicts $\nu \cap (\lambda_1 \cup \lambda_2) \neq 0$. Hence $\lambda_1 \cup \lambda_2$ is S-C_3-connected.

Theorem 4.3

If λ_1 and λ_2 are intersecting S-C_4-connected fuzzy sets in a bfts (X, τ_1, τ_2) then $\lambda_1 \cup \lambda_2$ is S-C_4-connected.

Proof

Since $\lambda_1 \cup \lambda_2 \subseteq \mu \cup \nu$ implies that $\lambda \subseteq \mu \cup \nu$ and $\subseteq \mu \cup \nu$ and $(\mu \cap \nu) \subseteq (\lambda_1 \cup \lambda_2)^c = \lambda_1{}^c \cap \lambda_2{}^c$ implies that $\mu \cap \nu \subseteq \lambda_2{}^c$. Then the proof follows the same steps as in Theorem 4.2.

The following example illustrates that the above theorems are not valid for disjoint (non-intersecting) fuzzy sets.

Example 4.4

Let $X = [0, 1]$ and define fuzzy sets μ and ν as follows:

$$\mu(x) = \begin{cases} 0 \text{ if } 2/3 \leq x < 1 \\ 2/3 \text{ if } 0 \leq x \leq 2/3 \end{cases}, \quad \nu(x) = \begin{cases} 2/3 \text{ if } 2/3 \leq x \leq 1 \\ 0 \text{ if } 0 \leq x \leq 2/3 \end{cases}$$

Then $\tau_1 = \{0, 1, \mu\}$ and $\tau_2 = \{0, 1, \nu\}$ are fuzzy topologies on X.

Define fuzzy sets λ_1 and λ_2 as follows:

$$\lambda_1(x) = \begin{cases} 1/3 \text{ if } 2/3 \leq x \leq 1 \\ 0 \text{ if } 0 \leq x \leq 2/3 \end{cases}, \quad \lambda_2(x) = \begin{cases} 0 \text{ if } 2/3 < x \leq 1 \\ 1/3 \text{ if } 0 \leq x \leq 2/3 \end{cases}$$

It is clear that $\lambda_1 \cap \lambda_2 = 0$, λ_1 and λ_2 are S-C_3-connected (S-C_4-connected) because $\lambda_1 \cap \mu = 0$ and $\lambda_2 \cap \mu = 0$ but $\lambda_1 \cup \lambda_2 = 1/3$ is an S-C_3-disconnected because $(\lambda_1 \cap \lambda_2) \subseteq \mu \cup \nu$, $(\lambda_1 \cup \lambda_2) \cap (\mu \cap \nu) = (1/3) \cap 0 = 0$, $(\lambda_1 \cap \lambda_2) \cap \mu \neq 0$ and $(\lambda_1 \cap \lambda_2) \cap \nu \neq 0$.

Theorem 4.5

Let $\{\lambda_i : i \in \Delta\}$ be a family of S-C_3-connected fuzzy sets in (X, τ_1, τ_2) such that $i, j \in \Delta$, $i \neq j$, the fuzzy sets λ_i and λ_j are intersecting. Then $\cup \{\lambda_i : i \in \Delta\}$

Proof

Let $\{\lambda_i : i \in \Delta\}$, where λ_i is as stated in the above theorem ($i \in \Delta$). To prove λ is S-C_3-connected, suppose not. Then there exists $\mu, \nu \in \tau_1 \cup \tau_2$ such that $\lambda \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap \lambda = 0$ $\mu \cap \lambda \neq 0$ and $\nu \cap \lambda \neq 0$. Now fix $k \in \Delta$. Since λ_k is S-C_3-connected and we have clearly $\lambda_k \subseteq \mu \cup \nu$, $(\mu \cap \lambda) \cap \lambda_k = 0$, therefore $\mu \cap \lambda_k = 0$ or $\mu \cap \lambda_k = 0$. We shall deal with deal with the first case only because the second case can be treated similarly. So we may assume that $\mu \cap \lambda_k = 0$. We claim that $\nu \cap \lambda_i \neq 0$ for all $i \in \Delta - \{k\}$. To prove our claim, suppose note, *i.e.*, $\nu \cap \lambda_i = 0$ for some $i \in \Delta - \{k\}$. Let Let $\Delta_1 = \{i \in \Delta - \{k\} : \nu \cap \lambda_i = 0\}$, then $\Delta_1 \neq \phi$. Now let $i \in \Delta_1$. Then $\lambda_k \cap \lambda_i \neq 0$. So there exists $x \in X$ such that $(\lambda_k \cap \lambda_i)(x) \neq 0$. This implies that $\lambda_k(x) \neq 0$ and $\lambda_i(x) \neq 0$. Since $\mu \cap \lambda_i = 0$, therefore $\mu(x) = 0$. Since $\nu \cap \lambda i = 0$ and $\lambda_i(x) \neq 0$, therefore $\nu(x) = 0$. Consequently $(\mu \cup \nu)(x) = 0$. The fact that $\lambda \subseteq \mu \cup \nu$ implies that $\lambda(x) = 0$ and this implies that $\lambda_i(x) = 0$ for all $i \in \Delta$ which is a contradiction. This completes the proof of our claim that $\nu \cap \lambda_i \neq 0$ for all $i \in \Delta - \{k\}$. For $i \in \Delta - \{k\}$ we have $\lambda_i \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap \lambda_i = 0$ and λ_i is S-C_3-connected. This implies that $\mu \cap \lambda_i = 0$ or $\nu \cap \lambda_i = 0$. Combing this result with above claim we conclude that $\mu \cap \lambda_i = 0$ for all $i \in \Delta - \{k\}$. But we know that $\mu \cap \lambda_k = 0$. This implies $\mu \cap \lambda_i = 0$ for all $i \in \Delta$. Consequently $\mu \cap \lambda = 0$ and this is absurb. Hence $\lambda = \cup \{\lambda_i : i \in \Delta\}$ is S-C_3-connected.

Theorem 4.6

. Let $\{\lambda_i : i \in \Delta\}$ be a family of S-C_4-connected fuzzy sets in (X, τ_1, τ_2) such that for $i, j \in \Delta$, $i \neq j$, the fuzzy set λ_i and λ_j are intersecting. Then $\cup \{\lambda_i : i \in \Delta\}$ is S-C_4-connected.

Proof

The proof follows the same steps as Theorem 4.5.

Corollary 4.7

If $\{\lambda_i : i \in \Delta\}$ is a family of S-C_3-connected (S-C_4-connected) fuzzy sets in (X, τ_1, τ_2) and $\cap \{\lambda_i : i \in \Delta\} \neq 0$ where $i \in \Delta$, then $\cup\lambda_1$ is an S-C_3-connected (S-C_4-connected).

Proof

Since $\cap \{\lambda_i : i \in \Delta\} \neq 0$, then $\lambda_i \cap \lambda_j \neq 0$ for all $i \neq j$, i.e. λ_i and λ_j are intersecting for $i \neq j$.

The following example shows that Theorem (3.4.12) fails for S-C_1-connectedness S-C_2-connectedness).

Theorem 4.8

Let $X = [0, 1]$ and define fuzzy sets μ and ν as follows:

$$\mu(x) = \begin{cases} 6/7 \text{ if } 2/3 < x \leq 1 \\ 2/7 \text{ if } 0 \leq x \leq 2/3 \end{cases}, \quad \nu(x) = \begin{cases} 2/7 \text{ if } 2/3 < x \leq 1 \\ 6/7 \text{ if } 0 \leq x \leq 2/3 \end{cases}$$

Then $\tau_1 = \{0, 1, \mu\}$ and $\tau_2 = \{0, 1, \nu\}$ are fuzzy topologies on X.

Define fuzzy sets λ_1 and λ_2 as follows:

$$\lambda_1(x) = \begin{cases} 1/7 \text{ if } 2/3 < x \leq 1 \\ 2/7 \text{ if } 0 \leq x \leq 2/3 \end{cases}, \quad \lambda_2(x) = \begin{cases} 2/7 \text{ if } 2/3 < x \leq 1 \\ 1/7 \text{ if } 0 \leq x \leq 2/3 \end{cases}$$

It is clear that $\lambda_1 \cap \lambda_2 \neq 0$, λ_1 and λ_2 are S-C_2-connected because $\mu \subseteq \lambda_1{}^c$ and $\nu \subseteq \lambda_2{}^c$ but $\lambda_1 \cup \lambda_2$ is an S-C_2-connected because $2/7 = (\lambda_1 \cup \lambda_2) \subseteq (\mu \cup \nu) = 6/7$, $2/7 = (\mu \cap \nu) \subseteq (\lambda_1 \cup \lambda_2)^c = 5/7$, $\mu \not\subset (\lambda_1 \cup \lambda_2)^c$ and $\nu \not\subset (\lambda_1 \cup \lambda_2)^c$.

Theorem 4.9

If λ_1 and λ_2 are overlapping S-C_1-connected fuzzy sets in a bfts (X, τ_1, τ_2) then $\lambda_1 \cup \lambda_2$ is S-C_1-connected.

Proof

Suppose that $\lambda_1 \cup \lambda_2$ is S-C_1-disconnected, then there exist non zero fuzzy sets $\mu, \nu \in \tau_1 \cup \tau_2$ such that; $(\lambda_1 \cup \lambda_2) \subseteq (\mu \cup \nu)$, $(\mu \cup \nu) \cap (\lambda_1 \cup \lambda_2) = 0$, $\mu \not\subset (\lambda_1 \cup \lambda_2)^c$ and $\mu \in (\lambda_1 \cup \lambda_2)^c$. (1)

Since $(\lambda_1 \cup \lambda_2) \subseteq (\mu \cup \nu)$, then it is clear that $\lambda_1 \subseteq \mu \cup \nu$ and $\lambda_2 \subseteq \mu \cup \nu$. Since $(\mu \cup \nu) \cap (\lambda_1 \cup \lambda_2) = 0$, then we have $[(\mu \cap \nu) \cap \lambda_1] \cup [(\mu \cap \nu) \cap \lambda_2] = 0$ which implies that $(\mu \cup \nu) \cap \lambda_1 = 0$ and $(\mu \cup \nu) \cap \lambda_2 = 0$. Since λ_1 and λ_2 are S-C_1-connected then $(\lambda_1 \subseteq \mu^c$ or $\lambda_1 \subseteq \mu^c)$ and $(\lambda_2 \subseteq \mu^c$ or $\lambda_2 \subseteq \mu^c)$. Since λ_1 and λ_2 are overlapping, then there exists $y \in X$ such that

$$\lambda_1(y) > 1 - \lambda_2(y) \tag{2}$$

Now consider the following:

Case I. Suppose $\lambda_1 \subseteq \mu^c$, then by (2) we have

$$\lambda(y) \leq 1 - \lambda_1(y) < \lambda_2(y) \tag{3}$$

We claim that $\lambda_2 \not\subset \nu^c$. Suppose, if possible $\lambda_2 \subseteq \nu^c$ which gives

$$\nu(y) \leq 1 - \lambda_2(y) < \lambda_1(y) \tag{4}$$

Now by (3) and (4), $(\mu \cup \nu)(y) < (\lambda_1 \cup \lambda_2)(y)$ which implies that $(\lambda_1 \cup \lambda_2) \not\subset (\mu \cup \nu)$, this contradicts (1). Hence $\lambda_2 \not\subset \nu_c$ and so $\lambda_2 \subseteq \mu^c$. Therefore $\mu \leq \lambda_1{}^c \cap \lambda_2{}^c = (\lambda_1 \cup \lambda_2)^c$.

Case II. Suppose $\lambda_1 \subseteq \nu^c$. Here we can show as in case I that $\mu \not\subset 1 - \lambda_2$. Therefore $\nu \subseteq 1 - \lambda_2$. Hence $\nu \leq \lambda_1{}^c \cap \lambda_2{}^c = (\lambda_1 \cup \lambda_2)$. Therefore $\mu \subseteq (\lambda_1 \cup \lambda_2)^c$ and $\nu \subseteq (\lambda_1 \cup \lambda_2)^c$. This contradicts (1). Hence $\lambda_1 \cup \lambda_2$ is S-C_1-connected.

Theorem 4.10

If λ_1 and λ_2 are overlapping S-C_2-connected fuzzy sets in a bfts (X, τ_1, τ_2) then $\lambda_1 \cup \lambda_2$ is S-C_2-connected.

Proof

Since $\lambda_1 \cup \lambda_2 \subseteq \mu \cup \nu$ implies that $\lambda_1 \subseteq \mu \cup \nu$ and $\lambda_2 \subseteq \mu \cup \nu$ and $(\mu \cap \nu) \subseteq (\lambda_1 \cup \lambda_2)^c = \lambda_1{}^c \cap \lambda_2{}^c$ implies that $\mu \cap \nu \subseteq \lambda_1{}^c$ and $\mu \cap \nu \subseteq \lambda_2{}^c$. Then the proof follows the same steps as in Theorem 4.9.

Theorem 4.11

Let $\{\lambda_i : i \in \Delta\}$ be a family of S-C_1-connected fuzzy sets in X such that for $i, j \in \Delta$, $i \neq j$, the fuzzy sets λ_i and λ_j are overlapping. Then $\cup \{\lambda_i : i \in \Delta\}$ is an S-C_1-connected.

Proof

Let $\lambda = \cup \{\lambda_i : i \in \Delta\}$ and λ_k be any fuzzy set of the given family and so λ_k is an S-C_1-connected. Suppose that λ is an S-C_1-disconnected. Then there exist non zero fuzzy sets μ, $\nu \in \tau_1 \cup \tau_2$ such that

$$\lambda \subseteq \lambda \cup \nu,\ (\mu \cap \nu) \cap \lambda = 0\ ((\mu \cap \nu) \subseteq \lambda^c),\ \mu \not\subset \lambda^c \text{ and } \nu \not\subset \lambda^c. \quad (1)$$

Since $\lambda \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap \lambda = 0$, then $\lambda_k \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap \lambda_k = 0$. Since λ_k is S-C_1-connected, then $\mu \subseteq \lambda^c$. Since λ_1 and λ_2 are overlapping, then there exists $y \in X$ such that

$$\lambda_k(y) > 1 - \lambda_i(y) \quad (2)$$

Now consider the following:

Case I. Suppose $\lambda_k \subseteq \mu^c$, then by (2) we have

$$\mu(y) \leq 1 - \lambda_k(y) < \lambda_i(y), \quad (3)$$

We claim that $\lambda_i \not\subset \nu^c$. Suppose, if possible $\lambda_i \subseteq \nu^c$ which gives

$$\nu(y) \leq 1 - \lambda_i(y) < \lambda_k(y), \quad (4)$$

Now by (3) and (4), $(\mu \cup \nu)(y) < (\lambda_k \cup \lambda_i)(y)$ which implies that $(\lambda_k \cup \lambda_i) \not\subset (\mu \cup \nu)$, this contradicts (1). Hence $\lambda_i \not\subset \nu^c$ and so $\lambda_i \subseteq \mu^c$. Therefore $\mu \subseteq \lambda_i{}^c = \lambda^c$.

Case II. Suppose $\lambda_k \subseteq \nu^c$. Here we can show as in case I that $\mu \not\subset 1 - \lambda_i$. Hence $\nu \subseteq \cap \lambda_i{}^c = \lambda^c$. This contradicts (1). Hence λ is S-C_1-connected.

Theorem 4.12

Let $\{\lambda_i : i \in \Delta\}$ be a family of S-C_2-connected fuzzy sets in X such that for $i, j \in \Delta$, $i \neq j$, the fuzzy sets λ_i and λ_j are overlapping. Then $\cup \{\lambda_i : i \in \Delta\}$ is an S-C_2-connected.

Proof

The proof follows the same steps as Theorem 4.11.

Corollary 4.13

Let $\{\lambda_i : i \in \Delta\}$ is a family of an S-C_1-connected (S-C_2-connected) fuzzy sets in (X, τ_1, τ_2) and p be a fuzzy point with support x and value 1/2 such that $p(x) \in \cap \{\lambda_i : i \in \Delta\}$, then $\cup \{\lambda_i : i \in \Delta\}$ is an S-C_1-connected (S-C_2-connected) fuzzy sets in X.

Proof

Since $p(x) \in \cap \{\lambda_i : i \in \Delta\}$, then λ_i and λ_j are overlaping for all $i, j \in \Delta$.

REFERENCES

1. A.S. Au Safiya, A.A. Fora and M.W. Warner "*Fuzzy Sets and Systems*" 62(1994): 367–373.
2. N, Ajmal, and J.K.Kohle "Connectedness in fuzzy topological spaces" *Fuzzy Sets and Systems* 31(1989): 369–388.
3. T.Birsan "Compacite dans les espaces bitopologiques" An. st. Univ. Lasi, S.I.a. Matematica 15(1969): 317–328.
4. U.V. Fatteh and D.S. Bassan "Fuzzy connectedness and its stronger forms" *J. Math. Anal. Appl. III* (1985): 449–464.
5. B.Hutton. "Products of fuzzy topological spaces "Topology. *Appl*. 11(1980): 59–67.
6. J.C.Kelly "Bitopological spaces", *Proc. London Math. Soc.*, **13**(1963): 71–89.
7. W.J. Pervin "Connectedness in bitopological spaces" Akad. Van Weten, *Amsterdam Proc. Series A* **70**(1967): 369–372.
8. J. Swart "Total disconnectedness in bitopological spaces and product bitopological spaces "Nederl. Akad, Welensch. Proc. Ser. A 74, *Indag-Math.* **33** (1971): 135–145.
9. C.K. Wong (1974) "Fuzzy Points and Local Properties of Fuzzy Topology" *J. Math. Appl.* **46**: (1974) 316–328.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 105 - 111

THE ERROR ANALYSIS OF THE IMPLICIT TRAPEZOIDAL RULE FOR LINEAR DIFFERENTIAL-ALGEBRAIC SYSTEMS OF INDEX μ

Sennur Somali
Dokuz Eylul Universitesi,
Department of Mathematics, 1420 sok. No: 79
Kahramanlar, IZMIR, TURKIYE

ABSTRACT

The local truncation error and its global propagation associated with the application of the Implicit Trapezoidal rule to the solution of constant coefficient differential-algebraic systems of index μ are studied.

1. INTRODUCTION

We are interested in initial value problems for the differential-algebraic systems in Kronecker canonical form

$$Ey'(t) = y(t) + g(t)$$

$$y(t_0) = y_0 \quad y \in \mathbb{R}^{\mu} \qquad (1.1)$$

where E has the property that there exists an integer μ such that $E^{\mu} = 0$ and $E^{\mu-1} \neq 0$. The parameter μ is the nilpotency index of E. Our purpose in this paper is to examine, the properties of these problems which cause the implicit Tapezoidal method to fail, and the relationship between the order of the error of the implicit Trapezoidal method and the index of the system. The idea of using ODE methods for solving differential-algebraic systems directly was introduced in [2]. A good general discussion of the systems may be found in [1], [2], [3]. In Section 2, we give the application and error analysis with the orders of the implicit Trapezoidal rule for the system (1.1). The numerical results for the index 4 test problem are presented in Section 3.

2. FORMULATION OF THE IMPLICIT TRAPEZOIDAL RULE AND ERROR ANALYSIS

If we apply the implicit Trapezoidal rule to (1.1),

$$E(y_{n+1} - y_n) = \frac{h}{2}(y_{n+1} + y_n) + \frac{h}{2}(g_{n+1} + g_n)$$

We get

$$y_{n+1} = \left(E - \frac{h}{2}I\right)^{-1}\left(E + \frac{h}{2}I\right)y_n + \left(E - \frac{h}{2}I\right)^{-1}\frac{h}{2}(g_{n+1} + g_n) \quad (2.1)$$

$$\text{where } E = \begin{pmatrix} 0 & 0 & . & . & 0 \\ 1 & . & 0 & . & 0 \\ 0 & . & . & 0 & . \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{\mu\times\mu}, \quad E^{\mu} = 0.$$

Let us define $K = \left(E - \frac{h}{2}I\right)^{-1}\left(E + \frac{h}{2}I\right)$ and $M = \left(E - \frac{h}{2}I\right)^{-1}$ and write

$$y_{n+1} = K\, y_n + \frac{h}{2}M\,(g_{n+1} + g_n)$$

Since E is nilpotent matrix, noting that,

$$\left(E - \frac{h}{2}I\right)^{-1} = \sum_{i=0}^{\mu-1}\left(\frac{2}{h}\right)^{i+1} E^i$$

K and M can be expanded as

$$K = -\left\{I + \frac{4}{h}E + \frac{8}{h^2}E^2 + \frac{16}{h^3}E^3 + \dots + \frac{2^{\mu}}{h^{\mu-1}}E^{\mu-1}\right\}$$

$$M = -\left\{\frac{2}{h}I + \frac{4}{h^2}E + \frac{8}{h^3}E^2 + \frac{16}{h^4}E^3 + \dots + \frac{2^{\mu}}{h^{\mu}}E^{\mu-1}\right\}$$

The local truncation error of the method is defined by

$$T_{n+1} = y(t_{n+1}) - K\,y(t_n) - \frac{h}{2}M\,\left(g(t_{n+1}) + g(t_n)\right)$$

and after some calculations, we get;

$$T_{n+1} = \sum_{k=a}^{\infty}(-1)^k\frac{h^k}{k!}\left(\frac{k}{2} - 1\right)M\,E\,y^{(k)}\,(t_{n+1})$$

The exact solution satisfies equation (2.1) with the local truncation error

$$y(t_{n+1}) = K\,y(t_n) + \frac{h}{2} M\left(g\,(t_{n+1}) + g(t_n)\right) - T_{n+1}$$

From the definition of the error $e_{n+1} = y_{n+1} - y(t_{n+1})$, $e_{n+1} = K\,e_n + T_{n+1}$ is obtained. For consistent initial conditions, i.e. $e_o = 0$, and after the back substitution process, we have

$$e_n = \sum_{j=0}^{n-1} K^j\, T_{n-j} = \sum_{j=0}^{n-1} \sum_{k=a}^{\infty} (-1)^k K^j\, ME\left(\frac{k}{2} - 1\right)\frac{h^k}{k!} y^{(k)}\,(t_{n-j}).$$

If we substitute the Taylor expansion of $y^{(k)}\,(t_{n-j})$ at the point t_n and $K^j ME = (-1)^j A_j$ with

$$A_j = \left\{\frac{2}{h} E + (1 + 2j)\frac{4}{h^2} E^2 + (1 + 2j + 2j^2)\frac{8}{h^3} E^3 + \ldots\right.$$

$$\left.\ldots\ldots + (d_1 + d_2)^2 + \ldots + d_{\mu-2}j^{\mu-2})\frac{2^{\mu-1}}{h^{\mu-1}} E^{\mu-1}\right\},$$

then the error will be

$$e_n = \sum_{k=3}^{\infty} \sum_{l=0}^{\infty} a_{k,l}\, h^{k+l} \left\{\sum_{j=0}^{n-1} (-1)^j j^l A_j\right\} y^{(k+l)}\,(t_n)$$

where $a_{k,l} = (-1)^{k+l}\left(\frac{k}{2} - 1\right)\frac{1}{1!\,k!}$ and $d_1, \ldots, d_{\mu-2}$ are some constants which are found from the binomial expansion terms of K^j. For the Standard ODE the order of the error of implicit Trapezoidal rule is $o(h^2)$, but in our system (2.1) the term $K^j ME$ in error formula changes the order by decreasing with respect to the index μ. Now we will discuss how this term effects the order of the error.

For odd n;

$$\sum_{j=0}^{2n} (-1)^j j^l A_j = \sum_{j=0}^{n} (2j)^l A_{2j} - \sum_{j=0}^{n-1} (2j + 1)^l A_{2j+1}$$

$$= (2n)^l A_{2n} + \sum_{j=0}^{n-1} (2j)^l (A_{2j} - A_{2j+1}) + \sum_{j=0}^{n-1} \sum_{p=1}^{l-1} (2j)^p A_{2j+1} \quad (2.2)$$

Since the degree of the matrix polynomial $A_{2j} - A_{2j+1}$ is $\mu-3$, the maximum degree of j in the first summation term of (2.2) will be

$l + \mu - 3$ and in the second summation term, j has maximum degree of $l + \mu - 3$.

Since $n = \dfrac{t_n - t_o}{h}$, we define $n = O(h^{-1})$ and

$$\sum_{j=0}^{n} j^l = b_{l+1} n^{l+1} + \ldots + b_1 n = O(h^{-l-1}) + O(h^{-l}) + \ldots + O(h^{-1}) \quad (2.3)$$

where $b_1, \ldots, b_{l+1}$ are some summation constants which are independent of n. Using the above results (2.3), we rewrite the summation term (2.2) as follows;

$$\begin{aligned}\sum_{j=0}^{2n} (-1)^j j^l A_j = {} & \left\{O(h^{-l-1}) + O(h^{-l}) + \ldots + O(h^{-1})\right\} E \\ & + \left\{O(h^{-l-3}) + O(h^{-l-2}) + \ldots + O(h^{-1})\right\} E^2 \\ & + \left\{O(h^{-l-5}) + O(h^{-l-4}) + \ldots + O(h^{-1})\right\} E^3 \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \left\{O(h^{-l-2\mu+3}) + O(h^{-l-2\mu+4}) + \ldots + O(h^{-1})\right\} E^{\mu-1}\end{aligned}$$

By similar consideration, we get the same res[illegible] for even n. So the error will be

$$\begin{aligned}\|e_n\| \le {} & \sum_{k=3}^{m}\sum_{l=0}^{m} \left\{\text{cons. } h^{k+l-1} + \ldots + \text{cons. } h^{k-1}\right\} \|E y^{(k+l)}(t_n)\| \\ & + \sum_{k=3}^{m}\sum_{l=0}^{m} \left\{\text{cons. } h^{k+l-1} + \ldots + \text{cons. } h^{k-3}\right\} \|E^2 y^{(k+l)}(t_n)\| \\ & + \sum_{k=3}^{m}\sum_{l=0}^{m} \left\{\text{cons. } h^{k+l-1} + \ldots + \text{cons. } h^{k-5}\right\} \|E^3 y^{(k+l)}(t_n)\| \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \sum_{k=3}^{m}\sum_{l=0}^{m} \left\{\text{cons.} h^{k+l-1} + \ldots + \text{cons.} h^{k-2\mu+3}\right\} \|E^{\mu-1} y^{(k+l)}(t_n)\| + \|R_{m+1}\|\end{aligned}$$

where $\|R_{m+1}\| \le \text{cons. } h^3$

For the sufficiently differentiable function $y(t)$ and $g(t)$, written in the components of index μ system with $y(t_n) = (y_1(t_n), ..., y_\mu(t_n))^T$, and using

$$E^i y = \begin{pmatrix} 0 \\ \cdot \\ \cdot y_1 \\ \cdot \\ \cdot \\ y_{\mu-i} \end{pmatrix}_{\mu \times 1}$$

we have;

$$\|e_{\mu,n}\| \leq \text{cons. } h^{\sigma-2\mu},$$

where the subscript μ denotes the μ^{th} component of the vector e_n. Here "cons" denotes a constant which is independent of h. For the first component we found an error. 118D−14 because it was solved exactly by the Trapezoidal rule.

3. NUMERICAL RESULTS

We consider the following linear constant coefficient index 4 differential algebraic system as a test example

$$0 = y_1 + t^3 + e^t + \sin(t)$$

$$y_1' = y_2$$

$$y_2' = y_3$$

$$y_3' = y_4$$

with the consistent initial conditions

$$y_1(0) = -1 \quad y_2(0) = -2 \quad y_3(0) = -1 \quad y_4(0) = -6$$

The numerical results using constant stepsize and observed orders for each component are listed in the following tables at $t = 1.2$. The observed orders are computed using $p = \log(e_n/e_{n+1})/\log(h_n/h_{n+1})$ where e_n and e_{n+1} are the global errors when the problem is solved with step sizes h_n and h_{n+1} respectively.

Table 1: Global error and observed orders for y_2.

h	ITR	Orders
0.2	.984D−02	--
0.1	.246D−02	2.
0.05	.616D−03	1.9976
0.025	.154D−03	2.

Table 2: Global error and observed orders y_3.

h	ITR	Orders
0.2	.238D+01	--
0.1	.239D+01	−0.0060
0.05	.240D+01	−0.0060
0.025	.240D+01	0.0

Table 3: Global error and observed orders y_4.

h	ITR	Orders
0.2	.145D+03	--
0.1	.577D+03	−1.9925
0.05	.230D+04	−1.9949
0.025	.922D+04	−2.0031

(ITR : Implicit Trapezoidal Rule)

It is seen that the predicted orders from the error formula as $e_2 = O(h^2)$, $e_3 = O(h^o)$ and $e_4 = O(h^{-2})$ agree with the results in the tables.

REFERENCES

1. Brenan, K.E., Campbell, S.L., Petzold, L.R: **Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations**, Elsevier Science Publishing Co., Inc., 1989.

2. Gear, C.W: The simultaneous numerical solution of differential/algebraic equations, IEEE *Trans. Circuit Theory* TC-18, (1), 1971, 89–95.

3. Somali, S.: Iterated defect correction methods for semi-explicit differential algebraic equations, Ph.D. thesis, Middle East Technical University, Department of Mathematics, 1990.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 112 - 117

ON FIXED POINT THEOREMS FOR KANNAN MAPS

Zeqing Liu

Department of Mathematics,
Liaoning Normal University,
Dalian, Liaoning, 116022,
P.R. China.

ABSTRACT

A few fixed point theorems for Kannan maps in complete and compact metric spaces are established which improve and extend corresponding results of Kannan, Khan and Kalinde.

KEY WORDS AND PHRASES: Complete metric space, compact metric space, Kannan map, fixed point.

1991 AMS subject classification. 54H25.

1. INTRODUCTION

Let f be a self map of a metric space (X, d). f is said to be a Kannan map if

$$d(fx, fy) \leq \frac{1}{2}[d(x, fx) + d(y, fy)] \tag{1}$$

for all $x, y \in X$. In a compact metric space, Kannan [1, 2] obtained the existence of fixed point for Kannan maps; In a complete metric space, Kalinda [3] gave a necessary and sufficient condition for Kannan maps to posses a fixed point.

The purpose of this paper is to establish a few fixed point theorems for Kannan maps in complete and compact metric spaces. Our results improve and extend results of Kannan [1, 2], Khan [4] and Kalinde [3]. An illustrative example is given in support of our result.

For $S \subset X$, $\bar{S}$ and $\delta(S)$ denote the closure and diameter of S respectively. N and ω denote the sets of positive integers and nonnegative integers respectively. For $x \in X$ and $k \in N$, define $O(x, \infty) = \{f^n x \mid n \in \omega\}$ and $o(x, k) = \{f^n x \mid O \leq n \leq k\}$.

In order to obtain our main results, we need the following.

Lemma

Let f be a Kannan map on a metric space (X, d). Then $d(f^n x, f^m x) \leq d(x, fx)$ for $n, m \in N$ and $x \in X$.

Proof

Let $x \in X$ and $n, m \in N$. It follows from (1) that

$$d(f^n x, f^{n+1} x) \leq \frac{1}{2}[d(f^{n-1}x, f^n x) + d(f^n x, f^{n+1} x)]$$

which implies

$$d(f^n x, f^{n+1} x) \leq d(f^{n-1}x, f^n x) \leq \ldots \leq d(x, fx)$$

By (1) and the above inequalities we have

$$d(f^n x, f^m x) \leq \frac{1}{2}[d(f^{n-1}x, f^n x) + d(f^{m-1}x, f^m x)] \leq d(x, fx)$$

This completes the proof.

Remark 1

Every Kannan map has at most one fixed point.

Our main results are as follows:

Theorem 1

Let f be a Kannan map on a comlete metric space (X, d). Then the following conditions are equivalent:

(a) f has a fixed point;

(b) $\inf\{\delta(O(x, k)) \mid x \in X\} = 0$ for all $k \in N$;

(c) $\inf\{\delta(O(x, k)) \mid x \in X\} = 0$ for some $k \in N$;

(d) $\inf\{\delta(O(x, \infty)) \mid x \in X\} = 0$.

Proof

The following implications are trivial (a) → (b) → (c) and (a) → (d). We now show that (c) implies (a). Define $M_i = \{x \mid x \in X$ and $\delta(O(x, k)) \leq \frac{1}{i}\}$ for $i \in N$. Clearly $M_{i+1} \subset M_i$ for $i \in N$. By (c) we obtain that $M_i \neq \Phi$ for $i \in N$. For any $x \in M_i$, we have by the Lemma

$$\delta(O(fx,k)) = \max\{d(f^{n+1}x, f^{m+1}x) \mid 0 \le n,m \le k\} \le d(x,fx) \le \delta(O(x,k)) \le \frac{1}{i}$$

i.e., $fx \in M_i$. Hence $fM_i \subset M_i$. For any $x, y \in M_i$, by the triangle inequality and (1) we get

$$\begin{aligned} d(x,y) &\le d(x,fx) + d(fx,fy) + d(y,fy) \le \frac{3}{2}[d(x,fx) + d(y,fy)] \\ &\le \frac{3}{2}[\delta(O(x,k)) + \delta(O(y,k))] \le \frac{3}{i} \end{aligned}$$

which implies

$$\delta(M_i) = \sup\{d(x,y) \mid x, y \in M_i\} \le \frac{3}{i}$$

Note that $\delta(\overline{M_i}) = \delta(M_i)$. By the completeness of X and the above inequality we obtain $\cap_{i \in N} \overline{M}_i \neq \Phi$. Take $w \in \cap_{i \in N} \overline{M}_i$ and $a_i \in M_i$ for $i \in N$. Then

$$\begin{aligned} d(w,fw) &\le d(w,a_i) + d(a_i,fa_i) + d(fa_i,fw) \\ &\le d(w,a_i) + \frac{3}{2}d(a_i,fa_i) + \frac{1}{2}d(w,fw) \end{aligned}$$

which implies

$$d(w,fw) \le 2d(w,a_i) + 3d(a_i,fa_i)$$

Note that $fM_i \subset M_i$ and $w, a_i \in \overline{M}_i$. Then

$$d(w,fw) \le 2\delta(\overline{M}_i) + 3\delta(M_i) \le \frac{15}{i}$$

As $i \to \infty$, we conclude that $d(w,fw) = 0$. Hence w is a fixed point of f; i.e., (a) holds.

We next show that (d) implies (b). Note that $\delta(O(x,k)) \le \delta(O(x,\infty))$ for $k \in N$ and $x \in X$. Then

$$\inf\{\delta(O(x,k)) \mid x \in X\} \le \inf\{\delta(O(x,\infty)) \mid x \in X\} = 0$$

i.e., $\inf\{\delta(O(x,k)) \mid x \in X\} = 0$

for all $k \in N$. Hence (b) holds. This completes the proof.

Remark 2

Our Theorem 1 improves and extends Theorem of Kalinde [3].

Since $\delta(O(x, 1)) = d(x, fx)$, we have by Theorem 1.

Corollary 1

Let f be a Kannan map on a complete metric space (X, d). Then f has a fixed point if and only if $\inf \{d(x, fx) \mid x \in X\} = 0$.

Theorem 2

Let f be a Kannan map on a compact metric space (X, d). Suppose that for any closed subset K of X with $fK \subset K$ and $\delta(K) > 0$,

$$\text{(2)} \qquad \inf\{d(x, fx) \mid x \in K\} < \delta(K)$$

Then f has a unique fixed point.

Proof

Let $r = \inf \{d(x, fx) \mid x \in X\}$. Then there exists a sequence $\{x_n\}_{n \in N} \subset X$ such that $\lim_{n\to\infty} d(x, fx_n) = r$. By the compactness of X, there exists a convergent subsequence $\{f^{n_k} x_{n_k}\}_{k \in N}$ of $\{f^n x_n\}_{n \in N}$. Put $\lim_{n\to\infty} f^{n_k} x_{n_k} = u \in X$. Then by (1) and the Lemma we get

$$\begin{aligned} d(u, fu) &= \lim_{n\to\infty} d(f^{n_k} x_{n_k}, fu) \leq \limsup_{n\to\infty} d(f^n x_n, fu) \\ &\leq \frac{1}{2} \lim_{n\to\infty} \sup\, [d(f^{n-1}x, f^n x_n) + d(u, fu)] \\ &\leq \frac{1}{2} \lim_{n\to\infty} \sup\, [d(x_n, fx_n) + d(u, fu)] \\ &\leq \frac{1}{2} [r + d(u, fu)] \end{aligned}$$

which implies $d(u, fu) \leq r$. Consequently $r = d(u, fu)$ by the definition of r.

Let $K = \{x \mid x \in X \text{ and } d(x, fx) = r\}$. Then $u \in K \neq \Phi$. By the Lemma and the definition of r we conclude easily that $fK \subset K$. Thus $d(fu, f^2u) = r$. For any $x, y \in K$, we have by (1)

$$d(fx, fy) \leq \frac{1}{2} [d(x, fx) + d(y, fy)] = r$$

Consequently

$$r = d(fu, f^2u) \le \delta(fK) = \sup\{d(fx, fy) \mid x, y \in K\} \le r$$

i.e., $\delta(fK) = r$. Set $E = \overline{fK}$. Clearly $\delta(E) = \delta(fK) = r$ and $fu \in E$.

We now prove that $E \subset K$. For any $x \in E$, there is a sequence $\{y_n\}_{n\in N} \subset K$ such that $\lim_{n\to\infty} fy_n = x$. Using (1) we obtain

$$d(x, fx) \le d(x, fy_n) + d(fy_n, fx) \le d(x, fy_n) + \frac{1}{2}[d(y_n, fy_n) + d(x, fx)]$$

$$= d(x, fy_n) + \frac{1}{2}[r + d(x, fx)]$$

It is easy to show that $d(x,fx)=r$; i.e., $x\in K$ and hence $E\subset K$. This implies $fE\subset fK\subset \overline{fK}=E$. Hence E is a nonempty closed and f-invariant subset of X.

We next prove that $u = fu$. Otherwise $u \ne fu$. Then $\delta(E) = r = d(u, fu) > 0$. Using (2) we have

$$r = d(fu, f^2u) = \inf\{d(x, fx) \mid x \in E\} < \delta(E) = r$$

which is impossible. Hence $u = fu$ and it follows from Remark 1 that u is the only fixed point of f. This completes the proof.

Corollary 2

Let f be a Kannan map on a compact metric space (X, d). Suppose that for any closed subset K of X with $fK \subset K$ and $\delta(K) > 0$ there exists $y \in K$ such that

(3) $\qquad d(y, fy) < \sup\{d(x, fx) \mid x \in K\}$

Then f has a unique fixed point.

Proof

Clearly (3) implies (2). Corollary 2 follows from Theorem 2.

Remark 3

The following example verifies that our Corollary 2 does indeed generalize Theorem A of Kannan [2] and Theorem 4 of Khan [4].

Example

Let $X = [0, 1]$, $d(x, y) = |x - y|$. Define $f : X \to X$ by

$$fx = \begin{cases} \frac{1}{10} & x \in [0, \frac{1}{2}] \\ \frac{1}{20} & x \in (\frac{1}{2}, 1]) \end{cases}$$

It is a simple matter to show that f satisfies (1). For any closed subset K of X with $fK \subset K$ and $\delta(K) > 0$, we have $\Phi \neq fK \subset fX = \{\frac{1}{10}, \frac{1}{20}\}$. We now assert that $\frac{1}{10} \in fK$. Suppose $\frac{1}{20} \in fK$. Then $\frac{1}{20} \in K$. This implies $\frac{1}{10} = f\frac{1}{20} \in fK$. Hence $\frac{1}{10} \in fK$. Note that $\frac{1}{10} \in fK \subset K$. Consequently (3) holds. Thus the conditions of Corollary 2 are satisfied but Theorem A of Kannan [2] and Theorem 4 of Khan [4] are not applicable since f is not continuous at $x = \frac{1}{2}$ and f does not satisfy

$$d(fx, fy) \leq [d(x, fx)\, d(y, fy)]^{1/2}$$

for $x = \frac{1}{10}$ and $y = 1$.

REFERENCES

1. R. Kannan, Some results on fixed point IV, *Fund. Math.* **74**(1972), 181–187.
2. ____, Construction of fixed points of a class of nonlinear mappings, *J. Math. Anal. Appl.* **41**(1973), 430–438.
3. A.K. Kalinde, On a fixed point theorem for Kannan's type mappings, *Math. Japonica* **33**(1988), 721–723.
4. M.S. Khan, On fixed point theorems, *Math. Japonica* **23**(1978), 201–204.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 118 - 121

HÖLDER CONTINUITY OF CELLERIER'S NON-DIFFERENTIABLE FUNCTION

Ebrahim Esrafilian

and

Abdullah Shidfar

Department of Mathematics,
Iran University of Science and Technology,
Narmak, Tehran-16, Iran.

ABSTRACT

In this paper, we show that Cellerier's nowhere differentiable function is everywhere continuous in the Hölder sense. The result is used to construct a Lyapunov surface which is non-regular in the Kellogg sense at any of its points.

KEY WORDS: Lyapunov Surface, Cellerier's, Non-Differentiable Function, Holder.

1. INTRODUCTION

Hölder continuity has now replaced the strong condition of differentiability in many theorems of the theory of partial differential equations. This paper aims to demonstrate the advantage of such a repalcement by presenting a counterexample.

Let us turn our attention to the potential theory. It is seen that the surface S, over which the single-layer and double-layer potentials are defined, must hold certain properties. According to Kellogg [1], S must be twice differentiable. Weaker conditions have also been proposed in the literature, for example, S being smooth in the Lyapunov sense [2, 3].

It has been shown that $z = f(x, y)$ represents a Lyapunov surface if, and only if, f_x and f_y exist and are continuous in the Hölder sense [4]. It is not difficult to find a Hölder continuous function which is non differentiable at a finite number of points. But note that, when integrating the boundary conditions on the surface S, we can easily remove such exceptional points under integration. The problem of interest is therefore to find an everywhere Hölder continuous function which has no derivatives at any point. In this way, we will be able to construct a Lyapunov surface which is everywhere non-regular in the Kellogg sense.

In [5], it has been shown that van der Waerden's classical non-differentiable function defined by $f(x) = \sum_{n=0}^{\infty} 2^{-n}\psi(2^n x)$, where $\psi(x)$ is the distance from x to the nearest integer, is Hölder continuous of class s, at any point, for every s between 0 and 1.

In this paper, we consider Cellerier's nowhere differentiable function and discuss its continuity in the Hölder sense. Then we use the result to construct a non-regular Lyapunov surface.

2. CELLERIER'S NON-DIFFERENTIABLE FUNCTION

Cellerier's function is defined [6] by $f(x) = \sum_{n=1}^{\infty} \frac{1}{a^n} \sin a^n x$, where a is a sufficiently large even integer. By Weierstrass' M-test, this infinite series is absolutely convergent and hence $f(x)$ is continuous at any point. However, it has been shown that Cellerier's function has no finite differential coefficients anywhere [6].

In the following, we prove the Hölder continuity function at any point. For $0 < s \leq 1$, recall that a function f is Hölder continuous is Holder continuous of class s at x if there is a positive constant M_s such that

$$|f(x + t) - f(x)| \leq M_s \; |t|^s, \tag{1}$$

for any real t.

By substituting Cellerier's function, we obtain

$$|f(x + t) - f(x)| \;= 2 \; \Big| \sum_{n=1}^{\infty} \frac{1}{a^n} \cos a^n \, (x+t/2) \sin a^n t/2 \Big| \tag{2}$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{a^n} \; |\sin a^n t/2|$$

since $|\cos a^n(x + t/2)| \leq 1$.

It can easily be shown that, for $0 < s \leq 1$ and for any real u,

$$\frac{|\sin u|}{|u|^s} \leq 1 \tag{3}$$

The validity of (3) for $|u| \geq 1$ is evident. For $|u| < 1$, we have $|u|^s \geq |u|$ and $\frac{|\sin u|}{|u|^s} \leq \frac{|\sin u|}{u} \leq 1$.

Now, (3) yields

$$\frac{|\sin a^n t/2|}{|a^n t/2|^s} \leq 1 . \tag{4}$$

Recalling that $a > 1$, for $0 < s < 1$, we have $a^{-(1-s)} < 1$. The series $\sum_{n=1}^{\infty} a^{-n(1-s)}$ is clearly convergent, and hence by (4), there exists a positive constant M_s

$$M_s = \sum_{n=1}^{\infty} \frac{2^{1-s}}{a^{n(1-s)}} \tag{5}$$

such that

$$2 \sum_{n=1}^{\infty} \frac{1}{a^n} |\sin a^n t/2| \leq M_s |t|^s \tag{6}$$

From (2) and (6), we can conclude the validity of Hölder's condition (1) for any real numbers x and t. This implies that Cellerier's function is Hölder continuous of class s, at any point, for $0 < s < 1$.

3. A NON-REGULAR LYAPUNOV SURFACE

Now, we are able to construct a surface which is everywhere smooth in the Lyapunov sense, but nowhere regular in the Kellogg sense.

Let us consider the surface $z = f(x, y) = \sum_{n=1}^{\infty} a^{-2n} (\cos a^n x + \cos a^n y)$, where a is again a sufficiently large even integer. Weierstrass' M-test ensures the absolute convergence of this infinite series and hence the validity of differentiation term by term. Note that both f_x and f_y are equal to Cellerier's function which is everywhere Hölder continuous, as was shown above, but non-differentiable at any point. According to [4], this implies that $z=f(x,y)$ is a Lyapunov surface which is non-regular in the Kellogg sense at any of its opoints.

REFERENCES

1. O.D. Kellogg, *Foundatons of Potential Theory*, Springer-Verlag (1967).

2. S.L. Sobolev, *Partial Differential Equations of Mathematical Physics*, Addison-Wesley Publishing Company, Inc. (1964).

3. V.I. Smirnov, *Integral Equations and Partial Differential Equations*, Addison-Wesley Publishing Company, Inc. (1964).

4. A. Shidar, On the Lyapunov and Kellogg Surfaces, *Journal of Mathematics*, Vol. 13, No. 4 (1987) 567–571 Houston.

5. A. Shidfar and K. Sabetfakhri, On the Continuity of Van der Waerden's Function in the Holder Sense, *The American Mathematical Monthly*, **Vol. 93** (1986), 375–376.

6. E.W. Hobson, *The Theory of Functions*, Harren Press (1950).

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 122 – 131

SOME REMARKS ON COMMON FIXED POINT MAPINGS

H.K. Pathak

Department of Mathematics,
Kalyan Mahavidyalaya,
Bhilai Nagar - 490 006 (MP), India.

Brian Fisher

Department of Mathematics and Computer Science,
University of Leicester,
Leicester, LE1 7RH, England.

ABSTRACT

Some theorems are proved for orbitally continuous and compatiable mappings on a complete metric space, generalizing earlier results.

1991 AMS Mathematics Subject Classification: 47H10, 54H25.

Key words and phrases: Complete metric space, orbitally continuous mappings, compatible mappings, common fixed point.

In our first theorem a mapping T of a metric space (X, d) into itself is said to be *orbitally continuous* (Ciric [3]) if $u \in X$ is such that $u = \lim_{i\to\infty} T^{n_i}x$ for some $x \in X$, then $Tu = \lim_{i\to\infty} TT^{n_i}x$.

Theorem 1

Let (X, d) be a complete metric space and let S, T be two orbitally continuous mappings of X into itself. Suppose that there exist functions $\{\Phi_j : 1 \le j \le N\}$ of X into $[0, \infty)$ such that

$$d(Sx, Ty) + q \max\{d(x, Sx), d(y, Ty)\} \le pd(x, y) + q \max\{d(x, Ty), d(y, Sx)\} + \sum_{j=1}^{N} [a_j(\Phi_j(x) - \Phi_j(Sx)) + b_j(\Phi_j(y) - \Phi_j(Ty))] \quad (1)$$

for all $x, y \in X$, some fixed $p, q \in [0, 1)$ with $0 \le p + q < 1$ and some fixed $a_j, b_j \ge 0$. Then S, T have a unique common fixed point $x^* \in X$. Further, if $x \in X$ then $\lim_{n\to\infty} S^n x = \lim_{n\to\infty} T^n x = x^*$.

Proof

The technique of our proof relies on the method of Bhakta and Basu [1]. Let x_0, y_0 be arbitrary points in X. We consider the two sequences $\{x_n\}$, $\{y_n\}$ in X obtained recursively by

$$x_n = S^n x_0, \; y_n = T^n y_0 \quad (n = 1, 2, ...)$$

From (1) we have

$$d(x_i, y_i) + q \max\{d(x_{i-1}, x_i), d(y_{i-1}, y_i)\}$$
$$= d(Sx_{i-1}, Ty_{i-1}) + q \max\{d(x_{i-1}, Sx_{i-1}), d(y_{i-1}, Ty_{i-1})\}$$
$$\leq pd(x_{i-1}, y_{i-1}) + q \max\{d(x_{i-1}, y_i), d(y_{i-1}, x_i)\}$$
$$+ \sum_{j=1}^{N} [a_j (\Phi_j(x_{i-1}) - \Phi_j(x_i)) + b_j(\Phi_j(y_{i-1}) - \Phi_j(y_i))]$$
$$\leq pd(x_{i-1}, y_{i-1}) + q \max\{d(x_{i-1}, y_{i-1}) + d(y_{i-1}, y_i), d(y_{i-1}, x_{i-1})$$
$$+ d(x_{i-1}, x_i)\} + \sum_{j=1}^{N} [a_j (\Phi_j(x_{i-1}) - \Phi_j(x_i))$$
$$+ b_j(\Phi_j(y_{i-1}) - \Phi_j(y_i))] \tag{2}$$

In either of the cases $d(x_{i-1}, x_i) \geq d(y_{i-1}, y_i)$ or $d(y_{i-1}, y_i) \geq d(x_{i-1}, x_i)$, equation (2) yields

$$d(x_i, y_i) \leq (p+q)\, d(x_{i-1}, y_{i-1}) + \sum_{j=1}^{N} [a_j (\Phi_j(x_{i-1}) - \Phi_i(x_i)) +$$
$$b_j (\Phi_j(y_{i-1}) - \Phi_j(y_i))]. \tag{3}$$

Adding inequalities (3) for $i = 1, ..., n+1$, we obtain

$$\sum_{i=1}^{n+1} d(x_i, y_i) \leq (p+q) \sum_{i=1}^{n+1} d(x_{i-1}, y_{i-1}) +$$
$$+ \sum_{j=1}^{N} \{a_j \sum_{i=1}^{n+1} [\Phi_j(x_{i-1}) - \Phi_j(x_i)] + b_j \sum_{i=1}^{n+1} [\Phi_j(y_{i-1}) - \Phi_j(y_i)]\}$$

and so

$$(1-p-q)\sum_{i=1}^{n} d(x_i, y_i) + d(x_{n+1}, y_{n+1}) \le (p+q)\, d(x_0, y_0) +$$

$$+ \sum_{j=1}^{N} [a_j\, \Phi_j(x_0) + b_j\Phi_j(y_0) - a_j\Phi_j(x_n) - b_j\Phi_j(y_n)],$$

from which it follows that

$$\sum_{i=1}^{n} d(x_i,y_i) \le \frac{p+q}{1-p-q} d(x_0, y_0) + \frac{1}{1-p-q}\sum_{j=1}^{N} [a_j\Phi_j(x_0)+b_j\Phi_j(y_0)] \quad (4)$$

We denote the right hand side of (4) by A. It is obvious that A is a fixed number in $[0, \infty)$.

Next, we consider

$$d(x_{i+1}, y_i) + q \max\{d(x_i, x_{i+1}), d(y_{i-1}, y_i)\}$$
$$= d(Sx_i, Ty_{i-1}) + q \max\{d(x_i, Sx_i), d(y_{i-1}, Ty_{i-1})$$
$$\le pd(xi, y_{i-1}) + q \max\{d(x_i, y_i), d(y_{i-1}, x_{i+1})\} +$$
$$+ \sum_{j=1}^{N} [a_j(\Phi_j(x_i)-\Phi_j(x_{i+1}))+b_j(\Phi_j(y_{i-1})-\Phi_j(y_i))]$$

By an argument analogous to the one used in obtaining inequality (4), we obtain

$$\sum_{i=1}^{n} d(x_{i+1},y_i) \le \frac{p+q}{1-p-q} d(x_1,y_0) + \frac{1}{1-p-q}\sum_{j=1}^{N} [a_j\Phi_j(x_1)+b_j\Phi_j(y_0)] \quad (5)$$

We denote the right hand side of (5) by B, which is a fixed number in $[0, \infty)$.

By the triangle inequality, we have

$$d(x_i, x_{i+1}) \le d(x_i, y_i) + d(y_i, x_{i+1})$$

and it follows from inequalities (4) and (5) that

$$\sum_{i=1}^{n} d(xi, x_{i+1}) \le A + B \quad (6)$$

Inequality (6) shwos that the series $\sum_{i=1}^{\infty} d(x_i, x_{i+1})$ is convergnet. Now let m, n be any positive integers with $m > n$. Then

$$d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \to 0 \text{ as } m, n \to \infty$$

and so the sequence $\{x_n\}$ is a Cauchy sequence in X.

Similarly, we can show that the sequence $\{y_n\}$ is also a Cauchy sequence in X. By the completeness of X, the sequences $\{x_n\}$ and $\{y_n\}$ converge to some points x^* and y^* respectively in X. Since S and T are orbitally continuous, $\lim_{n\to\infty} Sx_n = Sx^*$ & $\lim_{n\to\infty} Ty_n = Ty^*$. Hence, we have $Sx^* = x^*$ and $Ty^* = y^*$.

Now, we consider

$$\begin{aligned} d(x^*, y^*) &= d(Sx^*, Ty^*) + q \max\{d(x^*, Sx^*), d(y^*, Ty^*) \\ &\leq pd(x^*, y^*) + q \max\{d(x^*, Ty^*), d(y^*, Sx^*)\} \\ &\quad + \sum_{j=1}^{N} [a_j(\Phi_j(x^*) - \Phi_j(Sx^*)) + b_j(\Phi_j(y^*) - \Phi_j(Ty^*))] \end{aligned}$$

or $$(1 - p - q)\, d(x^*, y^*) \leq 0.$$

Since $p + q < 1$, it follows that $x^* = y^*$ and so x^* is a common fixed point of S and T.

The uniqueness of x^* follows easily using (1). This completes the proof of the theorem.

Remark 1

If $S = T$, $a_j = b_j = 0$ $(j = 1, \ldots, N)$ and $q = 0$ in Theorem 1, we obtain the celibrated Banach Contraction Theorem for a metric space.

Remark 2

If $a_j = b_j = 1$ $(j = 1, \ldots, N)$ and $q = 0$ in Theorem 1, we obtain Dien's Theorem 1.2, [5].

Remark 3

For $S = I$, the identity mapping and $b_1 = 1$, $b_j = 0$ $(j = 2, 3, \ldots, N)$, Theorem 1 reduces to Caristi's Theorem [2] under the continuity of T instead of the lower semiconinuity of Φ.

Under suitable conditions on the parameters p, q, a_j, b_j, we have the following corollaries:

Corollary 1 [1]

Let (X, d) be a complete metric space and let S, T be two orbitally continuous mappings of X into itself. Suppose that there exist two functions Φ and ϕ of X into $[0, \infty)$ *such that*

$$d(Sx, Ty) \leq \Phi(x) - \Phi(Sx) + \phi(y) - \phi(Ty)$$

for all $x, y \in X$. *Then S, T have a unique common fixed point* $x^* \in X$. *Further, if* $x \in X$ *then* $\lim_{n\to\infty} S^n x = \lim_{n\to\infty} T^n x = x^*$.

Corollary 2 [4]

Let (X, d) be a complete metric space and let S, T be two orbitally continuous mappings of X into itself. Suppose that there exists a function Φ of X into $[0, \infty)$ *such that*

$$d\,(Sx, Ty) \leq pd(x, y) + \Phi(x) - \Phi(Sx) + \Phi(y) - \Phi(Ty)$$

for all $x, y \in X$ *and some* $p \in [0, 1)$. *Then S, T have a unique common fixed point* $x^* \in X$. *Further, if* $x \in X$ *then* $\lim_{n\to\infty} S_n x = \lim_{n\to\infty} T^n x = x^*$.

Remark 4

The following example shows that there is a mapping which satisfies neither the condition of the Banach Contraction Theorem nor the conditions of Corollary 1. However, it satisfies all the conditions of Theorem 1 for $S = T$, $q = 0$ and hence has a fixefd point.

Let $X = \{1, 2, 3\}$ and let d be the metric defined by

$$d(1, 2) = 5,\ d(2, 3) = 4,\ d(3, 1) = 8.$$

Define $T : X \to X$ by $T1 = 1$, $T2 = 3$, $T3 = 1$. Then with $x = 1, y = 2$ we have $d(T1, T2) > d(1, 2)$ and so it is seen that the conditions of Banach's contraction theorem do not hold.

Now consider $\Phi : X \to [0, \infty)$ defined by $\Phi(1) = 0$, $\Phi(2) = 1$, $\Phi(3) = 2$. Then Corollary 1 does not hold. However, by putting $q = \frac{1}{2}$, $a = 9, b = 1$, the condition of Theorem 1 is satisfied for all $x, y \in X$.

In our next theorem two mappings S and I of a metric space (X, d) into itself are said to be *compatible* (Jungck [6]) if

$$\lim_{n\to\infty} d(SIx_n, ISx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ix_n = z$ for some $z \in X$.

Theorem 2

Let (X, d) be a complete metric space and let S, T, I, J be four continuous mappings of X into itself. Suppose that

(1°) $S(X) \subseteq I(X)$ and $T(X) \subseteq J(X)$,

(2°) *the pairs (S, I) and (T, J) are compatible,*

(3°) *there exist functions $\{\Phi_j : 1 \le j \le N\}$ of X into $[0, \infty)$ such that*

$$d(Sx, Ty) + q \max\{d(Ix, Sx), d(Jy, Ty)\}$$

$$\le pd(Ix, Jy) + q \max\{d(Ix, Ty), d(Jy, Sx)\} +$$

$$+ \sum_{j=1}^{N} [a_j (\Phi_j(Ix) - \Phi_j(Sx)) + b_j (\Phi_j(Jy) - \Phi_j(Ty))] \quad (7)$$

for all $x, y \in X$, some fixed $p, q \in [0, 1)$ with $0 \le p + q < 1$ and some fixed $aj, bj \ge 0$. Then S, T, I, J have a unique common fixed point $x^* \in X$.

Proof

Let x_0, y_0 be arbitrary points in X. In view of (1°) we can define two sequences $\{Ix_n\}$, $\{Jy_n\}$ recursively as follows:

$$Ix_0, Ix_1 = Sx_0, Ix_2 = Sx_1, \ldots, Ix_n = Sx_{n-1}, \ldots$$

$$Jy_0, Jy_1 = Ty_0, Jy_2 = Ty_1, \ldots, Jy_n = Ty_{n-1}, \ldots$$

By an argument analogous to that used in the proof of Theorem 1, we see that the sequences $\{Ix_n\}$ and $\{Jy_n\}$ are convergent with limits x^* and y^* respectively. Moreover, the definitions of Ix_n and Jy_n imply that $\lim_{n\to\infty} Sx_n = x^*$ and $\lim_{n\to\infty} Ty_n = y^*$.

Now consider

$$d(Sx_i, Ty_i) + q \max\{d(Ix_i, Sx_i), d(Jy_i, Ty_i)\}$$

$$\le pd(Ix_i, Jy_i) + q \max\{d(Ix_i, Ty_i), d(Jy_i, Sx_i)\} +$$

$$+ \sum_{j=1}^{N} [a_j (\Phi_j(Ix_i) - \Phi_j(Sx_i)) + b_j(\Phi_j(Jy_i) - \Phi_j(Ty_i))]$$

from which it follows as above that

$$\sum_{i=1}^{n} d(Ix_i, Jy_i) \le \frac{p+q}{1-p-q} d(Ix_0, Jy_0) + \frac{1}{1-p-q} \sum_{j=1}^{N} [a_j \Phi_j(Ix_0) + b_j \Phi_j(Jy_0)]$$

The series $\sum_{n=1}^{\infty} d(Ix_n, Jy_n)$ is therefore convergent and so

$$\lim_{n\to\infty} d(Ix_n, Jy_n) = d(x^*, y^*) = 0$$

on using the continuity of I and J. Thus $x^* = y^*$. Using the compatability of S and I we have $\lim_{n\to\infty} d(ISx_n, SIx_n) = 0$ and since S and I are continuous, we have

$$\lim_{n\to\infty} SIx_n = Sx^* = \lim_{n\to\infty} ISx_n = Ix^*$$

giving $Sx^* = Ix^*$.

Similarly, we can show that $Tx^* = Jx^*$.

Using (7) again, we have

$$d(Sx^*, Tx^*) + q \max \{d(Ix^*, Sx^*), d(Jx^*, Tx^*)\} \le$$

$$\le pd(Ix^*, Jx^*) + q \max \{d(Ix^*, Tx^*), d(Jx^*, Sx^*)\} +$$

$$+ \sum_{j=1}^{N} [a_j (\Phi_j (Ix^*) - \Phi_j(Sx^*)) + b_j(\Phi_j(Jx^*) - \Phi_j(Tx^*))]$$

which gives

$$(1-p-q)\, d(Sx^*, Tx^*) \le 0.$$

Since $p + q < 1$, it follows that

$$Sx^* = Tx^* = Ix^* = Jx^* .$$

Finally, we consider

$$d(Sx^*, Ty_n) + q \max \{d(Ix^*, Sx^*), d(Jy_n, Ty_n)\} \le$$

$$\le pd(Ix^*, Jy_n) + q \max \{d(Ix^*, Ty_n), d(Jy_n, Sx^*))\} +$$

$$+ \sum_{j=1}^{N} [a_j(\Phi_j(Ix^*) - \Phi_j(Sx^*)) + b_j(\Phi_j(Jy_n) - \Phi_j(Ty_n))]$$

which yields

$$\sum_{i=1}^{n} d(Sx^*, Ty_n) \le \frac{p+q}{1-p-q} d(Sx^*, Jy_0) + \frac{1}{1-p-q} \sum_{j=1}^{N} b_j \Phi_j(Jy_0)$$

The series $\sum_{n=1}^{\infty} d(Sx^*, Ty_n)$ is therefore convergent and so

$$\lim_{n\to\infty} d(Sx^*, Ty_n) = d(Sx^*, x^*) = 0,$$

providing that $Sx^* = x^*$. This completes the proof of the theorem.

Remarks 5

If $q = 0$, $a_j = b_j = 1$ $(j = 1, ..., N)$, we obtain Theorem 2.1 of Dien [5] as a corollary to our Theorem 2.

Corollary 2

Let (X, d) be a complete metric space and let S, T, I, J be four continuous mappings of X into itself. Suppose that

(1°) $S(X) \subseteq I(X)$ *and* $T(X) \subseteq J(X)$,

(2°) *the pairs (S, I) and (T, J) are compatible,*

(3°) *there exists a function Φ of X into $[0, \infty)$ such that*

$$d(Sx,Ty) \leq pq(Ix,Jy) + a[\Phi(Ix)-\Phi(Sx)] + b[\Phi(Jy) - \Phi(Ty)]$$

for all $x, y \in X$, some fixed $q \in [0, 1)$ and some $a, b \geq 0$. Then S, T, I, J have a unique common fixed point.

Remarks 6

If $q = 0$, $a = b = 1$, we obtain Corollary 2.1 of Dien [5] as a particular case of our Corollary 2.

Theorem 3

Let (X, d) be a metric space, let S, I be compatible continous mappings of X into itself such that $S(X) \subseteq I(X)$ and let Φ be a functioon of X into $[0, \infty)$. Suppose there exists $z \in X$ such that

$$d(Sx,z) + qd(Ix, Sx) \leq pd(Ix, z) + q \max\{d(Ix, z), d(z,Sx)\} + + a[\Phi(Ix) - \Phi(Sx)] \qquad (8)$$

for all $x \in X$, some fixed $p, q \in [0, 1)$ with $p + q < 1$ and some fixed $p \geq 0$. Then z is the unique common fixed point of S and I.

Proof

Let x_0 be an arbitrary point in X. Since $S(X) \subseteq I(X)$ we can define a sequence $\{x_n\}$ recursively by $Ix_n = Sx_{n-1}$ for $n = 1, 2, \ldots$ Using (8) we have

$$d(Ix_i,z) + qd(Ix_{i-1},Ix_i) \le pd(Ix_{i-1}, z) + q \max \{d(Ix_{i-1}, z), d(z, Ix_i)\} + a [\Phi (Ix_{i-1}) - \Phi(Ix_i)]$$

$$\le pd(Ix_{i-1}, z) + q [d(z, Ix_{i-1}) + d(Ix_{i-1}, Ix_i)] + a [\Phi (Ix_{i-1}) - \Phi(Ix_i)]$$

and also

$$d(Ix_i, z) \le (p + q)\, d(Ix_{i-1}, z) + a[\Phi(Ix_{i-1}) - \Phi(Ix_i)]$$

for $i = 1, 2, \ldots$. It follows that

$$\sum_{i=1}^{n} d(Ix_i, z) \le \frac{p+q}{1-p-q} d(Ix_0, z) + \frac{a}{1-p-q} [\Phi(Ix_0) - \Phi(Ix_i)]$$

$$\le \frac{p+q}{1-p-q} d(Ix_0, z) + \frac{a}{1-p-q} \Phi(Ix_0).$$

The series $\sum_{n=1}^{\infty} d(Ix_n, z)$ is therefore convergent and so

$$\lim_{n\to\infty} d(Ix_n, z) = 0,$$

providing that the sequence $\{Ix_n\}$ converges to z. Using the compatibility and continuity of S and I it follows easily that $Iz = Sz$.

Using (8) again with $x = z$, we see that

$$d(Sz, z) \le pd\,(Sz, z)$$

and so $Sz = Tz = z$. The uniqueness of z follows easily using (8) again. This completes the proof of the theorem.

Remark 7

If $q = 0$ and $a = 1$, we obtain Theorem 2.2 of Dien [5] as a corollary of our Theorem 3.

Open Question

To what extent can we weaken the continuity requirements of the mappings in Theorem 1 and 2?

REFERENCES

1. P.C. Bhakta and T. Basu, "Some fixed point theorems on metric spaces", J. Indian Math. Soc. **45**(1981), 399-404.

2. J. Caristi, "Fixed point theorem for mappings satisfying inwardness conditions", Trans. Amer. Math. Soc., **215**(1976), 241-251.

3. L.B. Ciric, "A generalized contraction and fixed point theorem", Publ. Inst. Math. (Beograd), **12**(1971), 19-26.

4. N.H. Dien, "Fixed point and fuzzy mappings", Ph.D. Dissertation, Institute of Mathematics, Sofia, Bulgaria, 1986.

5. N.H. Dien, "Some remarks on common fixed point theorems", J. Math. Anal. Appl., **187**(1994), 76-90.

6. G. Jungck, "Compatible mappings and common fixed points", Internat. J. Math. Math. Sci. **9**(1986), 771-779.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 132 – 138

ON THE NEUTRIX CONVOLUTION PRODUCT

$\ln x_- \boxed{*} \ln x_+$.

Adem Kilicman & Brian Fisher

Department of Mathematics and
Computer Science, Leicester University,
LE1 7RH, U.K.

Serpil Pehlivan

S.D. University, Faculty of Science,
Department of Mathematics,
32001, Isparta, Turkey.

ABSTRACT

Let f and g be distributions in D' and let

$$f_n(x) = f(x)\,\tau_n(x),\; g_n(x) = f(x)\,\tau_n(x),$$

where $\tau_n(x)$ is a certain function which converges to the identity function as n tends to infinity. Then the neutrix convolution product $f \boxed{*} g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided the limit h exists in the sense that

$$\underset{n\to\infty}{N\text{-}\lim}\,(f_n * g_n, \phi) = (h, \phi)$$

for all ϕ in D. The neutrix convolution product $\ln x_- \boxed{*} \ln x_+$ *is evaluated, from which other neutrix convolution products are deduced.*

AMS Mathematics Subject Classification (1991): 46F10.

Key words and phrases: Distribution, neutrix, neutrix limit, neutrix convolution product.

In the following we let D be the space of infinitely differentiable functions with compact support and let D' be the space of distributions defined on D. The convolution product $f * g$ of two distributions f and g in D' is then usually defined as in the following definition, see Gel'fand and Shilov [5].

Definition 1

Let f and g be distributions in D'. Then the convolution product $f * g$ is defined by the equation

$$\big((f * g)(x), \phi \big) = \big(f(y), (g(x), \phi(x + y)) \big)$$

for orbitrary ϕ in D provided f and g satisfy either of the conditions and all functions which converge to zero in the usual sense as n tends to infinity.

Note that in this definition the convolution product $f_n * g_n$ is defined as in Definition 1, the distribution f_n and g_n having bounded support. It follows easily that if $f \boxed{*} g$ exists then $g \boxed{*} f$ exists and the two are equal. However, $(f \boxed{*} g)'$ is not necessarily equal to $f \boxed{*} g'$ or $f' \boxed{*} g$.

The following theorem was provded in [2] showing that the neutrix convolution product is a generalization of the convolution product.

Theorem 1

Let f and g be distributions in D' satisfying either condition (a) or condition (b) of Definition 1. Then the neutrix convolution product $f \boxed{*} g$ exists and

$$f \boxed{*} g = f * g.$$

A number of neutrix convolution products were considered in [2], [3] and [4]. In the following we will consider the neutrix convolution product $\ln x_- \boxed{*} \ln x_+$. First of all we have

Theorem 2

The neutric convolution product $\ln x_- \boxed{*} \ln x_+$ *exists and*

$$\ln x_- \boxed{*} \ln x_+ = -(\pi^2/6 + 1)\,|x| + |x| \ln|x| - \tfrac{1}{2}|x| \ln^2|x| \quad (3)$$

Proof

Putting

$$(\ln x_-)_n = \ln x - \tau_n(x), \quad (\ln x_+)_n = \ln x + \tau_n(x),$$

we have

$$((\ln x_-)_n * (\ln x_+)_n, \phi(x)) = ((\ln y_-)_n, ((\ln x_+)_n, \phi(x+y)))$$

$$= \int_{-n-n^{-n}}^{0} \ln(-y)\, \tau_n(y) \int_a^b \ln(x-y) + \tau_n\,(x-y)\, \phi(x)\, dx\, dy$$

$$= \int_a^b \phi(x) \int_{-n}^{0} \ln(-y) \ln\,(x-y) + \tau_n\,(x-y)\, dy\, dx +$$

$$+ \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\, \tau_n(y) \ln(x-y)\, \tau_n\,(x-y)\, dy\, dx \qquad (4)$$

for $n > -a$ and arbitrary ϕ in D with support contained in the interval $[a, b]$.

When $x < 0$ and $-n \le y \le 0$, $\tau_n(x-y) = 1$ on the support of ϕ and so in this case we have

$$\int_{-n}^{0} \ln(-y) \ln(x-y) + \tau_n(x-y)\, dy = \int_{-n}^{x} \ln(-y) \ln(x-y)\, dy$$

$$= \int_{-x}^{n} \ln t \ln\,(x+t)\, dt$$

$$= \int_{-x}^{n} \ln t[\ln t + \ln\,(1+x/t)]\, dt, \quad (5)$$

on making the substituti ı $y = -t$.

Now

$$\int_{-x}^{n} \ln^2 t\, dt = n\, \text{lı} \quad n - 2n \ln n + 2n + x \ln^2(-x) - 2x \ln\,(-x) + 2x$$

and it follows that

$$N\text{-}\lim_{n\to\infty} \int_{-x}^{n} \ln^2 t\, dt = -2x_- + 2x_- \ln x_- - x_- \ln^2 x_- \tag{6}$$

Further

$$\int_{-x}^{n} \ln t \ln(1+x/t)\, dt = -\sum_{i=1}^{\infty} \frac{(-x)^i}{i} \int_{-x}^{n} t^{-i} \ln t\, dt$$

$$= \tfrac{1}{2}x[\ln^2 n - \ln^2(-x)] - \sum_{i=2}^{\infty} \frac{(-x)^i}{i}\Big[\frac{n^{1-i}\ln n}{1-i} - \frac{n^{1-i}}{(1-i)^2}\Big] +$$

$$-\sum_{i=2}^{\infty}\Big[\frac{x\ln(-x)}{i\,(1-i)} - \frac{x}{i(1-i)^2}\Big]$$

and it follows that

$$N\text{-}\lim_{n\to\infty} \int_{-x}^{n} \ln t \ln(1+x/t)\, dt = \tfrac{1}{2}x_- \ln^2 x_- + \sum_{i=2}^{\infty}\Big[\frac{x\ln(-x)}{i\,(1-i)} - \frac{x_-}{i(1-i)^2}\Big]$$

$$= (1-\pi^2/6)x_- - x_- \ln x_- + \tfrac{1}{2}x_- \ln^2 x_- \tag{7}$$

since

$$\sum_{i=2}^{\infty} \frac{1}{i(1-i)} = -1, \quad \sum_{i=2}^{\infty} \frac{1}{i(1-i)^2} = \pi^2/6 - 1.$$

It now follows from equations (5), (6) and (7) that when $x < 0$.

$$N\text{-}\lim_{n\to\infty} \int_{-n}^{0} \ln(-y) \ln(x-y) + \tau_n(x-y)\, dy = -(\pi^2/6+1)x_-$$

$$+ x_- \ln x_- + -\tfrac{1}{2}x_- \ln^2 x_- \tag{8}$$

When $n > 2x > 0$ and $-n \le y \le 0$, we have

$$\int_{-n}^{0} \ln(-y) \ln(x-y) + \tau_n(x-y)\, dy = \int_{x-n}^{0} \ln(-y) \ln(x+t)\, xy$$

$$+ \int_{x-n-n^{-n}}^{x-n} \ln(-y) \ln(x-y)\, \tau_n(x-y)\, dy \tag{9}$$

Making the substitution $y = -t$ we have

$$\int_{x-n}^{0} \ln(-y)\ln(x-y)\,dy = \int_{0}^{n-x} \ln t \ln(x+t)\,dt$$

$$= \int_{0}^{x} \ln t\,[\ln x + \ln(1+t/x)]\,dt + \int_{x}^{n-x} \ln t\,[\ln t + \ln(1+x/t)]\,dt \quad (10)$$

Now $\ln x \int_{0}^{x} \ln t\,dt = x\ln^2 x - x\ln x$

and $$\int_{0}^{x} \ln t \ln(1+t/x)\,dt = -\sum_{i=1}^{\infty} \frac{(-1)^i}{ix^i} \int_{0}^{x} t^i \ln t\,dt$$

$$= -\sum_{i=1}^{\infty} \left[\frac{(-1)^i x \ln x}{i(i+1)} - \frac{(-1)^i x}{i(i+1)^2}\right]$$

$$= (2\ln 2 - 1)\,x\ln x + (2 - 2\ln 2 - \pi^2/12)x,$$

since $\sum_{i=1}^{\infty} \frac{(-1)^i}{i(i+1)} = 1 - 2\ln 2$, $\sum_{i=1}^{\infty} \frac{(-1)^i}{i(i+1)^2} = 2\ln 2 - 2 + \pi^2/12$

Thus $$\int_{0}^{x} \ln t \ln(x+t)\,dt = (2 - 2\ln 2 - \pi^2/12)x + (2\ln 2 - 2)x\ln x. \quad (11)$$

Next we have

$$\int_{x}^{n-x} \ln^2 t\,dt = (n-x)\ln^2(n-x) - 2(n-x)\ln(n-x)$$
$$+ 2(n-x) + -x\ln^2 x + 2x\ln x - 2x.$$

Since

$$\ln(n+x) = \ln n + \ln(1+x/n) = \ln n - \sum_{i=1}^{\infty} \frac{(-x)^i}{in^i},$$

when $n > x$, it is easily seen that

$$N\text{–}\lim_{n\to\infty} (n-x)\ln^2(n-x) = 0, \quad N\text{–}\lim_{n\to\infty} (n-x)\ln(n-x) = -x$$

and so

$$N\text{-}\lim_{n\to\infty} \int_x^{n-x} \ln^2 t\, dt = -2x + 2x\ln x - x\ln^2 x. \tag{12}$$

Further

$$\int_x^{n-x} \ln t \ln(1 + x/t)\, dt = -\sum_{i=1}^{\infty} \frac{(-x)^i}{i} \int_x^{n-x} t^{-i} \ln t\, dt$$

$$= \tfrac{1}{2}x\,[\ln^2(n-x) - \ln^2 x] - \sum_{i=2}^{\infty} \frac{(-x)^i}{i}\Big[\frac{(n-x)^{1-i}\ln(n-x)}{1-i} - \frac{(n-x)^{1-i}}{(1-i)^2}\Big] +$$

$$+ \sum_{i=2}^{\infty}\Big[\frac{(-1)^i x\ln x}{i(1-i)} - \frac{(-1)^i x}{i(1-i)^2}\Big]$$

and it follows that

$$N\text{-}\lim_{n\to\infty} \int_x^{n-x} \ln t \ln(1+x/t)\,dt = -\tfrac{1}{2}x\ln^2 x + \sum_{i=2}^{\infty}\Big[\frac{(-1)^i x\ln x}{i(1-i)} - \frac{(-1)^i x}{i(1-i)^2}\Big]$$

$$= (2\ln 2 - \pi^2/12)x + (1 - 2\ln 2)x\ln x + -\tfrac{1}{2}x\ln^2 x, \tag{13}$$

since

$$\sum_{i=2}^{\infty} \frac{(-1)^i}{i(i-1)^2} = 1 - 2\ln 2, \quad \sum_{i=2}^{\infty} \frac{(-1)^i}{i(1-i)^2} = -2\ln 2 + \pi^2/12.$$

Finally, it is easily seen that

$$\Big|\int_{x-n-n^{-n}}^{x-n} \ln(-y)\ln(x-y)\,\tau_n(x-y)\,dy\Big| = o(n^{-n}\ln^2 n) \tag{14}$$

and it now follows from equations (9), (10), (11), (12) and (13) that when $x > 0$,

$$N\text{-}\lim_{n\to\infty} \int_{x-n}^{0} \ln(-y)\ln(x-y) + \tau_n(x-y)dy = -(\pi^2/6+1)\,x_+ + x_+\ln x_+ + -\tfrac{1}{2}x_+\ln^2 x_+ \tag{15}$$

Equation (3) now follows from equations (8) and (15).

Corollary

The neutrix convolution products $\ln|x| \boxed{*} \ln x_+$, $\ln|x| \boxed{*} \ln x_-$ and $\ln|x| \boxed{*} \ln|x|$ *exist and*

$$\ln|x| \boxed{*} \ln x_+ = -(\pi^2/3-1)x - \tfrac{1}{2}\pi^2 x_- - x\ln|x| + \tfrac{1}{2}x\ln^2|x|, \tag{16}$$

$$\ln|x| \boxed{*} \ln x_- = -(\pi^2/3-1)x - \tfrac{1}{2}\pi^2 x_+ + x\ln|x| - \tfrac{1}{2}x\ln^2|x|, \tag{17}$$

$$\ln|x| \boxed{*} \ln|x| = -\tfrac{1}{2}\pi^2|x|. \tag{18}$$

Proof

The convolution product $\ln x_+ * \ln x_+$ exists by Definition 1 and it is easily proved that

$$\begin{aligned}\ln x_+ * \ln x_+ &= (2-\pi^2/6)x_+ - 2x_+\ln x_+ + x_+\ln^2 x_+ \\ &= \ln x_+ \boxed{*} \ln x_+ .\end{aligned} \tag{19}$$

Since the neutrix convolutionl product is clearly distributive with respect to addition, it follows that

$$\ln x_- \boxed{*} \ln x_+ + \ln x_+ \boxed{*} \ln x_+ = \ln|x| \boxed{*} \ln x_+ .$$

Equation (16) now follows frm equations (3) and (19). Equation (17) follows from equation (16) on replacing x by $-x$. Equation (18) follows from equations (16) and (17) on noting that

$$\ln|x| \boxed{*} \ln x_+ + \ln|x| \boxed{*} \ln x_- = \ln|x| \boxed{*} \ln|x| .$$

REFERENCES

1. J.G. van der Corput, "Introduction to the neutrix calculus", *J. Analyse Math.*, **7**, 291-398, 1959-60.
2. B. Fisher and Li Chen Kuan, "A commutative neutrix convolution product of distributions", *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.*, **23**(1993), 13-27.
3. B. Fisher and E. Ozcag, "A result on the commutative neutrix convolution product of distributions", *Doga - Tr. J. Math.*, **16**, 1-11, 1992.
4. B. Fisher and E. Ozcag, "Results on the commutative neutrix convolution product of distributions", *Arch. Math.*, **29**, 105-117, 1993.
5. I.M. Gel'fand and G.E. Shilov, *Generalized Functions*, Vol. I, New York, Academic Press, 423p., 1964.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 139 - 146

ON THE NON-COMUTATIVE NEUTRIX PRODUCT $x_+^{-r} \circ x_+^{-s}$

Brian Fisher and Adem Kilicman

Department of Mathematics
and Computer Science,
Leicester University, Leicester,
LE1 7RH, England.

ABSTRACT

The non-commutative neutrix product of the distributions x_+^{-r} and x_+^{-s} is evaluated for $r, s = 1, 2, \ldots$ Further non-commutative neutrix products are deduced.

KEYWORDS: Distribution, delta-function, neutrix, neutrix limit, neutrix product.

CLASSIFICATION: 46F10.

In the following, we let N be the neutrix, see van der Corput [1], having domain $N' = \{1, 2 \ldots, n, \ldots\}$ and range the real numbers, with negligible functions finite linear sums of the functions.

$$n^{\lambda} ln^{r-1} n, \;\; ln^r n: \;\; \lambda > 0, \;\; r = 1, 2, \ldots$$

and all functions which converage to zero in the normal sense as n tends to infinity.

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$.

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1.$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functins converging to the Dirac delta-function $\delta(x)$.

Now let D be the space of infinitely differentiable functions with compact support and let D' be the space of distributions defined on D. Then if f is an arbitrary distribution in D', we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$ It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions convering to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2] or [3].

Definition 1

Let f and g be distributions in D' for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \, [Fg^{(i)}]^{(k-i)} .$$

The following definition for the neutrix product of two distributions was given in [4] and generalizes Definition 1.

Definition 2

Let f and g be distributions in D' and let $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to the distribution h on the interval (a, b) if

$$\underset{n\to\infty}{\text{N-lim}} \langle f(x)\, g_n(x), \phi(x) \rangle = \langle h(x), \phi(x) \rangle$$

for all functions ϕ in D with support contained in the interval (a, b).

Note that if

$$\lim_{n\to\infty} \langle f(x)\, g_n(x), \phi(x) \rangle = \langle h(x), \phi(x) \rangle ,$$

we simply say that the *product* $f.g$ exists and equals h, see [3].

It is obvious that if the product $f.g$ exists then the neutrix product $f \circ g$ exists and the two are equal.

The following theorem holds, see [7].

Theorem 1

Let f and g be distributions in D′ and suppose that the neutrix products $f \circ g^{(i)}$ (or $f^{(i)} \circ g$) exist on the interval (a, b) for i = 0, 1, 2...,r. Then the neutrix products $f^{(k)} \circ g$ (or $f \circ g^{(k)}$) exists on the interval (a, b) for k = 1, 2, ..., r and

$$f^{(k)} \circ g = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \, [f \circ g^{(i)}]^{(k-i)} \tag{1}$$

or

$$f \circ g^{(k)} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \, [f^{(i)} \circ g]^{(k-i)}$$

on the interval (a, b) for k = 1, 2, ..., r.

In the following two theorems, which were proved in [5] and [6] respectively, the distributions x_+^{-r} and x_-^{-r} are defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!} (\ln x_+)^{(r)}, \; x_-^{-r} = -\frac{1}{(r-1)!} (\ln x_-)^{(r)},$$

for $r = 1, 2, \ldots$ and is distinct from the definition given by Gel'fand and Shilv [8].

Theorem 2

The neutrix products $x_+^{-r} \circ x_-^{-s}$ and $x_-^{-s} \circ x_+^{-r}$ exist and

$$x_+^{-r} \circ x_-^{-s} = \frac{(-1)^r c_1}{(r+s-1)!} \delta^{(r+s-1)}(x),$$

$$x_-^{-s} \circ x_+^{-r} = \frac{(-1)^{r-1} c_1}{(r+s-1)!} \delta^{(r+s-1)}(x),$$

for $r, s = 1, 2, \ldots$

Theorem 3

The neutrix products $\ln x_+ \circ x_+^{-s}$, $x_+^{-s} \circ \ln x_+$ *and* $x_+^{-r} \circ x_+^{-s}$ *exist for* $r, s = 1, 2, \ldots$ *In particular,*

$$\ln x_+ \circ x_+^{-1} = x_+^{-1} \ln x_+, \tag{2}$$

$$\ln x_+ \circ x_+^{-2} = x_+^{-2} \ln x_+ + (c_1 - 1)\, \delta'(x),$$

$$x_+^{-1} \circ \ln x_+ = x_+^{-1} \ln x_+, \tag{3}$$

$$x_+^{-2} \circ \ln x_+ = x_+^{-2} \ln x_+ + (c_1 - 1)\, \delta'(x),$$

$$x_+^{-1} \circ \ln x_+^{-1} = x_+^{-2} + (c_1 - 1)\, \delta'(x),$$

where

$$c_1 = \int_0^1 \rho(t) \ln t \, dt.$$

It was in fact proved in [6] that

$$\ln x_+ \circ x_+^{-s} = x_+^{-s} \ln x_+ - \frac{\Lambda_s + \psi_1(s-1)}{(s-1)!} (-1)^s \, \delta^{(s-1)}(x), \tag{4}$$

where

$$\Lambda_s = -c_1\, \psi(s-1) + \tfrac{1}{2} [\chi(s-1) - \psi^2(s-1)]$$

for $s = 1, 2, \ldots$ and

$$\psi(s) = \begin{cases} 0, & s = 0, \\ \sum_{i=1}^{s} 1/i, & s \geq 1, \end{cases} \qquad \psi_1(s) = \begin{cases} 0, & s = 0, \\ \sum_{i=1}^{s} \psi(i)/i, & s \geq 1, \end{cases}$$

$$\chi(s) = \begin{cases} 0, & s = 0, \\ \sum_{i=1}^{s} 1/i^2, & s \geq 1, \end{cases}$$

However, although the existence of $x_+^{-r} \circ x_+^{-s}$ was proved for $r, s = 1, 2, \ldots$, no general formula was obtained for it. In the following, we are going to prove that

$$x_+^{-r} \circ x_+^{-s} = x_+^{-s-r} + M_{rs}\, \delta^{(r+s-1)}(x), \qquad (5)$$

where

$$M_{rs} = \sum_{i=0}^{r-1} \binom{r-1}{i} \left[\frac{c_1}{s+i} - \frac{1}{(s+i)^2}\right] \frac{(-1)^{r+s+i}}{(r-1)!\,(s-1)!}$$

but first of all we will prove that

$$x_+^{-1} \circ x_+^{-s} = x_+^{-s-1} - \left[\frac{c_1}{s} - \frac{1}{s}\right] \frac{(-1)^s}{(s-1)!} \delta^{(s)}(x) \qquad (6)$$

Differentiating equation (4) we get

$$x_+^{-1} \circ x_+^{-s} - s \ln x_+ \circ x_+^{-s-1} = x_+^{-s-1} - s x_+^{-s-1} \ln x_+ \frac{\Lambda_s + \psi_1(s-1)}{(s-1)!} (-1)^s \delta^{(s)}(x),$$

from which it follows that

$$x_+^{-1} \circ x_+^{-s} = x_+^{-s-1} + \frac{\Lambda_{s+1} - \Lambda_s + \psi_1(s) - \psi_1(s-1)}{(s-1)!} (-1)^s\, \delta^{(s)}(x)$$

It is easily seen that

$$\psi_1(s) - \psi_1(s-1) = \psi(s)/s,$$

$$\Lambda_{s+1} - \Lambda_s = -\frac{c_1}{s} - \frac{\psi(s-1)}{s}$$

and equation (6) follows.

Putting $r = 1$ in equation (5) gives equation (6) and so equation (5) holds in the case $r = 1$ and $s = 1, 2, \ldots$ Assume that equation (5) holds for some r and $s = 1, 2, \ldots$. Then on differentiating equation (5) we get

$$-r x_+^{-r-1} \circ x_+^{-s} - s x_+^{-r} \circ x_+^{-s-1} = -(r+s)\, x_+^{-r-s-1} + M_{rs} \delta^{(r+s)}(x)$$

and from our assumption it follows that

$$r x_+^{-r-1} \circ x_+^{-s} - r x_+^{-r-s-1} = -(s M_{r,s+1} + M_{rs})\, \delta^{(r+s)}(x)$$

$$= \sum_{i=0}^{r-1} \binom{r-1}{i} \left[\frac{c_1}{s+i+1} - \frac{1}{(s+i+1)^2}\right] \frac{(-1)^{r+s+i}}{(r-1)!\,(s-1)!} \delta^{(r+s)}(x) +$$

$$-\sum_{i=0}^{r-1}\binom{r-1}{i}\left[\frac{c_1}{s+i}-\frac{1}{(s+i)^2}\right]\frac{(-1)^{r+s+i}}{(r-1)!\,(s-1)!}\delta^{(r+s)}(x)$$

$$=\sum_{i=0}^{r}\binom{r}{i}\left[\frac{c_1}{s+i}-\frac{1}{(s+i)^2}\right]\frac{(-1)^{r+s+i+1}}{(r-1)!\,(s-1)!}\delta^{(r+s)}(x),$$

since

$$\binom{r-1}{i}+\binom{r-1}{i-1}=\binom{r}{i}.$$

Equation (5) now follows by induction.

Replacing x by $-x$ in equation (5) we get

$$x_-^{-r}\circ x_-^{-s}=x_-^{-r-s}-(-1)^{r+s}M_{rs}\delta^{(r+s-1)}(x).$$

We now consider the product $x_+^{-r}\circ\ln x_+$. It was proved in [6] that this product was of the form

$$x_+^{-r}\circ\ln x_+=x_+^{-r}\ln x_+ + M_r\delta^{(r-1)}(x), \tag{7}$$

for $r=1, 2, \ldots$, but no expression was found for M_r except for the cases $r=1, 2$.

Using equation (1), we have

$$(-1)^{r-1}(r-1)!\,x_+^{-r}\circ\ln x_+=\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\,[x_+^{-1}\circ(\ln x_+)^{(i)}]^{(r-i-1)}$$

$$=(x_+^{-1}\circ\ln x_+)^{(r-1)}-\sum_{i=0}^{r-1}\binom{r-1}{i}(i-1)!\,[x_+^{-1}\circ x_+^{-i}]^{(r-i-1)}$$

Using equations (3) and (6) and picking out the coefficient of $\delta^{(r-1)}(x)$, we see that

$$(-1)^{r-1}(r-1)!\,M_r=\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\left[\frac{c_1}{i}-\frac{1}{i^2}\right]$$

for $r=2, 3, \ldots$.

Replacing x by $-x$ in equation (7) we get

$$x_-^{-r}\circ\ln x_-=x_-^{-r}\ln x_- -(-1)^r M_r\delta^{(r-1)}(x),$$

for $r = 1, 2, \ldots$

We finally note that since the neutrix product is clearly distributive with respect to addition then

$$x^{-r} \circ x_-^{-s} = [x_+^{-r} + (-1)^r x_-^{-r}] \circ x_-^{-s}$$

$$= (-1)^r x_-^{-r-s} + \Big[\frac{(-1)^r c_1}{(r+s-1)!} - (-1)^s M_{rs}\Big] \delta^{(r+s-1)}(x),$$

$$x_-^{-r} \circ x^{-s} = x_-^{-r} \circ [x_+^{-s} + (-1)^r x_-^{-s}]$$

$$= (-1)^s x_-^{-r-s} - \Big[\frac{(-1)^r c_1}{(r+s-1)!} + (-1)^r M_{rs}\Big] \delta^{(r+s-1)}(x).$$

Replacing x by $-x$ in these equations we get

$$x^{-r} \circ x_+^{-r} = x_+^{-r-s} - \Big[\frac{(-1)^{r+s} c_1}{(r+s-1)!} - M_{rs}\Big] \delta^{(r+s-1)}(x),$$

$$x_+^{-r} \circ x^{-s} = x_+^{-r-s} + \Big[\frac{(-1)^{r+s} c_1}{(r+s-1)!} + M_{rs}\Big] \delta^{(r+s-1)}(x)$$

and it now follows that

$$x^{-r} \circ x^{-s} = x^{-r} \circ [x_+^{-s} + (-1)^s x_-^{-s}] = x^{-r-s},$$

for $r, s = 1, 2, \ldots\ldots$

REFERENCES

1. J.G. van der Corput, "Introduction to the neutrix calculus" J. Analyse Math., **7**(1959-60), 291-398.

2. B. Fisher, "The product of distributions", Quart. J. Math. Oxford (2), **22**(1971), 291-298.

3. B. Fisher, "On defining the product of distributions", Math. Nachr., **99**(1980), 239-249.

4. B. Fisher, "A non-cmmutative neutrix product of distributions", Math. Nachr., **108**(1982), 117-127.

5. B. Fisher and A. Kilicman, "The non-commutative neutrix product of the distributions x_+^{-r} and x_-^{-s}", Math. Balkanica, 8(1994), 251-258.

6. B. Fisher, A. Kilicman, B. Damyanov and C.J. Ault, "On the non-commutative neutrix product $\ln x_+ \circ x_+^{-s}$", Comment. Math. Univ Carolinae, to appear.

7. B. Fisher, E. Savas, S. Pehlivan and E. Özcağ, "Results on the non-commutative neutrix product of distributions", Math. Balkanica, 7(1993), 347-356.

8. I.M. Gel'fand and G.E. Shilov, "Generalized Functions", Vol. I, Academic Press, 1964.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 147 – 153

ON COMMON FIXED POINTS IN UNIFORM SPACES

Zeqing Liu
Department of Mathematics
Liaoning Normal University,
Dalian, Liaoning,
116022, P.R. China.

ABSTRACT

In this paper we introduce the concept of compatible mappings in uniform spaces and establish common fixed point theorems by using the concept. Our results extend the results of Acharya [1], Mishra [2] and Jungck [3].

KEY WORDS AND PHRASES. Compatible mappings, uniform spaces, common fixed point.

AMS (1991) Subject Classification. 54H25.

1. INTRODUCTION

Acharya [1] and Mishra [2] established fixed point theorems in uniform spaces. Jungck [3] gave a necessary and sufficient condition of the existence of fixed point for a self mapping on metric spaces. Jungck [4] introduced the concept of compatible mappings in metric spaces.

In this paper we extend the concept of compatible mappings in metric spaces to the setting of uniform spaces and prove common fixed point theorems for compatible mapping in uniform spaces. Our results generalize some theorems of Acharya [1], Mishra [2] and Jungck [3].

Let $(X, \mathcal{U})$, be a uniform space. For any pseudometric p on X and $r > 0$, we write $V_{(p,r)} = \{(x, y) : x, y \in X \text{ and } p(x, y) < r\}$. Let $\mathcal{P}$ be a family of pseudometrics on X generating the uniformity $\mathcal{U}$. Denote by $\mathcal{V}$ the family of all sets of the form $\cap_{i=1}^{n} V_{(p_i,r_i)}$, where $p_i \in \mathcal{P}$ and $r_i > 0$, $i = 1, 2, ..., n$ (the integer n is not fixed).

Obviously, $\mathcal{V}$ is a base for the uniformity $\mathcal{U}$. For $V = \cap_{i=1}^{n} V_{(p_i, r_i)}$, let

$$rV = \begin{cases} \cap_{i=1}^{n} V_{(p_i, rr_i)}, & \text{if } r > 0; \\ \Delta(\text{the diagonal}), & \text{if } r = 0. \end{cases}$$

Acharya [1] proved the following results.

Lemma 1

If $V \in \mathcal{V}$ and $a, b > 0$, then

(i) $aV \circ bV \subset (a + b) V$.

(ii) $aV \subset bV$ for $a < b$.

Lemma 2

Let $V \in \mathcal{V}$. Then there is a pseudometric p on X such that $V = V_{(p,1)}$. This p is called a Minkowsiki's pseudometric of V.

2. COMMON FIXED POINTS

In this section we assume that $(X, \mathcal{U})$ is a sequentially complete Hausdorff uniform space. Further we suppose that $\mathcal{P}$ is a fixed family of pseudometrics on X which generates the uniformity $\mathcal{U}$. We denote by $\mathcal{V}$ the family of all sets of the form $\cap_{i=1}^{n} V_{(p_i, r_i)}$, $p_i \in \mathcal{P}$, $r_i > 0$ and the integer n is not fixed. N and ω denote the sets of positive integers and nonnegative integers respectively.

Definiton 1

Let S and T be self mappings of X. S and T are said to be compatible if for every $V \in \mathcal{V}$ there exists $k \in N$ such that $(TSx_n, STx_n) \in V$ for $n > k$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Theorem 1

Let A, B, S and T be self mappings of X satisfying.

(1) one of A, B, S and T is continuous,

(2) the pairs A, S and B, T are compatible,

(3) $AX \subset TX$ and $BX \subset SX$,

(4) there exists functions $a_i : X \times X \to [0, 1)$, $i = 1, ..., 5$, such that for any $x, y \in X$, any $v_i \in \mathcal{V}$ $(i = 1, ..., 5)$,

$$(Ax, By) \in a_1V_1 \circ a_2V_2 \circ a_3V_3 \circ a_4V_4 \circ a_5V_5$$

if $(Sx, Ax) \in V_1$, $(Ty, By) \in V_2$, $(Sx, By) \in V_3$, $(Ty, Ax) \in V_4$ and $(Sx, Ty) \in V_5$, where $a_i = a_i(x, y)$, $i = 1, ..., 5$, $a = \sup\{\sum_{i=1}^{5} a_i(x, y) : x, y \in X\} < 1$ and $a_3(x, y) = a_4(x, y)$.

Then A, B, S and T have a unique common fixed point in X.

Proof

Let $V \in \mathcal{V}$ be arbitrary. Denote by p a Minkowski's pseudometric of V. For any $x, y \in X$, set $p(Sx, Ax) = r_1$, $p(Ty, By) = r_2$, $p(Sx, By) = r_3$, $p(Ty, Ax) = r_4$ and $p(Sx, Ty) = r_5$. Take $\varepsilon > 0$. Then, $(Sx, Ax) \in (r_1 + \varepsilon)V$, $(Ty, By) \in (r_2 + \varepsilon)V$, $(Sx, By) \in (r_3 + \varepsilon)V$, $(Ty, Ax) \in (Sx, Ty) \in (r_4 + \varepsilon)V$, $(Sx, Ty) \in (r_5 + \varepsilon)V$. From (4) and Lemma 1 (i) we have

$$(Ax,By) \in a_1(r_1 + \varepsilon)V \circ a_2(r_2 + \varepsilon)V \circ a_3(r_3 + \varepsilon)V \circ a_4(r_4+\varepsilon)V$$
$$\circ\, a_5(r_5 + \varepsilon)\, V \subset \Big[\sum_{i=1}^{5} a_i(r_i + \varepsilon)\Big] V.$$

It follows that $p(Ax, By) < \sum_{i=1}^{5} a_i(r_i + \varepsilon)$. Letting $\varepsilon \to 0$, we have

(5) $$p(Ax,By) \le a_1p(Sx,Ax) + a_2p(Ty,By) + a_3p(Sx,By) + a_4p(Ty,Ax) + a_5p(Sx,Ty$$

Let $x_0 \in X$. By (3) we can easily choose a sequence $\{y_n\}$ in X such that $y_{2n} = Tx_{2n+1} = Ax_{2n}$, $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$, $n \in \omega$. Put $p_n = p(y_n, y_{n+1})$. Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (5) we have

$$p_{2n} \le a_1p_{2n-1} + a_2p_{2n} + a_3p(y_{2n-1}, y_{2n+1}) + a_4p(y_{2n}, y_{2n}) = a_5p_{2n-1}$$

which implies

$$p_{2n} \le \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3} p_{2n-1} \le \frac{a - a_2 - a_4}{1 - a_2 - a_3} p_{2n-1} \le ap_{2n-1}$$

where $a_i = a_i(x_{2n}, x_{2n+1})$, $i = 1, ..., 5$. Similarly $p_{2n-1} \leq ap_{2n-2}$. It follows that $p_n \leq ap_{n-1} \leq ... \leq a^n p_0$ for $n \in N$. Now, for $m, n \in N$ and $m > n$, we have

$$p(y_n, y_m) \leq \sum_{i=n}^{m-1} p_i \leq \sum_{i=n}^{m-1} a^i p_0 \leq \frac{a_n}{1-a} p_0$$

Since $a < 1$, there exists $k \in N$ satisfying $a^n p_0 < 1 - a$ for $n > k$. By Lemma 1 (ii) and Lemma 2 we have $(y_n, y_m) \in V$ for $m > n > k$. Hence $\{y_n\}$ is a Cauchy sequence. Since X is sequentially complete, $y_n \to u$ for some $u \in X$. Consequently, $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ also converge to u.

Now, suppose that T is continuous. Then

(6) $$Ty_{2n+1} = TBx_{2n+1} \to Tu, \; Ty_{2n} = TTx_{2n+1} \to Tu.$$

Since B and T are compatible and $\{Bx_{2n+1}\}$, $\{Tx_{2n+1}\}$ converge to u, then by (6) for any $W \in \mathcal{V}$, there exist $k \in N$ such that (BTx_{2n+1}, TBx_{2n+1}), $(TBx_{2n+1}, Tu) \in \frac{1}{2} W$ for $n > k$. By Lemma 1 (i) it follows that $(BTx_{2n+1}, Tu) \in \frac{1}{2} W \circ \frac{1}{2} W \subset W$ for $n > k$. This implies that

(7) $$BTx_{2n+1} = By_{2n} \to Tu$$

Take V and p as above. Using (5) we get

$$\begin{aligned} p(Ax_{2n}, By_{2n}) &\leq a_1 p(Sx_{2n}, Ax_{2n}) + a_2 p(Ty_{2n}, By_{2n}) + a_3 p(Sx_{2n}, By_{2n}) \\ &\quad + a_4 p(Ty_{2n}, Ax_{2n}) + a_5 p(Sx_{2n}, Ty_{2n}) \\ &\leq a[p(Sx_{2n}, Ax_{2n}) + p(Ty_{2n}, By_{2n})] = a \,.\, \max \\ &\quad \{p(Sx_{2n}, By_{2n}), p(Ty_{2n}, Ax_{2n}), p(Sx_{2n}, Ty_{2n})\} \end{aligned}$$

where $a_i = a_i(x_{2n}, y_{2n})$, $i = 1, ..., 5$. Letting $n \to \infty$, by (6) and (7) we obtain

$$p(u, Tu) \leq ap(u, Tu)$$

which implies $p(u, Tu) = 0$ and hence $(u, Tu) \in V$. Since V is arbitrary and X is a Hausdorff space, $u = Tu$. Again, using (5) we obtain

$$\begin{aligned} p(Ax_{2n}, Bu) &\leq a_1 p(Sx_{2n}, Ax_{2n}) + a_2 p(Tu, Bu) + a_3 p(Sx_{2n}, Bu) + \\ &\quad a_4 p(Tu, Ax_{2n}) + a_5 p(Sx_{2n}, Tu) \end{aligned}$$

$\leq a\,[p(Sx_{2n}, Ax_{2n}) + p(u, Ax_{2n}) + p(Sx_{2n}, u)] + a\,.\max\{p(u, Bu), p(Sx_{2n}, Bu)\}$

where $a_i = a_i\,(x_{2n}, u)$, $i = 1, \ldots, 5$. As $n \to \infty$ in the above inequality we have

$$p(u, Bu) \leq a[3p(u, u) + p(u, Bu)] = ap(u, Bu)$$

Similarly we deduce that $u = Bu$. Note that $BX \subset SX$. Then there is a point $w \in X$ satisfying $Bu = Sw = u$. By using (5) again we get

$$\begin{aligned} p(Aw, Bu) &\leq a_1 p(Sw, Aw) + a_2 p(Tu, Bu) + a_3 p(Sw, Bu) \\ &\quad + a_4 p(Tu, Aw) + a_5 p(Sw, Tu) \\ &= (a_1 + a_4)\, p(u, Aw) \leq ap(u, Aw) \end{aligned}$$

where $a_i = a_i\,(w, u)$, $i = 1, \ldots, 5$. It is easy to show that $Aw = u$. Put $z_n = w$ for $n \in N$. Note that $Aw = Sw = u$. Then $Az_n, Sz_n \to u$. By the compatibility of A and S, for every $V \in \mathcal{V}$, there exists $k \in N$ such that $(SAz_n, ASz_n) = (SAw, ASw) \in V$ for $n > k$. This implies that $SAw = ASw$ and hence $Au = AAw = ASw = SAw = SSw = Su$. From (5) we get

$$\begin{aligned} p(Au,Bu) &\leq a_1 p(Su,Au) + a_2 p(Tu,Bu) + a_3 p(Su,B_i) + a_4 p(Tu,Au) \\ &\quad + a_5 p(Su,Tu) \\ &= (a_3 + a_4 + a_5)\, p(Au, u) \leq ap(Au, u) \end{aligned}$$

where $a_i = a_i\,(u, u)$, $i = 1, \ldots, 5$. It follows that $u = Au$. Hence u is a common fixed point of A, B, S and T.

To show that u is the unique common fixed point of A, B, S and T, let us suppose that w is a second common fixed point of A, B, S and T. From (5) we have

$$p(u, w) = p(Au, Bw) \leq (a_3 + a_4 + a_5)\, p(u, v) \leq ap(u, v)$$

where $a_i = a_i(u, w)$, $i = 1, \ldots, 5$. It follows that $u = w$, providing the uniqueness of u.

Similarly, we can also complete the proof when A or B or S is continuous. This completes the proof.

Corollary 1

Let A and B be self mappings of X. Assume there exist functions $a_i : X \times X \rightarrow [0, 1)$, $i = 1, ..., 5$, such that for any $x, y \in X$, any $v_i \in \mathcal{V}$ $(i = 1, ..., 5)$.

$$(Ax, By) \in a_1V_1 \circ a_2V_2 \circ a_3V_3 \circ a_4V_4 \circ a_5V_5$$

if $(x, Ax) \in V_1$, $(y, By) \in V_2$, $(x, By) \in V_3$, $(y, Ax) \in V_4$ and $(x,y) \in V_5$, where $a = \sup \left\{ \sum_{i=1}^{5} a_i(x, y) : x, y \in X \right\} < 1$ and $a_3(x, y) = a_4(x, y)$. Then A and B have a unique common fixed point.

Proof

Let S and T be the identity mapping on X. Corollary 1 follows from Theorem 1.

Remark

Theorems 3.1, 3.2 and 3.4 of Acharya [1] are special cases of the above Corollary 1. In case $a_i(x, y)$ is a constant, $i = 1, ..., 5$, our Corollary 1 is due to Mishra [2].

Theorem 2

Let S and T be self mappings of X such that either S or T is continuous. Then S and T have a common fixed point in X if and only if there exist self mappings A and B of X satisfying (2), (3) and (4).

Proof

By Theorem 1 it is sufficient to prove the necessity of the condition. Suppose that S and T have a common fixed point u. Define self mappings A and B of X by putting $Ax = Bx = u$ for all $x \in X$. Let $a_i(x, y) = 1/6$ for all $x, y \in X$, $i = 1, ..., 5$. It is easy to see that (2) and (3) hold. For any $x, y \in X$, any $V_i \in \mathcal{V}$ $(i = 1, ..., 5)$ we have

$$(Ax, By) = (u, u) \in \Delta \subset a_1V_1 \circ a_2V_2 \circ a_3V_3 \circ a_4V_4 \circ a_5V_5$$

if $(Sx, Ax) \in V_1$, $(Ty, By) \in V_2$, $(Sx, By) \in V_3$, $(Ty, Ax) \in V_4$ and $(Sx, Ty) \in V_5$; i.e., (4) holds also. This completes the proof.

As a particular case of Theorem 2, we get the following result which extends the Theorem of Jungck [3].

Corollary 2

Let S be a continuous self mapping of X. Then S has a fixed point in X if and only if there exists $r \in [0, 1)$ and a self mapping A of X such that $AX \subset SX$, A and S are compatible and for any $x, y \in X$, any $V \in \mathcal{V}$.

$$(Ax, Ay) \in rV$$

if $(Sx, Sy) \in V$.

REFERENCES

1. S.P. Acharya, Some results on fixed points in uniform spaces, *Yokohama Math. J.* **22(1-2)** (1974), 105-116.
2. S.N. Mishra, On common fixed point in uniform spaces II, *Indian J. pure appl. Math.* **9(1)** (1978), 26-31.
3. G. Jungck, Commuting mapping and fixed points, *Amer, Math. Monthly,* **83**(1976), 261-263.
4. G. Iungck, Compatible mappings and common fixed points, *Internat. J. math. & Math. Sci.* **9(4)** (1986), 771-779.

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 154 – 158

SYMMETRIC PRESENTATIONS: $U_3(5)$

Abdul Jabbar

Department of Mathematics
Government College Samanabad,
Faisalabad, Pakistan.

INTRODUCTION

Let G be a group, and $\tau = \{t_1, t_2, ..., t_n\}$ be a subset of G. For $i = 1, 2, 3, ..., n$ let $T_i = <t_i>$, and $\bar{\tau} = \{T_1, T_2, T_3,..., T_n\}$. We define

$$N = N_G(\bar{\tau})$$

i.e., N is the normalizer in G of the cyclic subgroups $\bar{\tau}$. Following Curtis [2, 3] we refer to N as the control subgroup and define τ to be symmetric generating set for G if, and only if, $G = <\tau>$ and N permutes $\bar{\tau}$ transitively by conjugation.

A presentation of G given explicitly or implicitly in terms of τ is called a *symmetric presentation*.

STANDARD PROCEDURE

We first choose a group N together with a permutation action of N on n letters. Take a free group F_n on n generators $\tau = \{t_1, t_2, ..., t_n\}$ and extend this by a group of automorphisms isomorphic to N which acts on τ by conjugation, permuting the free generators (possibly normalizing the cyclic subgroups they generate). Moreover we shall take a monomial p-modular representation of N. This leads to a well-defined group

$$F_n : N$$

Let free generators be of order m, then we obtain

$$P = m^* : N$$

where m^* denote the free product of n copies of cyclic groups of order m. So

$$P = (T_1 * T_2 * T_3 * ... * T_n) : N$$

where $\qquad T_i = <t_i> = C_n$ for $i = 1, 2, 3, \ldots, n$.

The group P will be called the *progenitor* for the family of finite homomorphic images of itself. We are interested in the finite images of these progenitors which are generated by the images of symmetric generators, so we add one or two relations and do cosets enumeration over N. If coset enumeration does not work we do coset enumeration over larger groups.

e.g. $\qquad <t_i, i = 1, 2, 3, \ldots>$ or $<N, (t_i^{-1} t_j)^k>$

Now we give two symmetric representations of group $U_3(5)$ by taking distinct control subgroups.

Control Subgroup $N \cong 2 . A_5$

If we take the double cover $2 . A_5$ as a control subgroup we must seek a *faithful*, monomial p-modular representation of $2 . A_5$. In particular the central element will be represented by minus the identity matrix. i.e. it will invert all the symmetric generators. What is the lowest dimension we can consider? Five symmetric generators are not possible: $(2 . A_4)' \cong Q_8$, and so $2 . A_4$ has $|(2 . A_4) : Q_8| = 3$ linear representations each of which has the central element in its kernel. Thus there does not exist a faithful monomial 5-dimensional representation of $2 . A_5$.

However a subgroup of index 6 in $2 . A_5$ has shape

$H = <u, v \mid u^5 = v^4 = 1 = u^v u> \cong (2 \times 5) : 2$ where the derived group $H' = <u> \cong C_5$. Thus there are four linear representations, two of which map the central element to -1. We induce one of these up to $2 . A_5$ to obtain the required representation. Since $H/H' \cong C_4$ we must choose a finite field with fourth roots of unity, i.e. $4 \mid (p-1) \mid$; the smallest example is thus GF_5. This leads to the representation.

$$x \sim \left(\begin{array}{c|cccccc} 1 & & & & & & \\ \hline & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & -1 & \\ & 1 & & & & & \end{array}\right)$$

$$y \sim \left(\begin{array}{c|ccccc} & 1 & & & & \\ \hline & & -1 & & & \\ -1 & & & & & \\ & & & & & \\ & & & & -2 & \\ & & & & & 2 \end{array}\right)$$

i.e. $\quad x : (t_\infty)\,(t_0, t_1, t_2, t_3, t_4),\ \ y : (t_\infty, t_0, t_1^{-1})\,(t_2, t_4, t_3^{2})$

which satisfy the presentation.

$$< x, y \mid x^5 = y^3 = (xy)^4 = 1 = [(xy)^2\,,\ x] >$$

and give $\quad xy : \quad (t_\infty, t_0, t_\infty^{-1}, t_0^{-1})\,(t_1, t_4, t_1^{-1}, t_4^{-1})$

$$(t_2, t_2^2, t_2^{-1} t_2^{-2})\ (t_3, t_3^2, t_3^{-1}, t_3^2)$$

Note: Bold face letters t_i represent symmetric generators and Italic t_i show the action of the control subgroup on the symmetric generators. The progenitor

$$\mathbf{5^{*6} : (2 : A_5)}$$

thus has presentation

$$<x, y, \mathbf{t} \mid x^5 = y^3 = (xy)^4 = \mathbf{1} = [(xy)^2,x] = \mathbf{t}^5 = [x,\mathbf{t}] = \mathbf{t}^{yx^3y}\,(\mathbf{t}^{yx^2})^3>$$

where the last relation says that $\mathbf{t}_2^{(xy)} = \mathbf{t}_2^2$.

We note that

$$[(t_\infty, t_0^{-1}, t_1^{-1}, t_\infty^{-1}, t_0, t_1)\,(t_2, t_4^{-1}, t_3^{-2}, t_2^{-1}, t_4, t_3^2)\,t_\infty]^5 = 1.$$

Coset enumeration over control subgroup gives index 1050, which shows

$$\frac{\mathbf{5^{*6} : (2 : A_5)}}{[(t_\infty, t_0^{-1}, t_1^{-1}, t_\infty^{-1}, t_0, t_1)\,(t_2, t_4^{-1}, t_3^{-2}, t_2^{-1}, t_4, t_3^2)\,t_\infty]^5} \cong U_3(5)$$

i.e.

$$<x, y, t \mid x^5 = y^3 = (xy)^4 = 1 = [(xy)^2, x] = t^5 = [x, t] = t^{yx^3y}\,(t^{yx^2})^3 = (y(xy)^2t)^5>$$

is isomorphic to

$U_3(5)$.

In this case the six symmetric generators form ten pairs of triplets such that in each pair one triple generates A_7 and the other generate $U_3(5)$.

Control Subgroup $N \cong 2 \,.\, S_5$

We can also produce symmetric presentation of $U_3(5)$ by taking control subgroup as $2 \,.\, S_5$ with progenitor

$$5^{*(6+6)} : (2 \,.\, S_5)$$

where the 5-modular monomial representation of $\mathbf{2 \,.\, S_5}$ is defined by

$$x \sim (t_0)\,(t_1, t_2, t_3, t_4, t_5)$$
$$(s_0)\,(s_1, s_2, s_3, s_4, s_5)$$
$$y \sim (t_0, s_1, t_1, s_2^{-1}, s_0^{-1}, s_1^{-1}, t_1^{-1}, s_2)$$
$$(t_2, s_5, t_2^2, s_5^2, t_2^2, s_5^{-1}, t_2^{-2}, s_5^{-2})$$
$$(t_3, s_4^{-2}, t_4^{-1}, s_0^{-1}, t_3^{-4}, s_4^2, t_4, s_0)$$
$$(t_5, s_3^2, t_5^{-2}, s_3, t_5^{-1}, s_3^{-2}, t_5^2, s_3^{-1})$$

Note: Bold face letters $\mathbf{t_i}$ represent symmetric generators and Italic t_i show the action of the control subgroup on the symmetric generators.

A presentation for the control subgroup is given by:

$$<x, y \mid x^5 = y^8 = (xy)^2 = [x, y^4] = [x, xy]^3 = 1>$$

Then a presentation for the progenitor is

$$<x, y, t \mid x^5=y^8 = (xy)^2 = [x, y^4] = [x, xy]^3 = 1 = t^5 = [t, x] = t^{-2}t^{(yx^{-2}yx^2y^2)}>$$

We noted that

$$<t_0, s_0> \cap N = 2^2$$
$$(yx)^{y^3x} = t_0^2\, s_1\, t_0^2$$
$$(y\, t_0)^4 = 1$$

Coset enumeration over control subgroup gives an index 525, which shows

$$\frac{5^{*(6+6)} : (2 : S_5)}{(yt)^4} \cong U_3(5)$$

$$(y\,x)^{y^3x} = t_0^2\, s_1\, t_0^2$$

i.e.

$$<x, y, t \mid x^5 = y^8 = (xy)^2 = [x, y^4] = [x, xy]^3 = 1 = [t, x] = t^{-2}t^{(yx^{-2}yx^2y^2)} = (y\,t)^4,\ (yx)^{y^3x} = t^2t^yt^2>$$

is isomorphic to

$U_3(5)$.

It is remarked that without the relator $(y\,t)^4$ we get the group $3 : U_3(5)$ and also the relator $t^{-2}\,t^{yx^{-2}yx^2y^2}$ can be replaced by $t\,t^{y^4}$

For the interested reader we give here some maximal subgroups of G.

$A_7 = <t_0, t_1, t_2>$, has index 50 in G and is the largest maximal subgroup of G, it also contains s_0, s_1, s_5.

$\mathbf{5^{1+2} : 8} = <\mathbf{y}, \mathbf{t}_5>$, is a maximal subgroup and also contains the symmetric generator s_3.

$M_{10} = <y, \mathbf{t}_0>$, has index 175 in **G** and contains symmetric generators $\mathbf{t}_1, \mathbf{s}_1, \mathbf{s}_2$

$2 . S_5 = <x, y>$, is maximal and is control subgroup.

REFERENCES

1. Cannon J., Bosma W., Gayley quick reference guide (1991).
2. Curtis R. T., Symmetric Presentations I : Introduction with particular reference to Mathieu groups M_{24} and M_{12}. Proceedings of The LMS Durham Conference on graphs and combinatorix (1990).
3. Curtis R.T. Symmetric Presentation II: The Janko group J_1, unpublished.
4. Jabbar A., Symmetric Presentation of subgroups of the Conway group Co_1 and related topics, Ph.D. Thesis Birmingham Univ. (1992).

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 159 - 170

POSITIVITY PRESERVING USING RATIONAL CUBICS

Malik Zawwar Hussain
Department of Mathematics
University of the Punjab,
Lahore, Pakistan.

Sohail Butt
Departmentof Computer Science,
University of Engineering & Technology,
Lahore, Pakistan.

ABSTRACT

Sarfraz [10] has described and analysed a C^1 interpolatory rational cubic function. This rational cubic function involve two free parameters and does not preserve the shape of the positive data. This paper develop necessary and sufficient conditions under which the positive interpolation is successfully achieved. Moreover, user has a freedom to play further with the curve untill a desired shape is achieved. A simple case of constraint interpolation where the curve is forced to lie on the same side of a given straight line is also discussed.

1. INTRODUCTION

The problem of shape preserving curve is considered by various authors. For brevity user is refered to [1-6]. A particular shape problem is positivity problem which is introduced by Schmidt and Hess[7]. They discuss both quadratic and rational quadratic spline and develop necessary and sufficient conditions under which positive interpolation is of success. These conditiios may fail for quadratic spline while for rational quadratic spline they can always be satisfied. Schmid and Hess[8] use cubic Hermite spline and give necessary and sufficient conditions for a cubic polynomial to be positive in a given interval. Their algorithm works by estimating the slopes at given data points and therefore is not applicable when slopes are given. This problem occur in plotting the computed solution of an ordinary differential equation. Butt and Brodlie[9] show how it is possible to preserve positivity by inserting one, or possibly two, intermediate knots. Their algorithm produces positive curve through positive data but does not allow user to refine the curve interactively to desired shape.

In this paper we use a rational cubic Hermite spline of Sarfraz[10] which has cubic denominator and produces a positive curve through positive data and allows user to fine tune tue curve to desired shape. The refinement is necessary, as a curve drawn by a skillful draftsman usually looks more pleasing than the one generated by an automated algorithm. To prdouce a curve which would possibly look like the one produced by a draftsman we need some interaction. Rational cubic functions have been used to produce pelasing curves through given data, Sarfraz[11], Butt et al[12]. This paper is an addition to the methods given in these papers. In Section 2, we establish the necessary and sufficient conditions under which the rational cubic Hermite spline produces a positive curve through data. However, simpler sufficient conditions are used to visualize positive data in Section 4. This section includes various examples of curves produced by the method described in Section 2 & 3. The figures produced clearly describe the usefullness and power of interaction in curve drawing and designing. In Section 3, we consider a particular case of a constraint interpolation where the curve is required to lie on the same side of a given straight line. We have developed sufficient conditions which produce curves above a straight line. A particular example is also include in Section 4.

2. PRESERVING POSITIVITY IN INTERPOLATION

The problem of positive interpolation can be described as follows: For given points (t_0, f_0), (t_1, f_1), ..., (t_n, f_n), with $t_0 < t_1 < \dots < t_n$ and $f_0 \geq 0, f_1 \geq 0, \dots, f_n \geq 0$, construct an interpolant S which is positive on the whole interval $[t_0, t_n]$, that is $S(t) \geq 0$. Let

$$S_i(t) = \frac{p_i(t)}{q_i(t)} \tag{2.1}$$

where

$$p_i(t) = f_i(1-\theta)^3 + (v_i f_i + h_i d_i)\,\theta(1-\theta)^2 + (w_i f_{i+1} - hid_{i+1})\theta^2(1-\theta) + f_{i+1}\theta^3,$$

$$q_i(t) = (1-\theta)^3 + v_i\theta(1+\theta)^2 + w_i\theta^2(1-\theta) + \theta^3,$$

f_i and d_i are, respectively, the data values and the first derivative values at the knots t_i, $t = 0, 1, \ldots, n$ with $h_i = t_{i+1} - t_i$, $\theta = \frac{t-t_i}{h_i}$, $\Delta_i = \frac{f_{i+1} - f_i}{h_i}$ and $v_i, w_i > 0$ are free parameters.

The function $S(t)$ has the Hermite interpolation properties, that is,

$$S(t_i) = f_i, \quad \text{and} \quad S^{(1)}(t_i) = d_i, \quad i = 0, 1, \ldots, n \tag{2.2}$$

The v_i's and w_i's will be used as shape parameters to control and fine tune the shape of the curve. The case $v_i = w_i = 3$, $i = 0, 1, \ldots, n-1$, is that of cubic Hermite interpolation. The restriction $v_i, w_i > 0$ ensures a positive denominator $q_i(t)$ and so first condition on v_i, w_i is

$$v_i > 0, \quad w_i > 0. \tag{2.3}$$

Since $q_i(t) > 0$ for all $v_i, w_i > 0$, so we just need to talk about the positivity of $p_i(t)$. But first we express $p_i(t)$ into the form used in Schmidt and Hess[8] to develop the positivity conditions. We have after simplification:

$$p_i(t) = \kappa_i\theta^3 + \tau_i\theta^2 + \chi_i\theta + \psi_i, \tag{2.4}$$

where

$$\kappa_i = (1 - w_i) f_{i+1} - (1 - v_i) f_i + (d_{i+1} + d_i) h_i$$

$$\tau_i = w_i f_{i+1} + (3 - 2v_i) f_i - (d_{i+1} + 2d_i) h_i$$

$$\chi_i = (v_i - 3) f_i + d_i h_i$$

$$\psi_i = f_i$$

Now according to Schmidt and Hess[8], $p_i(t) \geq 0$ if and only if

$$(p_i'(0), p_i'(1)) \in R_1 \cup R_2 \tag{2.5}$$

where

$$R_1 = \left\{(a, b) : a \geq \frac{-3f_i}{h_i}, \ b \leq \frac{-3f_{i+1}}{h_i}\right\},$$

$$R_2 = \{(a, b) : 36f_if_{i+1}(a^2 + b^2 + ab - 3\Delta_i (a + b) + 3\Delta_i^2)$$
$$+ 3(f_{i+1}a - f_ib)(2h_iab - 3f_{i+1} a + 3f_ib) + 4h_i (f_{i+1} a^3 - f_ib^3)$$
$$- h_i^2a_2b_2 \geq 0\}.$$

We have $$p_i'(0) = \frac{(v_i - 3) f_i}{h_i} + d_i,$$

$$p_i'(1) = \frac{(3 - w_i) f_{i+1}}{h_i} + d_{i+1}.$$

Now (2.5) is true when

$$(p_i'(0), p_i'(1)) \in R_1$$

$$p_i'(0) \geq \frac{-3 f_i}{h_i} \text{ and } p_i'(1) \leq \frac{3 f_{i+1}}{h_i}.$$

This gives $$v_i > \text{Max}\left\{0, \frac{-h_i d_i}{h_i}\right\}, w_i > \text{Max}\left\{0, \frac{h_i d_{i+1}}{f_{i+1}}\right\}. \quad (2.6)$$

Further $$(p_i'(0), p_i'(1)) \in R_2$$

If $$\begin{aligned} &36 f_i f_{i+1} [\phi_1^2(v_i) + \phi_1^2(w_i) + \phi(v_i)\, \phi_2(w_i) - 3\Delta_i(\phi_1(v_i) + \phi_2(w_i)) \\ &+ 3\Delta_i^2] + 3\, [f_{i+1}\, \phi_1(v_i) - f_i \phi_2(w_i)]\, [2h_i \phi_1(v_i)\, \phi_2(w_i) \\ &- 3f_{i+1}\phi_1(v_i) + 3f_i\phi_2(w_i)] + 4h_i[f_{i+1}\phi_1^3(v_i) - f_i\phi_2^3(w_i)] \\ &- h_i^2\phi_1^2\,(v_i)\, \phi_2^2(w_i) \geq 0, \end{aligned} \quad (2.7)$$

where $\phi_1(v_i) = p_i'(0)$, $\phi_2(w_i) = p_i'(1)$.

This leads to the following:

Theorem 2.1

The rational cubic polynomial (2.1) preserves positivity if and only if either (2.6) or (2.7) is satisfied.

Remark 1

The conditions (2.7) are quite combersome so for implemention, (2.6) are used in Section 4.

Remark 2

This method can be used in both cases when either d_i's are particularly specified or estimated by some method.

3. CONSTRAINED INTERPOLATION

The problem of constrained interpolation may be described as follows; Given data points in a plane lying on one side of one or more given lines, how a curve interpolating the given data points and lying on same side of the given lines as data points can be determied? In this section, we consider the data lying above a straight line and develop a scheme to generate a curve which is also lying above the given line.

Let (t_i, f_i), $i = 0, 1, ..., n$ be given data points which lies above any straight line $f = mt + c$ provided given data also lies above this straight line, i.e.

$$f_i \geq mt_i + c \quad \text{for all} \quad i = 0, 1, ..., n.$$

We require $$S(t) = \frac{p_i(t)}{q_i(t)} \geq mt + c \quad \forall t \in [t_i, t_{i+1}].$$

We assume that $m > 0$. The case $m < 0$ can be handeled in a similar way. In each interval, $mt + c$ can be expressed as

$$a_i (1 - \theta) + b_i\theta,$$

where $a_i = mt_i + c$

$b_i = mt_{i+1} + c$

We thus require

$$S(t) = \frac{p_i(t)}{q_i(t)} \geq a_i(1 - \theta) + b_i\theta_i, \quad i = 1, 2, 3, ..., n - 1.$$

As $q_i(t) > 0$ for all v_i, $w_i > 0$, so we require:

$$Ui\ (t) = p_i(t) - \{a_i (1 - \theta) + b_i\theta\}\, q_i(t),$$

where $U_i(t)$ is a polynomial of degree 4.

It can be expressed as:

$$U_i(t) = \lambda_i(1-\theta)^4 + \mu_i\theta(1-\theta)^3 + \gamma_i\theta^2(1-\theta)^2 + \delta_i\theta^3(1-\theta) + \sigma_i\theta^4,$$

where

$$\lambda_i = f_i - a_i$$

$$\mu_i = v_i (f_i - a_i) + (f_i - b_i) + h_i d_i$$

$$\gamma_i = w_i(f_{i+1} - a_i) + v_i(f_i - b_i) + h_i(d_i - d_{i+1})$$

$$\delta_i = (f_{i+1} - a_i) + w_i (f_{i+1} - b_i) - h_i d_{i+1}$$

$$\sigma_i = (f_{i+1} - b_i)$$

As $\lambda_i \geq 0$, $\sigma_i \geq 0$, so $U_i(t) \geq 0$ if and only if $\mu_i \geq 0$, $\gamma_i \geq 0$ and $\delta_i \geq 0$. We so obtain:

$$v_i > Max \left\{0, \frac{b_i - f_i - h_i d_i}{f_i - a_i}\right\}, \qquad (3.1)$$

$$w_i > Max \left\{0, \frac{a_i - f_{i+1} + h_i d_{i+1}}{f_{i+1} - b_i}, \frac{h_i(d_{i+1} - d_i) - v_i(f_i - b_i)}{f_{i+1} - a_i}\right\}, \qquad (3.2)$$

The above discussion leads to the following theorem.

Theorem 3.1

The rational cubic (2.1) lies above the given straight line if and only if v_i and w_i satisfy (3.1) or (3.2) respectively.

4. NUMERICAL EXAMPLES

We begin this section by considering the data in Table 1.

Table 1

x	1	2	3	8	10	11	12	14
y	14	8	3	0.8	0.5	0.45	0.40	0.37

The curve in Figure 1, produced by cubic Bassel method, does not preserve positivity as it goes below the x-axis. We now apply piecewise rational cubic of Section 2 to the same data. The Figure 2 is produced by applying the global tensions given below:

$$v_i = 1.005\, L_{v,i}, \quad w_i = 1.005\, L_{w,i},$$

where $L_{v,i}$ and $L_{w,i}$, respectively represent the smallest value of which v_i and w_i can take.

The Figure 3 is produced by changing some parameters, locally:

$$v_3 = 1.2\, L_{v,3}, \; w_3 = 1.5\, L_{w,3}, \; v_4 = 10\, L_{v,4}, \; w_4 = 5\, L_{w,4}.$$

Next we consider the data given in Table 2.

Table 2

x	0	2	4	10	28	30	32
y	20.8	8.8	4.2	0.5	3.9	6.2	9.6

Once again we note that Figure 4 generated by cubic Bessel method looses positivity. To recover the positivity loss we apply cubic rational function of Section 2. Figure 5 is produced by applying the global tension:

$$v_i = 1.005\, L_{v,i}\,,\quad w_i = 1.005\, L_{w,i}\,,$$

However, the Figure 6 is produced by making the following change locally:

$$v_2 = 1.1\, L_{v,2},\, w_2 = 1.3\, L_{w,2},\, v_3 = 1.1\, L_{v,3}\,,\; w_3 = 1.0\, L_{w,3}\,,$$

We note that the Figures 2,3 and 5,6 produced by the piecewise cubic rational functions of Section 2 not only preserve positivity but also provide interaction to made the positive curve to desired shape.

Finally we consider data given in Table 3.

Table 3

x	0	2	4	10	28	30	32
y	22.8	12.8	10.2	12.5	33.9	38.9	43.6

This data always lie above the straight line $f = t + 2$. The curve shown in Figure 7 is generated by cubic Bessel method and it does not inherit the shape of data, that is, the curve lies below the straight line $f = t + 2$. However, the curve generated by the piecewise rational cubic function of Section 2 remains above the straight line and thus preserve the inherit shape of the data. This curve is shown in Figure 8.

5. CONCLUSIONS AND SUGGESTIONS

This paper has shown how positivity can be preserved using piecewise rational cubic Hermit spline. We have derived data dependent shape constraints on two shape parameters to assure the positivity of the positive data. The choice of the derivative parameters is left at the wish of the user.

A simple case of the problem of constraint interpolation where a curve is required to the same side of a given line is also solved. The general case requires the extension of the method for the planer curve which will be presented in a subsequent paper.

For the rational cubic of this paper monotonicity and convexity is discussed in Sarfraz[11]. This paper has added another feature of positivity to the same rational cubic. Future work will look at interpolation in two dimension where volume rendering techniques incloude interpolation as a key process with positivity as a typical requirement.

REFERENCES

1. Delbourgo, R. and Gregory, J. A. (1985), Shape preserving piecewise rational interpolation, *SIAM J. Stat. Computer* **6**, 967-976.
2. Fritsch, F.N. and Carlson, R.E. (1980), Monotone piecewise cubic interpolation, *SIAM J. Numer. Anal.*, **17(2)**, 238-246.
3. Goodman, T.N.T. and Unsworth, K. (1988), Shape preserving interpolation by parametrically defined curves, *SIAM J. Numer. Anal.* **25**, 1-13.
4. McAllister, D.F., Passow, E. and Roulier, J.A. (1977), Algorthms for computing shape preserving spline interpolations to data, *Math. Comp*. **31**, 717-725.
5. Sarfraz, M. (1993), Designing of curves and surfaces using rational cubics, *Computer and Graphics*, **17(5)**, 529-538.
6. R. W. Brankin, and I. Gladwell (1989), Shape preserving local interpolation for plotting solution of ODEs, *IMA J. Numer. Anal.*, **9**, 555-566.
7. Schmidt, J.W. and Hess, W. (1987), Positivity interpolation with rational quadratic splines interpolation, *Computing* **38**, 261-267.
8. Schmidt, J.W. and Hess, W. (1988), Positivity of cubic polynomial on intervals and positive spline interpolation. *BIT* **28**, 340-352.
9. Butt, S and Brodlie, K.W. (1993), Preserving positivity using piecewise cubic interpolation, *Computer and Graphics* **17(1)**, 55-64.
10. Sarfraz, M. (1992), Interpolatory rational cubic spline with biased, pint and interval tension, *Computer and Graphics* **16(4)**, 427-430.
11. Sarfraz, M. (1993) Shape preserving rational interpolation, *Extracta mathematicae* **28(2-3)**, 106-11.
12. Butt, S., M. Sarfraz and Hussain, M.Z. (1994), Pleasing curves to the positive data using rational cubics, TR#4, KFUPM, Saudi Arabia.

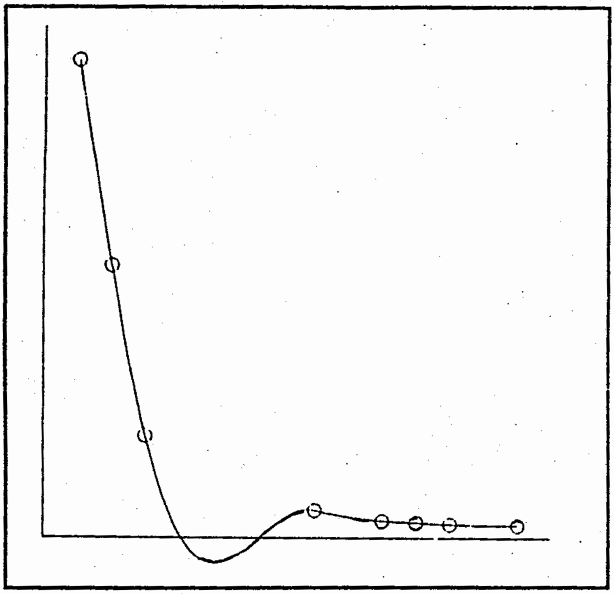


Figure 1: Cubic Osculatary Method

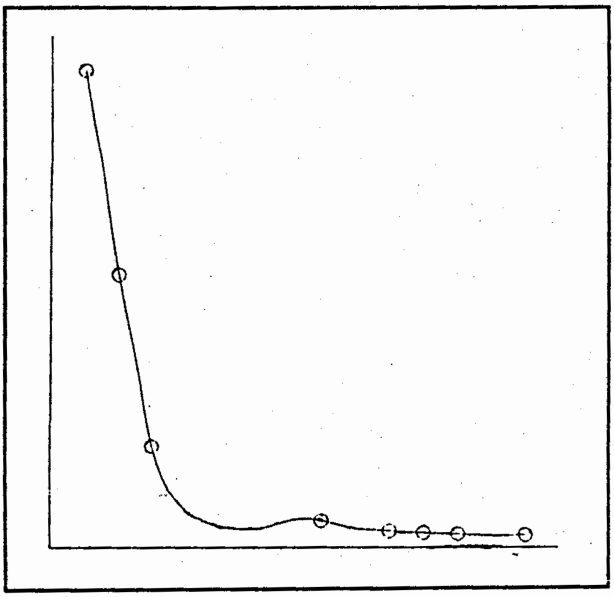


Figure 2: $v_i = 1.005L_{v,i}$, $w_i = 1.005L_{w,i}$

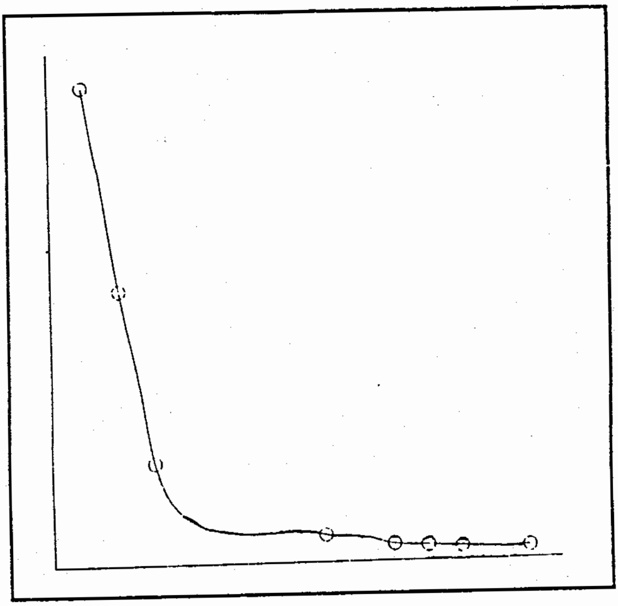


Figure 3: $v_3 = 1.2L_{v,3}$, $w_3 = 1.5L_{w,3}$, $v_4 = 10L_{v,4}$, $w_4 = 5L_{w,4}$

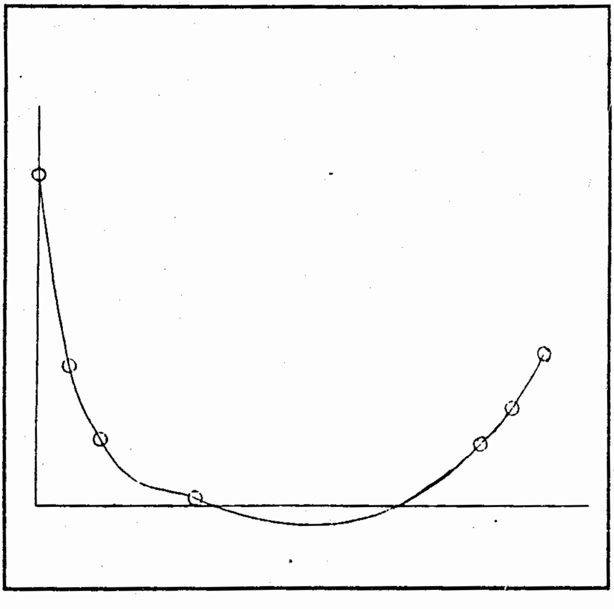


Figure 4: Cubic Osculatary Method

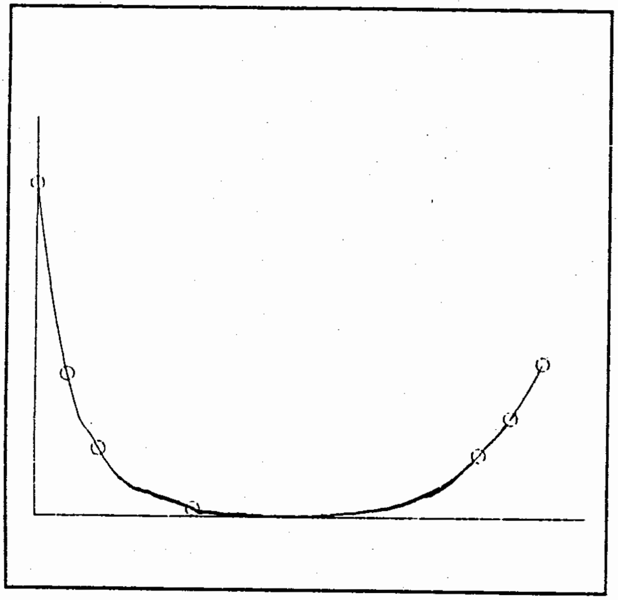


Figure 5: $v_1 = 1.005L_{v,1}$, $w_1 = 1.005L_{w,1}$

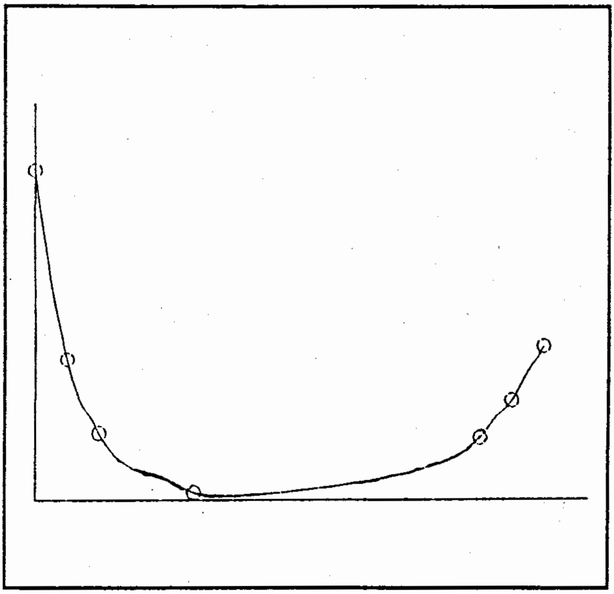


Figure 6: $v_2 = 1.1L_{v,2}$, $w_2 = 1.3L_{w,2}$, $v_3 = 1.1L_{v,3}$, $w_3 = L_{w,3}$

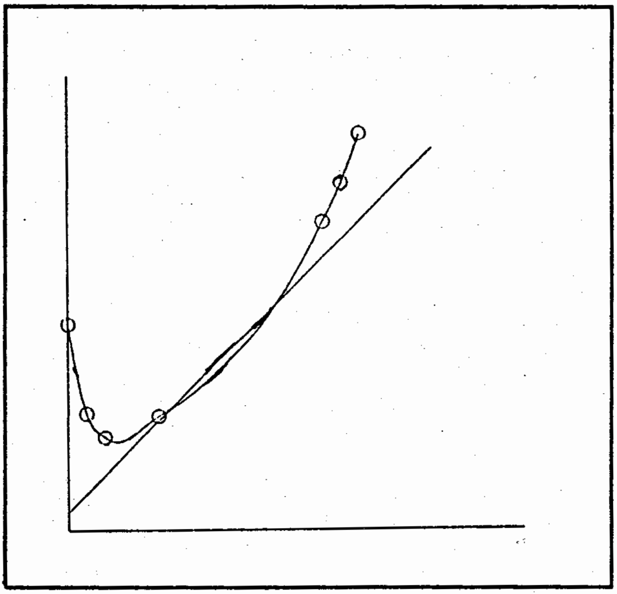


Figure 7: Cubic Osculatary Method

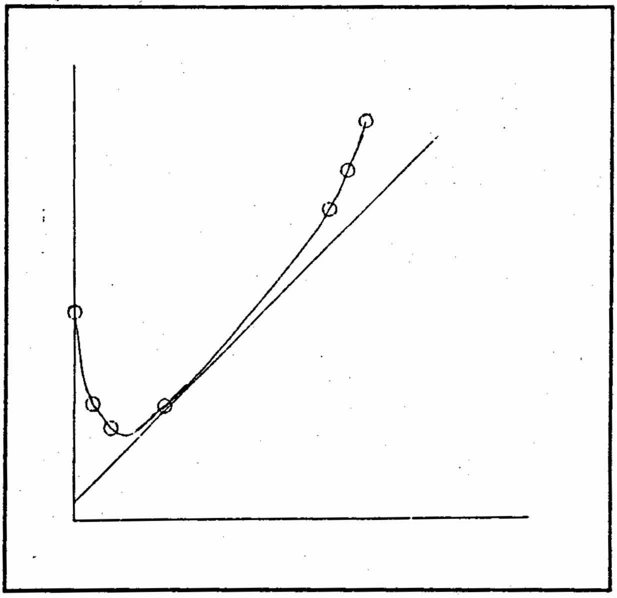


Figure 8: Rational Cubic Curve

Punjab University
Journal of Mathematics
Vol. XXVIII, (1995), pp 171 - 189

CROSSING TIME AND RENEWAL NUMBERS RELATED WITH 2-STAGES ERLANG PROCESS AND POISSON PROCESS

Mir G.H. Talpur
Department of Statistics
University of the Sindh,
Jamshoro

ABSTRACT

In this paper, it is considered how to find some joint distrinbutions and their margianl distributions of crossing time and renewal numbers related with 2-stages Erlang proocess and Poisson process by constructing and absorbing Markov proocess. The obtained results show that one-dimension marginal distributions are 2(N + 1) order PH-distributions.

KEYWORDS: Erlang process, Absorbing Markov process, Poisson process and PH-distribution.

1. INTRODUCTION

ASSUMPTION: Let N be a constant, $\{X_i\}$ and $\{Y_j\}$ be two sequences of random variables. Suppose that $\{X_i\}$, $i = 1, 2, ...$; are independently and identically distributed (i.i.d) $F(t)$ with finite mean and $\{Y_j\}, j = 1\ 2, 3 ...$; are i.i.d. $G(t)$ with mean μ^{-1}

$N_1(t)$ = a Erlang process associated with $\{X_i\}$ in which the distribution of X_i is a 2-stage Erlang distribution.

$N_2(t) = \sup\{n \mid T_n \leq t\}$ is the counting process associated with $\{Y_j\}$

where $T_0 = 0$ and $T_n = \sum_{j=1}^{\infty} Y_j$.

We assume that X_i, Y_j are mutually independent. Consider the same problem in Ref. [1, 2].

$$\xi_N = \inf\{n \mid T_n \geq S_N\} \text{ its taking values are } j = 1, 2 ...$$

$$T_{\xi_N} = \sum_{i=1}^{\xi_N} Y_j, \text{ its taking values are } t \geq 0$$

and $\quad \eta_N = N_1(T_{\xi_N})$, its taking values are $i = N, N + 1 ....$

We are interested in finding the joint and marginal distributions of T_{ξ_N}, ξ_N and η_N. In Ref. [1, 2] we have obtained explicit expression for the case of two Poisson process and first one is Poisson processs and other one 2-stage Erlang process. In this paper we consider first one is 2-stage Erlang process and other one is Poisson process.

2. JOINT DISTRIBUTION OF T_{ξ_N}, ξ_N AND η_N

2.1 Absorbing Markov Process and Absorbing Time Distribution

We considerf a Markov process $\{X(t),\ t \geq 0\}$ on the state space E. If E_0 and E_1 are two non null sub-sets of E and they satisfy:

1) $E_0 \cup E_1 = E$, $E_0 \cap E_1 = \phi$ in this case E_0, E_1 are called a partition of E;

2) E_0 is the absorbing state set and E_1 is the transient state set;

3) For given initial condition a_{E_1} the absorption of Markov process is certain, then $\{X(t),\ t \geq 0\}$ is called an absorbing Markov process (AMP).

We denote the distribution function of the absorbing time of the AMP $(X(t),\ t \geq 0\}$ by $H(t)$.

Theorem 2.1

Let $P_i(t) = P\{X(t) = i\}$, then

$$H'(t) = \sum_{\substack{i \in E_1 \\ j \in E_0}} P_i(t)\ q_{ij}$$

Proof

See Ref. [3]

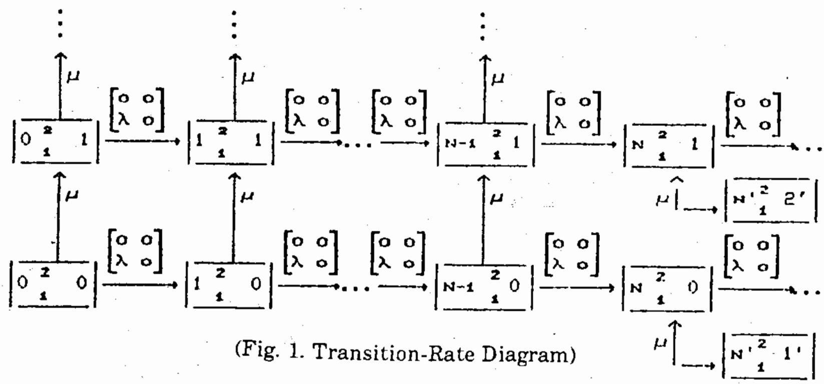
2.2 Construct an Absorbing Markov Process to Analyze the Problem

Now we construct an absorbing Markov process (AMP) to analyze our problem.

Consider the AMP { $N_1(t)$, $N_2(t)$, $I(t)$ in which $N_1(t)$ and N2(t) see section one, but $I(t)$ represent the phase of X_i at time t respectively. Its state space

$$E = \{(i,j,k),\ (i',j')\ /\ i,j = 0,1\ldots;k = 1,.2;\ i' = N',\ N' + 1,\ldots;j' = 1',2',\ldots;\}$$

where (i', j') are absorbing states. Their transition of states are shown in fig. 1.



(Fig. 1. Transition-Rate Diagram)

Let $\qquad P_{ij}(k, t) = P\{N(t)_1 = i, N(t)_2 = j, I(t) = k\}$

Then

$P_{ij}(t) = \{P_{ij}(1,1,t), P_{ij}\ (2,1,t)\}$ and suppose that $P_{-1}j(t) = P_{i-1}(t) = 0$. By the transition-rate diagram we can get the following system of differential equations.

$$P'_{ij}(t) = P_{ij}(t)\left(\begin{pmatrix} \lambda & -\lambda \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}\right) + P_{i-1,j}(t)\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$$

$$+ \mu P_{i,j-1}(t)$$

$$i = 0, \ldots N-1;\ j = 0, 1 \ldots; \qquad (2.1)$$

$$P'_{ij}(t) = P_{ij}(t)\left(\begin{pmatrix} \lambda & -\lambda \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}\right) + P_{i-1,j}(t)\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$$

$$i = N, \ldots N+1;\ j = 0, 1 \ldots; \qquad (2.2)$$

The initial conditions are $P_{00}(0) = [1, 0]$ and all others are equal to zero. We use the Z and L transforms to solve above system of equations.

Let

$P_i(t, u) = \sum_{j=0}^{\infty} P_{ij}(t)\, u^j,\ |u| \le 1$, then we have

$$P_i'(t, u) = P_i(t, u) \begin{pmatrix} \lambda+\mu & -\lambda \\ 0 & \lambda+\mu \end{pmatrix} + P_{i-1}(t,u) \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} + \mu u P_i(t, u)$$

$$i = 0, 1 \ldots N-1; \tag{2.3}$$

$$P_i'(t,u) = P_i(t,u) \begin{pmatrix} \lambda+\mu & -\lambda \\ 0 & \lambda+\mu \end{pmatrix} + P_{i-1}(t,u) \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, \quad i = N, N+1, \ldots; \tag{2.4}$$

Here initial conditions become $P_0(0, u) = [1, 0]$ and all others are equal to zero.

Let $\quad P(t,u,z) = \sum_{i=1}^{\infty} P_i(t, u)\, z_i,\ |z| \le 1,$

We get

$$P'(t,u,z) = P(t,u,z) \begin{pmatrix} \lambda+\mu & -\lambda \\ 0 & \lambda+\mu \end{pmatrix} + P(t,u,z) z \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} + \sum_{i=1}^{N-1} \mu\mu P_i(t,u) z_i, \tag{2.5}$$

The initial condition is $P(0,u,z) = [1, 0]$

Finally, let

$$P^*(s,u,z) = \int_0^{\infty} \exp(-st)\, P(t,u,z)\, dt, \quad \mathrm{Re}(s) > 0,$$

Then

$$sP^*(s,u,z) - [1,0] = P^*(s,u,z) \begin{pmatrix} \lambda+\mu & -\lambda \\ 0 & \lambda+\mu \end{pmatrix} + P^*(t,u,z) z \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$$

$$+ \sum_{i=1}^{N-1} \mu u P_i^*(s,u) z^i$$

It implies

$$P^*(s,u,z) = \left([1, 0] + \sum_{i=1}^{N-1} \mu u P_i^*(s,u) z^i \right) \begin{pmatrix} s+\lambda+\mu & -\lambda \\ -\lambda z & s+\lambda+\mu \end{pmatrix}^{-1} \tag{2.6}$$

The remaining problem is to determine the value of $\left(\sum_{i=1}^{N-1} P_i^*(s,u) z^i \right)$.

Using L-transform in eq. (2.3), we easily obtain.

$$P_0^*(s, u) = [1, 0]\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}$$

and

$$P_i^*(s, u) = P_{i-1}^*(s, u)\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} \quad i=1, 2, \ldots, N-1.$$

Therefore

$$\sum_{j=0}^{N-1} P_i^*(s,u)z^i$$

$$= [1,0]\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}\Bigg[I+\begin{pmatrix} 0 & 0 \\ z\lambda & 0 \end{pmatrix}\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}$$

$$+\ldots+\left\{\begin{pmatrix} 0 & 0 \\ z\lambda & 0 \end{pmatrix}\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}\right\}^N\Bigg]$$

$$= [1,0]\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}\Bigg[I+\begin{pmatrix} 0 & 0 \\ z\lambda & 0 \end{pmatrix}\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}$$

$$\left(\left\{\begin{pmatrix} 0 & 0 \\ z\lambda & 0 \end{pmatrix}\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}\right\}^{N-1}\right)$$

$$= [1,0]\Bigg[\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}-$$

$$\begin{pmatrix} 0 & 0 \\ z\lambda & 0 \end{pmatrix}\Bigg]^{-1}\left(I-\left\{\begin{pmatrix} 0 & 0 \\ z\lambda & 0 \end{pmatrix}\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}\right\}^N\right) \quad (2.7)$$

Theorem 2.2

The joint transform function and the joint distribution function of random variables T_{ξ_N}, ξ_N, and η_N are

$$f^*(s,u,z) = \frac{u\mu}{(s+\lambda+\mu)^2-\lambda^2 z} z^N\left(\frac{\lambda}{s+\lambda+\mu-u\mu}\right)^{2N}[(s+\lambda+\mu)+\lambda] \quad (2.8)$$

and

$$P\{T_{\xi_N}\le t,\xi_N=j,\eta_N=i\} = \binom{2N+j-2}{j-1}\left(\frac{\mu}{\lambda+\mu}\right)^j\left(\frac{\lambda}{\lambda+\mu}\right)^{2i+1}\sum_{r=2i+j}^{\infty}\frac{((\lambda+\mu)x)^r}{r!}$$

$$\exp\left(-(\lambda+\mu)x\right)^r + \binom{2N+j-2}{j-1}\left(\frac{\mu}{\lambda+\mu}\right)^j\left(\frac{\lambda}{\lambda+\mu}\right)^{2j+1} \sum_{r=2i+j+1}^{\infty}\frac{((\lambda+\mu)x)^r}{r!}\exp(-(\lambda+\mu)x)$$

Proof

By using Theorem (2.1) and eqs. (2.6), (2.7) we have

$$f^*(s,u,z) = \sum_{i=N}^{\infty}\sum_{j=0}^{\infty} P_{ij}^*(s)\, u^{j+1} z^i \begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

$$= u\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij}^*(s)\, u^j z^i - \sum_{i=0}^{N-1}\sum_{j=0}^{\infty} P_{ij}^*(s)\, u^j z^i\right)\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

$$= u\left(P^*(s,u,z) - \sum_{i=0}^{N-1} P_i^*(s,u)\, z^i\right)\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

$$= u\left[\left(\left[[1,0]+u\mu\sum_{i=0}^{N-1} P_i^*(s,u)\, z^i\right]\right)\begin{pmatrix}s+\lambda+\mu & -\lambda\\ -\lambda z & s+\lambda+\mu\end{pmatrix}^{-1} - \sum_{i=0}^{N-1} P_i^*(s,u) z^i\right]\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

$$= u\quad [1,0]+u\mu\sum_{i=0}^{N-1} P_i^*(s,u)\, z^i - \sum_{i=0}^{N-1} P_i^*(s,u)\, z^i\ )\ \begin{pmatrix}s+\lambda+\mu & -\lambda\\ -\lambda z & s+\lambda+\mu\end{pmatrix}$$
$$\begin{pmatrix}s+\lambda+\mu & -\lambda\\ -\lambda z & s+\lambda+\mu\end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

$$= u\,[1,0] - \sum_{i=0}^{N-1} P_i^*(s,u)\, z^i \begin{pmatrix}s+\lambda+\mu-u\mu & -\lambda\\ -\lambda z & s+\lambda+\mu\end{pmatrix}\begin{pmatrix}s+\lambda+\mu & -\lambda\\ -\lambda z & s+\lambda+\mu-\mu u\end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

$$= u\left\{[1,0]-[1,0]\begin{pmatrix}s+\lambda+\mu-u\mu & -\lambda\\ -\lambda z & s+\lambda+\mu-u\mu\end{pmatrix}^{-1}\right.$$
$$\left.\left(I-\left\{z\begin{pmatrix}0 & 0\\ \lambda & 0\end{pmatrix}\begin{pmatrix}s+\lambda+\mu-u\mu & -\lambda\\ 0 & s+\lambda+\mu-u\mu\end{pmatrix}\right)^{-1}\right\}^N \begin{pmatrix}s+\lambda+\mu-u\mu & -\lambda\\ -\lambda z & s+\lambda+\mu-u\mu\end{pmatrix}\right\}$$
$$\begin{pmatrix}s+\lambda+\mu & -\lambda\\ -\lambda z & s+\lambda+\mu-u\mu\end{pmatrix}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

$$= u[1,0]\left\{\left(\begin{pmatrix}s+\lambda+\mu-u\mu & -\lambda\\ -\lambda z & s+\lambda+\mu-u\mu\end{pmatrix}^{-1}\left\{z\begin{pmatrix}0 & 0\\ \lambda & 0\end{pmatrix}^{-1}\begin{pmatrix}s+\lambda+\mu-u\mu & -\lambda\\ -\lambda z & s+\lambda+\mu-u\mu\end{pmatrix}^{-N}\right\}\right.$$
$$\begin{pmatrix}s+\lambda+\mu-u\mu & -\lambda\\ -\lambda z & s+\lambda+\mu-u\mu\end{pmatrix}\begin{pmatrix}s+\lambda+\mu & -\lambda\\ 0 & s+\lambda+\mu\end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

$$=u[1,0]\left(\left\{\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}^{-1}\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}\right\}\right)$$

$$z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}^{N-1}$$

$$\left(I-\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)^{-1}\begin{pmatrix} s+\lambda+\mu & -\lambda \\ -\lambda z & s+\lambda+\mu \end{pmatrix}\begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$=u[1,0]\left(I-\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)^{-1}$$

$$\left(\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)^{N}\left(I-\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)$$

$$\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ -\lambda z & s+\lambda+\mu-u\mu \end{pmatrix}\begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$=u[1,0]\left(I-\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)^{-1}$$

$$\left(\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)^{N}-\left(\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)^{N+1}$$

$$\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}\begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$=u[1,0]\left(I-\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)^{-1}$$

$$\left(I-\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)\left(\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)^{N}$$

$$\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ -\lambda z & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}\begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$=u[1,0]\left(\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ 0 & s+\lambda+\mu-u\mu \end{pmatrix}^{-1} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)^{N}\begin{pmatrix} s+\lambda+\mu-u\mu & -\lambda \\ -\lambda z & s+\lambda+\mu-u\mu \end{pmatrix}^{-1}\begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$=u[1,0]\left(\begin{pmatrix} \dfrac{1}{s+\lambda+\mu-u\mu} & \dfrac{\lambda}{(s+\lambda+\mu-u\mu)^2} \\ 0 & \dfrac{1}{s+\lambda+\mu-u\mu} \end{pmatrix} z\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)^{N}$$

$$\begin{pmatrix} \frac{(s+\lambda+\mu)}{(s+\lambda+\mu)^2-\lambda^2 z} & \frac{\lambda}{(s+\lambda+\mu)^2-\lambda^2 z} \\ \frac{\lambda z}{(s+\lambda+\mu)^2-\lambda^2 z} & \frac{(s+\lambda+\mu)}{(s+\lambda+\mu)^2-\lambda^2 z} \end{pmatrix} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$= u(1,0) \begin{pmatrix} \left(\frac{\lambda}{s+\lambda+\mu-u\mu}\right)^2 & 0 \\ \left(\frac{\lambda}{s+\lambda+\mu-u\mu}\right) & 0 \end{pmatrix} \begin{pmatrix} \frac{(s+\lambda+\mu)}{(s+\lambda+\mu)^2-\lambda^2 z} & \frac{\lambda}{(s+\lambda+\mu)^2-\lambda^2 z} \\ \frac{\lambda z}{(s+\lambda+\mu)^2-\lambda^2 z} & \frac{(s+\lambda+\mu)}{(s+\lambda+\mu)^2-\lambda^2 z} \end{pmatrix} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$= u(1,0) z^N \begin{pmatrix} \left(\frac{\lambda}{s+\lambda+\mu-u\mu}\right)^{2N} & 0 \\ \left(\frac{\lambda}{s+\lambda+\mu-u\mu}\right)^{2N-1} & 0 \end{pmatrix} \begin{pmatrix} \frac{(s+\lambda+\mu)}{(s+\lambda+\mu)^2-\lambda^2 z} & \frac{\lambda}{(s+\lambda+\mu)^2-\lambda^2 z} \\ \frac{\lambda z}{(s+\lambda+\mu)^2-\lambda^2 z} & \frac{(s+\lambda+\mu)}{(s+\lambda+\mu)^2-\lambda^2 z} \end{pmatrix} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$= \mu u z^N \left[\left(\frac{\lambda}{s+\lambda+\mu-u\mu}\right)^{2N} \left(\frac{\lambda}{(s+\lambda+\mu)^2-\lambda^2 z}\right) \right.$$

$$\left. + \left(\frac{\lambda}{s+\lambda+\mu-u\mu}\right)^{2N} \left(\frac{\lambda}{(s+\lambda+\mu)2-\lambda 2z}\right) \right]$$

$$= \frac{\mu u z^N}{(s+\lambda+\mu)^2-\lambda^2 z} \left(\frac{\lambda}{s+\lambda+\mu-u\mu}\right)^{2N} [(s+\lambda+\mu)+\lambda]$$

By the definitions of the z and L transforms one can write

$$f^*(s,u,z) = \sum_{i=N}^{\infty} \sum_{j=1}^{\infty} \int_0^{\infty} \exp(-st)\, dP\,\{T_{\xi_N} \le t,\ \xi_N = i,\ \eta_N = i\}\, dz^{ji}$$

We have proved that

$$f^*(s,u,z) = \frac{\mu u z^N}{(s+\lambda+\mu)^2-\lambda^2 z} \left(\frac{\lambda}{s+\lambda+\mu-u\mu}\right)^{2N} [(s+\lambda+\mu)+\lambda]$$

So

$$\sum_{i=N}^{\infty} \sum_{j=1}^{\infty} \int_0^{\infty} \exp(-st)\, dP\,\{T_{\xi_N} \le t,\ \xi_N = j,\ \eta_N = i\}\, z^{ji}$$

$$= \frac{\mu u z^N}{(s+\lambda+\mu)^2-\lambda^2 z} \left(\frac{\lambda}{s+\lambda+\mu-u\mu}\right)^{2N} [(s+\lambda+\mu)+\lambda]$$

$$=\left(\frac{\mu u}{(s+\lambda+\mu)^2}1-\frac{\lambda^2 z}{(s+\lambda+\mu)^2}\right)^{-1}\left(\frac{\lambda}{(s+\lambda+\mu)^2}1-\frac{\mu u}{(s+\lambda+\mu)}\right)^{-1}[(s+\lambda+\mu)+\lambda]$$

$$=\frac{\mu u}{(s+\lambda+\mu)2}\sum_{k=0}^{\infty}\left(\frac{\lambda^2 z}{(s+\lambda+\mu)}\right)^k z^N\left(\frac{\lambda}{s+\lambda+\mu}\right)^{2N}\sum_{i=1}^{\infty}\binom{2N+l-1}{l}\left(\frac{\mu u}{s+\lambda+\mu}\right)^l[(s+\lambda+\mu)+\lambda]$$

$$=\left(\frac{\mu u}{s+\lambda+\mu}\right)\sum_{k=0}^{\infty}\left(\frac{\lambda}{s+\lambda+\mu}\right)^{2k} z^{N+K}\left(\frac{\lambda}{s+\lambda+\mu}\right)^{2N}\sum_{i=0}^{\infty}\binom{2N+i-1}{i}\left(\frac{\mu u}{s+\lambda+\mu}\right)\left(1+\frac{\lambda}{(s+\lambda+\mu)}\right)$$

$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{2N+l-1}{l}\left(\frac{\mu}{s+\lambda+\mu}\right)^{l+1}\left\{\left(\frac{\lambda}{s+\lambda+\mu}\right)^{2(k+N)}+\left(\frac{\lambda}{s+\lambda+\mu}\right)^{2(k+N)+1}\right\}u^{l+1}z^{N+k}$$

Let $i = N + k$ and $j = l + 1$ then we have

$$\sum_{i=N}^{\infty}\sum_{j=1}^{\infty}\int_0^{\infty}\exp(-st)\,dP\,\{T_{\xi_N}\le t,\ \xi_N=j,\ \eta_N=i\}\,dz^{ji}$$

$$=\sum_{i=N}^{\infty}\sum_{j=1}^{\infty}\binom{2N+i-2}{j-1}\left(\frac{\mu}{s+\lambda+\mu}\right)^j\left\{\left(\frac{\lambda}{s+\lambda+\mu}\right)^{2i}+\left(\frac{\lambda}{s+\lambda+\mu}\right)^{2l+1}\right\}u^j z^i$$

Comparing the coefficient of $u^j z^i$ and taking the inverse of L-transform we obtain the joint density function of T_{ξ_N}, ξ_N and η_N random variables

$$dP\{T_{\xi_N}\le t,\xi_N=j,\eta_N=i\}=\binom{2N+i-2}{j-1}\left(\frac{\lambda^{2i}\mu^j t^{2i+j-1}}{(2i+j-1)!}+\frac{\lambda^{2i+1}\mu^j t^{2i+j}}{(2i+j)!}\right)\times$$

$\exp\,(-(\lambda+\mu)\,t)\,dt.$

The distribution function can be obtained by integration, thus we complete the proof of theorem 3.2.2.

3. TWO DIMENSION MARGINAL DISTRIBUTIONS

Theorem 3.1

The joint probability generating and distribution functions of ξ_N and η_N random variables are;

a) $$f_1(z, u) = \frac{\mu u z^N}{(\lambda + \mu)^2 - \lambda^2 z}\left(\frac{\lambda}{s+\lambda-u\mu}\right)^{2N} [(\lambda + \mu) + \lambda] \quad (3.1)$$

and

$$P\{\xi_N=j, \eta_N = i\} = \binom{2N+i-2}{j-1}\left(\frac{\mu}{\lambda+\mu}\right)^i\left(\frac{\mu}{\lambda+\mu}\right)^{2i}$$

$$+ \binom{2N+i-2}{j-1}\left(\frac{\mu}{\lambda+\mu}\right)^j\left(\frac{\lambda}{\lambda+\mu}\right)^{2i+1} \quad (3.2)$$

b) The joint transform and joint distribution functions of T_{ξ_N} and ξ_N random variables are;

$$f_2^*(s, u) = \frac{\mu u}{(s + \lambda + \mu)^2 - \lambda^2}\left(\frac{\lambda}{s+\lambda+\mu-u\mu}\right)^{2N} [(s+\lambda+\mu)+\lambda] \quad (3.3)$$

and

$$P\{T_{\xi_N} \le t, \xi_N = j\} = \sum_{i=N}^{\infty}\binom{2N+i-2}{j-1}\left(\frac{\mu}{\lambda+\mu}\right)^i\left\{\left(\frac{\mu}{\lambda+\mu}\right)^{2i}\sum_{j=1}^{\infty}\frac{[(\lambda+\mu)x]^r}{r!} +\right.$$

$$\left.\left(\frac{\lambda}{\lambda+\mu}\right)^{2i+1}\sum_{r=2i+j+1}^{\infty}\frac{[(\lambda+\mu)x]^r}{r!}\right\} \exp(-(\lambda+\mu)x) \quad (3.4)$$

c) The joint transform and joint distribution functios of T_{ξ_N} and η_N random variables are

$$f_3^*(s, z) = \frac{\mu z^N}{(s + \lambda + \mu)^2 - \lambda^2 z}\left(\frac{\lambda}{s+\lambda}\right)^{2N} [(s+\lambda+\mu)+\lambda] \quad (3.5)$$

and

$$P\{T_{\xi_N} \le t, \eta_N = i\} = \left(\frac{\lambda}{\mu}\right)^{2i}\left\{\sum_{k=0}^{2N-1}\sum_{r=k+1}^{\infty}(-1)^k\binom{2(i-N)+k}{2(i-N)}\left(\frac{\mu}{\lambda}\right)^{k+1}\frac{(\lambda x)^r}{r!}\right.$$

$$\exp(-\lambda x) + \sum_{k=0}^{2(i-N)}\sum_{r=k+1}^{\infty}(-1)^{2N}\binom{2N+k-1}{2N-1}\left(\frac{\mu}{\lambda}\right)^{k+1}\frac{[(\lambda+\mu)x]^r}{r!}$$

$$\exp(-(\lambda+\mu)x) + \left(\frac{\lambda}{\mu}\right)^{2i+1}\left\{\sum_{k=0}^{2N}\sum_{r=k+1}^{\infty}(-1)^k\binom{2(i-N)+k}{2(i-N)}\left(\frac{\mu}{\lambda}\right)^{k+1}\frac{(\lambda x)^r}{r!}\times\right.$$

$$\exp(-\lambda x) + \sum_{k=0}^{2(i-N)} \sum_{r=k+1}^{\infty} (-1)^{2N} \binom{2N+k-1}{2N-1} \left(\frac{\mu}{\lambda}\right)^{k+1} \frac{[(\lambda+\mu)x]^r}{r!} \exp(-(\lambda+\mu)x) \Bigg\}$$

(3.6)

Proof

(a) Letting s close to 0^+ in Theorem (3.2.2) we can get

$$f_1(z,u) = \frac{\mu u^N}{(\lambda + \mu)^2 - \lambda^2 z} \left(\frac{\lambda}{\lambda+\mu-u\mu}\right)^{2N} [(\lambda + \mu) + \lambda)]$$

Making it in the series form and compare the coefficient of $u^j z^i$ we conclude the desired result.

b) Similarly letting z close to 1^- in Theorem 3.2.2 we obtain

$$f_2^*(s,u) = \frac{\mu u}{(s + \lambda + \mu)^2 - \lambda^2} \left(\frac{\lambda}{s + \lambda + \mu - u\mu}\right)^{2N} [(s+\lambda+\mu)+\lambda)]$$

By the same calculation procedure as theorem 3.2.2, comparing the coefficient of u^j we get the density function

$$dP\{T_{\xi_N} \le t, \xi_N = j\} = \sum_{i=N}^{\infty} \binom{2N+j-2}{j-1} \Bigg[\frac{\mu^j \lambda^{2i} t^{2i+j-1}}{(2i+j-1)!} \exp(-(\lambda+\mu)t) + \frac{\mu^j \lambda^{2i+1} t^{2i+j}}{(2i+j)!} \exp(-(\lambda+\mu)t) \Bigg]$$

Probability distribution function can be obtain by the integration, hence we complete the proof of theorem 3.1(b).

(c) Again letting u close to 1^- in Theorem 3.2(a) we can get

$$f_3^*(s,z) = \frac{\mu z^N}{(s + \lambda + \mu)^2 - \lambda^2 z} \left(\frac{\lambda}{s + \lambda}\right)^{2N} [(s+\lambda+\mu) + \lambda)]$$

Using same calculation procedure as in theorem 2.2, comparing the coefficient of z^i and taking inverse of L-transform we get the density function through integration we complete the proof of theorem 3.1(c).

4. ONE DIMENSION MARGINAL DISTRIBUTIONS AS PH-DISTRIBUTIONS

Theorem 4.1

Probability generating function and distribution functions of T_{ξ_N}, ξ_N and η_N random variables are

a) $$\phi^*_{T_{\xi_N}}(s) = \frac{\mu}{(s+\lambda+\mu)^2-\lambda^2}\left(\frac{\lambda}{s+\lambda}\right)^{2N}[(s+\lambda+\mu)+\lambda] \quad (4.1)$$

$$P\{T_{\xi_N} \le t\} = \left(\frac{\lambda}{\mu}\right)^{2i}\left\{\sum_{k=0}^{2N-1}\sum_{r=k+1}^{\infty}(-1)^k\binom{2(i-N)+k}{2(i-N)}\left(\frac{\mu}{\lambda}\right)^{k+1}\frac{(\lambda x)^r}{r!}\exp(-\lambda x)+\right.$$

$$\sum_{k=0}^{2(i-N)}\sum_{r=k+1}^{\infty}(-1)^{2N}\binom{2N+k-1}{2N-1}\left(\frac{\mu}{\lambda}\right)^{k+1}\frac{[(\lambda+\mu)x]^r}{r!}\exp(-(\lambda+\mu)x)+$$

$$\left(\frac{\lambda}{\mu}\right)^{2i+1}\left\{\sum_{k=0}^{2N}\sum_{r=k+1}^{\infty}(-1)^k\binom{2(i-N)+k+1}{2(i-N)+1}\left(\frac{\mu}{\lambda}\right)^{k+1}\frac{(\lambda x)^r}{r!}\exp(-\lambda x)+\right.$$

$$\sum_{k=0}^{2(i-N)+1}\sum_{r=k+1}^{\infty}(-1)^{2N}\binom{2N+k}{2N}\left(\frac{\mu}{\lambda}\right)^{k+1}\frac{[(\lambda+\mu)x]^r}{r!}\exp(-(\lambda+\mu)x). \quad (4.2)$$

b) $$G_{\xi_N}(u) = \frac{\mu z}{(\lambda+\mu)^2-\lambda^2}\left(\frac{\lambda}{\lambda+\mu-u\mu}\right)^{2N}[(\lambda+\mu)+\lambda)] \quad (4.3)$$

and

$$P\{\xi_N = j\} = \sum_{i=N}^{\infty}\binom{2N+j-2}{j-1}\left(\frac{\mu}{\lambda+\mu}\right)^j\left\{\left(\frac{\lambda}{\lambda+\mu}\right)^{2j}+\left(\frac{\lambda}{\lambda+\mu}\right)^{2j+1}\right\} \quad (4.4)$$

c) $$G_{\eta_N}(z) = \frac{\mu z^N}{(\lambda+\mu)^2-\lambda^2 z}[(\lambda+\mu)+\lambda)] \quad (4.5)$$

$$P\{\eta_N = i\} = \left(\frac{\mu}{\lambda+\mu}\right)\left\{\left(\frac{\lambda}{\lambda+\mu}\right)^{2(i-N)}+\left(\frac{\lambda}{\lambda+\mu}\right)^{2(i-N)+1}\right\} \quad (4.6)$$

Proof

a) Letting z and u close to 1^- in theorem 3.2.2 we get

$$\phi^*_{T_{\xi_N}}(s) = \frac{\mu}{(s+\lambda+\mu)^2-\lambda^2}\left(\frac{\lambda}{s+\lambda}\right)^{2N}[(s+\lambda+\mu)+\lambda]$$

$$= \frac{\mu}{(s+\lambda+\mu)^2}\frac{1}{1-\left(\frac{\lambda}{(s+\lambda+\mu)}\right)^2}\left(\frac{\lambda}{s+\lambda}\right)^{2N}[(s+\lambda+\mu)+\lambda]$$

$$= \left(\frac{\mu}{s+\lambda+\mu}\right)\sum_{k=0}^{\infty}\left(\frac{\lambda}{s+\lambda+\mu}\right)^{2k}\left(\frac{\lambda}{s+\lambda}\right)^{2N}\left\{1+\left(\frac{\lambda}{s+\lambda+\mu}\right)\right\}$$

$$= \sum_{k=0}^{\infty}\left\{\left(\frac{\lambda}{s+\lambda+\mu}\right)^{2k}+\left(\frac{\lambda}{s+\lambda+\mu}\right)^{2k+1}\right\}\left(\frac{\lambda}{s+\lambda}\right)^{2N}\left(\frac{\mu}{s+\lambda+\mu}\right)$$

Let $i = k + N$, and taking inverse of L-transform we have

$$dP\{T_{\xi_N}\le t\} = \left\{\sum_{j=1}^{2N}(-1)^{j-1}\binom{2(i-N)+j-1}{2(i-N)}\left(\frac{\lambda}{\mu}\right)^{2(i-N)+j}\mu\frac{(\lambda t)^{2N-j}}{(2N-j)!}\exp(-\lambda t)\right.$$

$$+\sum_{j=1}^{2(i-N)+1}(-1)^{2N}\binom{2N+j-2}{2N-1}\left(\frac{\lambda}{\mu}\right)^{2i}\left(\frac{\mu}{\lambda+\mu}\right)^{2(i-N)+1-j}\times$$

$$\mu\frac{[(\lambda+\mu)t]^{2(i-N)+1-j}}{(2(i-N)+1-j)!}\exp(-\lambda+\mu t)+\sum_{j=1}^{2N+1}(-1)^{j-1}\binom{2(i-N)+j}{2(i-N)+1}\times$$

$$\left(\frac{\lambda}{\mu}\right)^{2(i-N)+1-j}\mu\frac{(\lambda t)^{2N-j+1}}{(2N-j)!}\exp(-\lambda t)+\sum_{j=1}^{2(i-N)+2}(-1)^{2N+1}\binom{2N+j-1}{2N}\left(\frac{\lambda}{\mu}\right)^{2i}$$

$$\left.\left(\frac{\mu}{\lambda+\mu}\right)^{2(i-N)+2-j}\mu\frac{[(\lambda+\mu)t]^{2(i-N)+2-j}}{(2(i-N)+2-j)!}\exp(-(\lambda+\mu)t)\right\}$$

Through integration we complete the proof.

b) Similarly letting s close to 0^+ and z close to 1^- in theorem 2.2 we obtain the probability generating function

$$G_{\xi_N}(u) = \frac{\mu u}{(\lambda+\mu)^2-\lambda^2}\left(\frac{\lambda}{\lambda+\mu-u\mu}\right)^{2N}[(\lambda+\mu)+\lambda]$$

$$= \frac{\mu u}{(\lambda+\mu)^2}\frac{1}{1-\left(\frac{\lambda}{(\lambda+\mu)^2}\right)^2}\left\{\frac{\lambda}{\lambda+\mu}\frac{1}{1-\frac{\mu u}{\lambda+\mu}}\right\}^{2N}[(\lambda+\mu)+\lambda)]$$

$$\times\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\binom{2N+k-1}{k}\left(\frac{\mu u}{\lambda+\mu}\right)^{2(i+N)}\left(\frac{\mu u}{\lambda+\mu}\right)^{k+1}\left\{1+\frac{\lambda}{\lambda+\mu}\right\}$$

Let $i = l + N$ and $j = k + 1$, comparing the coefficient of u^j we prove theorem 4.1 (b).

c) Again letting s close to 0^+ and u close 1^- in theorem 2.2 we get the probability generating function of η_N random variable

$$G_{\eta_N}(z) = \frac{\mu u^N}{(\lambda + \mu)^2 - \lambda^2 z}[(\lambda + \mu) + \lambda]$$

$$= \frac{\mu}{(\lambda + \mu)^2} \frac{1}{1 - \left(\frac{\lambda^2 z}{(\lambda+\mu)^2}\right)^2} z^N [(\lambda+\mu)+\lambda)]$$

$$= \left(\frac{\mu}{\lambda+\mu}\right) \sum_{k=1}^{\infty} \left(\frac{\lambda}{\lambda+\mu}\right)^{2k} z^{k+N} \left\{1 + \frac{\lambda}{\lambda + \mu}\right\}$$

Let $i = k + N$ and comparing the coefficient of z we get

$$P\{\eta_N = i\} = \left(\frac{\mu}{\lambda+\mu}\right)\left\{\left(\frac{\lambda}{\lambda+\mu}\right)^{2(i-N)} + \left(\frac{\lambda}{\lambda+\mu}\right)^{2(i-N)+1}\right\}$$

Theorem 4.2

a) T_{ξ_N} is a $2(N + 1)$ order continuous PH-distribution with representation (α, T) where $\alpha = [[1, 0], 0, \ldots, 0]$, $\alpha_{2(N+1)+1} = 0$ and

$$T = \begin{pmatrix} \begin{pmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{pmatrix} & \begin{pmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{pmatrix} & \\ & & \\ & & \begin{pmatrix} -(\lambda+\mu) & \lambda \\ \lambda & -(\lambda+\mu) \end{pmatrix} \end{pmatrix}, \quad T^0 = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \begin{pmatrix} 0 \\ \mu \end{pmatrix} \end{pmatrix}$$

b) η_N is a $2(N + 1)$ order discrete PH-distributioon with representation (β, L) where $\beta = [[1, 0], 0, \ldots, 0]$, $\beta_{2(N+1)+1} = 0$ and

$$L = \begin{pmatrix} 0 & A & & & \\ & 0 & A & & \\ & & \ddots & \ddots & \\ & & & A & B \\ & & & & B \end{pmatrix}, \quad L^0 = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ AC \\ C \end{pmatrix},$$

where $A = \left(\begin{pmatrix} \lambda & -\lambda \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)$

$$B = \left(\begin{pmatrix} (\lambda+\mu) & -\lambda \\ 0 & (\lambda+\mu) \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right) \text{ and } C = \begin{pmatrix} (\lambda+\mu) & -\lambda \\ 0 & (\lambda+\mu) \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

c) ξ_N is a $2(N+1)$ order discrete PH-distribution with representation (γ, R) where $\gamma = [[1, 0], 0, \ldots, 0]$, $\gamma_{2(N+1)+1} = 0$ and

$$R = \begin{pmatrix} 0 & A & BA & B^2A & B^3A & \cdots & B^{N-1}A \\ 0 & A & BA & B^2A & B^3A & \cdots & B^{N-1}A \\ 0 & 0 & A & BA & B^2A & \cdots & B^{N-2}A \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & A \end{pmatrix},$$

$$R^0 = \begin{pmatrix} B^N[I-B]^{-1}C \\ B^N[I-B]^{-1}C \\ B^{N-1}[I-B]^{-1}C \\ \cdot \\ \cdot \\ B[I-B]^{-1}C \end{pmatrix}$$

where $A = \begin{pmatrix} (\lambda+\mu) & -\lambda \\ 0 & (\lambda+\mu) \end{pmatrix}^{-1} \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$, $B = \begin{pmatrix} (\lambda+\mu) & -\lambda \\ 0 & (\lambda+\mu) \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$

and $C = \begin{pmatrix} \lambda+\mu & -\lambda \\ 0 & \lambda+\mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$

Proof

a) For definition and property of PH-distribution see Ref. [4]. For continuous PH-distribution its L-transform is

$$\varphi^*_{T_{\xi_N}}(s) = \alpha_{2(N+1)+1} + \alpha[sI - T]^{-1}\, T^0$$

Let $K=[sI-T]$, $A=\begin{pmatrix} s+\lambda & -\lambda \\ 0 & s+\lambda \end{pmatrix}$, $B=\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$, $C=\begin{pmatrix} s+\lambda+\mu & -\lambda \\ -\lambda & s+\lambda+\mu \end{pmatrix}$, then

$$K^{-1} = \begin{pmatrix} A^{-1} & A^{-1}BA^{-1} & (A^{-1}B)^2A^{-1} & \cdots & (A^{-1}B)^N C^{-1} \\ 0 & A^{-1} & (-A^{-1}B)^2A^{-1} & \cdots & (-A^{-1}B)^{N-1}C^{-1} \\ 0 & 0 & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 \;\cdot\;\cdot & C^{-1} \end{pmatrix}$$

Thus

$$\varphi^*_{T_{\xi_N}}(s) = [[1, 0], 0, \ldots, 0]\, K^{-1}\, T^0$$

$$= [1, 0]\, [-A^{-1}B]^N\, C^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

Substituting the values of A, B and C we have

$$\varphi^*_{T_{\xi_N}}(s) = [1, 0] \left(\begin{pmatrix} s+\lambda & -\lambda \\ 0 & s+\lambda \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right)^N \begin{pmatrix} s+\lambda+\mu & -\lambda \\ -\lambda & s+\mu+\lambda \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$= [1, 0] \begin{pmatrix} \left(\dfrac{\lambda}{s+\lambda}\right)^{2N} & 0 \\ \left(\dfrac{\lambda}{s+\lambda}\right)^{2N-1} & 0 \end{pmatrix} \begin{pmatrix} \dfrac{(s+\lambda+\mu)}{(s+\lambda+\mu)^2-\lambda^2} & \dfrac{(s+\lambda+\mu)}{(s+\lambda+\mu)^2-\lambda^2} \\ \dfrac{\lambda}{(s+\lambda+\mu)^2-\lambda^2} & \dfrac{(s+\lambda+\mu)}{(s+\lambda+\mu)^2-\lambda^2} \end{pmatrix} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$= \frac{\mu}{(s + \lambda + \mu)^2 - \lambda^2} \left(\frac{\lambda}{s + \lambda}\right)^{2N} [(s + \lambda + \mu) + \lambda \qquad (4.7)$$

Obviously, this is the same as (4.1).

b) For discrete PH-dsitribution, its probability generating function is $G\eta_N(z) = \beta_{2(N+1)+1} + z\beta[I - zL]^{-1} L^0$.

On a similar plan, let $P = [I - zL]$, then we have

$$P^{-1} = \begin{pmatrix} I & zA & (zA)^2 & \ldots & (zA)^{N-1} & (zA)^N B(I-zB)^{-1} \\ 0 & I & zA & \ldots & (zA)^{N-2} & (zA)^{N-1} B(I-zB)^{-1} \\ \cdot & \cdot & \cdots & & \cdot & \cdot \\ \cdot & \cdot & \cdots & & \cdot & \cdot \\ \cdot & \cdot & \cdots & & I & zAB(I-zB)^{-1} \\ 0 & 0 & 0 & \ldots & \cdot & (I-zB)^{-1} \end{pmatrix}, \quad L^0 = e - Le = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ AC \\ C \end{pmatrix}$$

$$G_{\eta_N}(z) = \beta_{2(N+1)+1} + \beta z P^{-1} R^0$$

$$= [\,[1, 0], 0, 0, \ldots, 0]\, P^{-1} L^0$$

$$= [1, 0] z \left\{ (zA)^{N-1} A + (zA)^N B(I-B)^{-1} \right\} C$$

$$= [1, 0]\, [zA]^N\, [I - zB]^{-1}\, C$$

By putting the values of A, B and C we can get

$$= [1,0]\, z^N \left(\begin{pmatrix} \lambda & -\lambda \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right)^N \left(I - z \left(\begin{pmatrix} (\lambda+\mu) & -\lambda \\ 0 & (\lambda+\mu) \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right) \right)^{-1} \times$$

$$\begin{pmatrix} (\lambda+\mu) & -\lambda \\ 0 & (\lambda+\mu) \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$= [1,0]\, z^N \left(\begin{pmatrix} \lambda & -\lambda \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right)^N \begin{pmatrix} (\lambda+\mu) & -\lambda \\ 0 & (\lambda+\mu) \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$= [1,0]\, z^N \left(\begin{pmatrix} \lambda & -\lambda \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right)^N \begin{pmatrix} \lambda+\mu & -\lambda \\ -\lambda z & (\lambda+\mu) \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$= [1,0]\, z^N \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}^N \begin{pmatrix} \dfrac{\lambda+\mu}{(\lambda+\mu)^2-\lambda^2 z} & \dfrac{\lambda}{(\lambda+\mu)^2-\lambda^2 z} \\ \dfrac{\lambda}{(\lambda+\mu)^2-\lambda^2 z} & \dfrac{\lambda+\mu}{(\lambda+\mu)^2-\lambda^2 z} \end{pmatrix} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

$$= \mu z^N \left(\frac{\lambda+\mu}{(\lambda+\mu)^2-\lambda^2 z} + \frac{\lambda}{(\lambda+\mu)^2-\lambda^2 z} \right)$$

$$= \frac{\mu z^N}{(\lambda+\mu)^2-\lambda^2 z} \{(\lambda+\mu)+\lambda\} \qquad (4.8)$$

This is also same as (4.5).

c) For a discrete PH-distribution, its probability generating function is $G_{\xi_N}(u) = \gamma_{2(N+1)+1} + u\gamma\, [\mathrm{I} - uR]^{-1} R^0$

Let $Q = [I - uR]$, then

$$Q^{-1} = \begin{pmatrix} I & uA(I-uA)^{-1} & (I-uA)^{-1}BuA(I-uA)^{-1} & [(I-uA)^{-1}B]^2uA(I-uA)^{-1} & [(I-uA)^{-1}B]^3uA(I-uA)^{-1} & [(I-uA)^{-1}B]^{N-1}uA(I-uA)^{-1} \\ 0 & [I-uA(I-uA)^{-1}] & (I-uA)^{-1}BuA(I-uA)^{-1} & [(I-uA)^{-1}B]^2uA(I-uA)^{-1} & [(I-uA)^{-1}B]^3uA(I-uA)^{-1} & [(I-uA)^{-1}B]^{N-1}uA(I-uA)^{-1} \\ 0 & 0 & [I-uA(I-uA)^{-1}] & (I-uA)^{-1}BuA(I-uA)^{-1} & [(I-uA)^{-1}B]^2uA(I-uA)^{-1} & [(I-uA)^{-1}B]^{N-2}uA(I-uA)^{-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & [I-uA(I-uA)^{-1}] \end{pmatrix}, \quad R^0 = \begin{pmatrix} B^N(I-B)^{-1}C \\ B^N(I-B)^{-1}C \\ B^{N-1}(I-B)^{-1}C \\ \vdots \\ B(I-B)^{-1}C \end{pmatrix}$$

Hence

$G_{\xi_N}(u) = [\,[1, 0], 0, 0, ..., 0]\ uQ^{-1} R^0$

$= [1, 0]\, u\ \Big[B^N(I-B)^{-C} + uA(I-uA)^{-1}[B^N(I-B)^{-1}C] + (I-uA)^{-1} BuA(I-uA)^{-1}[B^{N-1}(I-B)^{-1}C] + [(I-uA)^{-1}B]^2 uA(I-uA)^{-1} [B^{N-2}(I-B)^{-1}C] + [(I-uA)^{-1}B]^3 uA (I-uA)^{-1} [B^{N-3}(I-B)^{-1}C] + ... + [(I-uA)^{-1}B]^{N-1}uA(I-uA)^{-1}[B(I-B)^{-1}C]\Big]$

$= [1, 0]\, u \Big[(I-uA)^{-1}B^N + (I-uA)^{-1}BuA(I-uA)^{-1}B^{N-1} + [(I-uA)^{-1}B]^2 uA(I-uA)^{-1} B^{N-2} + [(I-uA)^{-1}B]^3 uA(I-uA)^{-1}B^{N-3} + ... + [(I-uA)^{-1}B]^{N-1} uA(I-uA)^{-1}B\Big]\ (I-B)^{-1}C$

$= [1, 0]\, u \Big[[(I-uA)^{-1}B]^2 (I-uA)^{-1}B^{N-1} + [(I-uA)^{-1}B]^2 uA(I-uA)^{-1}B^{N-2} + [(I-uA)^{-1}B]^3 uA(I-uA)^{-1}B^{N-3} + ... + [(I-uA)^{-1}B]^{N-1} uA(I-uA)^{-1}B\Big]\ (I-B)^{-1}C$

$= [1, 0]\, u \Big[[(I-uA)^{-1}B]^2 (I-uA)^{-1}B^{N-2} + [(I-uA)^{-1}B]^3 uA(I-uA)^{-1}B^{N-3} + ... + [(I-uA)^{-1}B]^{N-1}uA(I-uA)^{-1}B\Big]\ (I-B)^{-1}C$

$G_{\xi_N} = [0, 1]\, u\, [(I - uA)^{-1} B]^N (I - B)^{-1} C$

Substituting the values of A, B and C into the last expression, we obtain

$$G_{\xi_N}(u) = [1,0]u\left(I-u\begin{pmatrix}(\lambda+\mu) & -\lambda\\ 0 & (\lambda+\mu)\end{pmatrix}^{-1}\begin{pmatrix}\mu & 0\\ \mu & 0\end{pmatrix}\right)^{-1}\begin{pmatrix}(\lambda+\mu) & -\lambda\\ 0 & (\lambda+\mu)\end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ \lambda & 0\end{pmatrix}^{N}$$

$$\left(I-\begin{pmatrix}(\lambda+\mu) & -\lambda\\ 0 & (\lambda+\mu)\end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ \lambda & 0\end{pmatrix}\right)^{-1}\begin{pmatrix}\lambda+\mu & -\lambda\\ 0 & \lambda+\mu\end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

$$= u[1, 0]\left(\begin{pmatrix}\lambda+\mu-u\mu & -\lambda\\ 0 & \lambda+\mu-u\mu\end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ \mu & 0\end{pmatrix}\right)^{N}\begin{pmatrix}\lambda+\mu & -\lambda\\ -\lambda & \lambda+\mu\end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

$$= u\ [1, 0]\begin{pmatrix}\left(\dfrac{\lambda}{\lambda+\mu-u\mu}\right)^{2N} & 0\\ \left(\dfrac{\lambda}{\lambda+\mu-u\mu}\right)^{2N-1} & 0\end{pmatrix}\begin{pmatrix}\dfrac{(\lambda+\mu)}{(\lambda+\mu)^2-\lambda^2} & \dfrac{\lambda}{(\lambda+\mu)^2-\lambda^2}\\ \dfrac{\lambda}{(\lambda+\mu)^2-\lambda^2} & \dfrac{(\lambda+\mu)}{(\lambda+\mu)^2-\lambda^2}\end{pmatrix}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

$$= u/u\left(\left(\frac{\lambda}{\lambda+\mu-u\mu}\right)^{2N}\frac{(\lambda+\mu)}{(\lambda+\mu)^2-\lambda^2}+\left(\frac{\lambda}{\lambda+\mu-u\mu}\right)^{2N}\frac{(\lambda+\mu)}{(\lambda+\mu)^2-\lambda^2}\right)$$

$$= \frac{\mu u}{(\lambda+\mu)^2-\lambda^2}\left(\frac{\lambda}{\lambda+\mu-u\mu}\right)^{2N}\{(\lambda+\mu)+\lambda\} \qquad (4.9)$$

This is also same as eq. (4.3), and hence we compelte the proof of the theorem 4.2.

REFERENCE

1. Talpur G.H. Mir and D.H. Shi, Some joint distribution of crossing time and renewal numbers related with two Poisson processes, *J. of Shanghai Univ. of Sci. & Tech.* 1994.
2. Talpur G.H.Mir and D.H. Shi Some joint distributions of crossing time and renewal numbers related with two different processes *J. of Shanghai Univ. of Sci & Tech.* 1994.
3. Shi D.H., Two counting processes in a Markoov chain with continuous time, (in chinese) Chinese *J. of App. Prob. & Stats*. 1994.
4. Neuts, M.F. Matrix-Geometric Solutions in Stochastic Models An algorthmic Approach, The John Hopkins University Press 1981.

Published by Chairman Department of Mathematics for the University of the Punjab, Lahore-Pakistan.

Printed at
Al-Hijaz Printers
Darbar market, Lahore,

Composed at
Combine Graphics
1-Turner Road, Lahore
Ph: 7244043

CONTENTS