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Local Convergence of Newton's Method Under a Weak Gamma Condition

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Abstract. We provide a local convergence analysis of Newton's method under a weak gamma condition on a Banach space setting. It turns out that under the same computational cost and weaker hypotheses than in [4], [5], [7], we can obtain a larger radius of convergence and finer estimates on the distances involved.

AMS (MOS) Subject Classification Codes: 65G99, 65B05, 47H17, 49M15.
Key Words: Local convergence, Newton's method, Banach space, radius of convergence, gamma condition, Newton–Kantorovich condition, Smale's analyticity.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1. 1}$$

where, F is a Fréchet-differentiable operator defined on a closed ball $\overline{U}(y, R)$ $(R > 0)$ which is a subset of a Banach space X with values in a Banach space Y. The most popular method for generating a sequence $\{x_n\}$ $(n \geq 0)$ approximating x^* is undoubtedly Newton's method given by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0) \quad (x_0 \in \overline{U}(y, R)). \tag{1. 2}$$

Local as well as semilocal convergence results for Newton's method have been provided by many authors under various assumptions. Historically Newton–Kantorovich type assumptions under Lipschitz continuity conditions on a domain D including $\overline{U}(y, R)$ have been used first to show convergence [1]–[3], [6] followed by Smale conditions using information only at a point [4], [5].

Here in particular we are motivated by the elegant work by Han and Wang [5], which follows Smale's theory, where they used the gamma condition (see Definition 2.1 that follows) to find the radius of convergence for Newton's method in this case.

It turns out that under the same computational cost and weaker hypotheses than before [4], [5], [7], we can obtain a larger radius of convergence and finer error estimates on the distances involved.

We also give a comparison between gamma-type results and the Kantorovich's. In particular we provide the connection between the gamma and Lipschitz parameters. Finally we show that under the Kantorovich hypotheses the radius of convergence is always larger than the one using the gamma hypotheses.

2. Local Convergence Analysis of Newton's Method

We need the definition of a gamma (γ_0, γ) condition at a point:

Definition 2.1. Let R, γ_0, and γ be positive parameters with $\gamma_0 \leq \gamma$. Let $x^* \in D$ with $\overline{U}(x^*, R) \subseteq D$. We say F satisfies the (γ_0, γ) condition at y if the following hold true for all $x \in \overline{U}(x^*, R)$ and all $t \in [0,1]$:

$$F'(x^*)^{-1} \in L(Y, X), \tag{2. 1}$$

$$\|F'(x^*)^{-1}[F'(x) - F'(y)]\| \leq \frac{1}{(1-\gamma_0\|x - x^*\|)^2} - 1 \tag{2. 2}$$

and

$$\|F'(x^*)^{-1}[F'(x^t) - F'(x)]\| \leq \frac{2\gamma\|x^t - x\|}{(1-\gamma\|x^t - x\|)^3}, \tag{2. 3}$$

where,

$$x^t = x + t(x^* - x). \tag{2. 4}$$

By the definition of γ_0 and γ there exists $a \in [0,1]$ such that $\gamma_0 = a\gamma$. Set $r_0 = \gamma_0\|x - x^*\|$, and $r = \gamma\|x - x^*\|$. Then we get $r_0 = ar$. It is convenient to define scalar functions $f(a,r)$, $g(a,r)$ and $h(a,r)$ on $[0,1]^2$ by

$$f(a,r) = 2a^2r^4 - (5a+4)ar^3 + (2a^2+10a+1)r^2 - (4a+3)r + 1, \tag{2. 5}$$

$$g(a,r) = 2a^2r^2 - 4ar + 1, \tag{2. 6}$$

and

$$h(a,r) = \left(\frac{1-ar}{1-r}\right)^2 \frac{r}{g(a,r)} \tag{2. 7}$$

provided that $r \neq 1$ and $g(a,r) \neq 0$.

For each fixed $a \in [0,1]$ function f has a zero $r^a \in (0,1)$ since $f(a,0)f(a,1) = -(a-1)^2 \leq 0$. We use the same symbol r^a for the minimal zero of f in $[0,1)$. It is also simple algebra to show for all $a \in [0,1)$, $r \in (0,1]$

$$f(a,r) > f(1,r), \tag{2. 8}$$

$$g(a,r) > g(1,r), \tag{2. 9}$$

$$h(a,r) < h(1,r), \tag{2. 10}$$

and

$$h(a,r) < 1 \quad \text{for all } r \in [0, r^a]. \tag{2. 11}$$

Indeed by (2.5)

$$f(a,r) - f(1,r) = [2(a+1)r^3 - (5a+9)r^2 + 2(a+6)r - 4](a-1)r,$$

which is clearly positive since the quantity in the bracket is negative. In view of (2.6)

$$g(a,r) - g(1,r) = 2[(a+1)r - 2](a-1)r,$$

which is positive. The difference $h(a,r) - h(1,r)$ will be negative if

$$\frac{(1-ar)^2}{g(a,r)} < \frac{(1-r)^2}{g(1,r)}$$

or if $(1+a)r < 2$ which is true. Finally (2.11) will be true if $f(a,r) < 0$ which holds for all $r \in [0, r^a]$. Note that for $a = 1$ estimates (2.8)–(2.10) hold as equalities. The estimate $f(a, r^1) > f_1(1, r^1) = 0$ leads to

$$r^1 < r^a \quad \text{for all } a \neq 1. \tag{2. 12}$$

We shall show below that r^1, r^a are the convergence radii for Newton's method (1.2) under Definition 2.1.

Theorem 2.2. *Let $x^* \in D$ be a simple zero of F, and let F satisfy the (γ_0, γ) condition at x^*. Then sequence $\{x_n\}$ $(n \geq 0)$ generated by Newton's method (1.2) is well defined, remains in $\overline{U}(x^*, \frac{r^a}{\gamma_0})$ for all $n \geq 0$ and converges to x^* provided that $x_0 \in U(x^*, \frac{r^a}{\gamma_0})$, and*

$$r^a \leq \left(1 - \frac{\sqrt{2}}{2}\right) \frac{1}{\gamma_0} \leq R.$$

Moreover the following estimate hold for all $n \geq 0$

$$\|x_n - x^*\| \leq h^{2^n - 1} \|x_0 - x^*\|, \tag{2. 13}$$

where,

$$h = h(a, r^a).$$

Proof. By hypothesis $x_0 \in U(x^*, \frac{r^a}{\gamma_0}) \subseteq \overline{U}(x^*, \frac{r^a}{\gamma_0})$. Let $x \in \overline{U}(x^*, \frac{r^a}{\gamma_0})$. Using (2.2) and the choice of r^a we get

$$\begin{aligned} \|F'(x^*)^{-1}[F'(x) - F'(x^*)]\| &\leq \frac{1}{(1 - \gamma_0\|x - x^*\|)^2} - 1 \\ &\leq \frac{1}{(1 - \gamma_0 r^a)^2} - 1 < 1. \end{aligned} \tag{2. 14}$$

It follows from (2.14) and the Banach Lemma on invertible operators [6] that $F'(x)^{-1} \in L(Y, X)$ and

$$\|F'(x)^{-1}F'(x^*)\| \leq \frac{(1 - ar^a)^2}{g(a, r^a)}. \tag{2. 15}$$

Let us assume $x_k \in \overline{U}(x^*, \frac{r^a}{\gamma_0})$ for $k = 0, 1, \ldots, n$. Using (1.2) we obtain the identity

$$\begin{aligned} x_{k+1} - x^* &= x_k - F'(x_k)^{-1}F(x_k) - x^* \\ &= F'(x_k)^{-1}[F(x^*) - F(x_k) - F'(x_k)(y - x_k)] \qquad (2.\ 16) \\ &= [F'(x_k)^{-1}F'(x^*)]F'(x^*)^{-1} \int_0^1 [F'(x^t) - F'(x_k)](x^* - x_k)dt. \end{aligned}$$

In view of (2.3), (2.7), (2.15), and (2.16) we get

$$\begin{aligned} \|x_{k+1} - x^*\| &< \frac{(1 - ar^a)^2}{g(a, r^a)} \int_0^1 \frac{2\gamma t}{(1 - \gamma t)^3} \|x_k - x^*\|^2 = h(a, r^a)\|x_k - x^*\| \\ &= h\|x_k - x^*\| < \|x_k - x^*\|, \end{aligned} \tag{2. 17}$$

which shows (2.13), $x_{k+1} \in \overline{U}(x^*, \frac{r^a}{\gamma_0})$ and $\lim_{k \to \infty} x_k = x^*$.

That completes the proof of Theorem 2.2.

Remark 2.3. If $a = 1$, i.e. if $\gamma_0 = \gamma$ and (2.2) and (2.3) are replaced by the stronger condition

$$\|F'(x^*)^{-1}F''(x)\| \leq \frac{2\gamma}{(1-\gamma\|x-x^*\|)^3} \tag{2. 18}$$

for all $x \in \overline{U}(x^*, R)$, then Theorem 2.2 reduces to Theorem 2.1 in [5, p. 98]. Otherwise due to (2.10) and (2.12) our theorem is an improvement since it provides a larger radius of convergence and a smaller ration h than Theorem 2.1 in [5]. Note that

$$r^1 = \frac{5-\sqrt{17}}{4\gamma} \tag{2. 19}$$

was found in [5].

We now investigate conditions for x_0 being an approximate zero of the adjoint zero x^*. Let us define scalar function $\overline{f}$ on $[0,1]^2$ by

$$\overline{f}(a,r) = 2a^2r^4 - 2(3a+2)ar^3 + (2a^2+12a+1)r^2 - 4(1+a)r + 1. \tag{2. 20}$$

As above for each fixed $a \in [0,1]$ function $\overline{f}$ has a zero $\overline{r}^a \in (0,1)$, since $\overline{f}(a,0)\overline{f}(a,1) = -2(a-1)^2 \leq 0$. We use the same $\overline{r}^a$ for the minimal zero of $\overline{f}$ in $(0,1)$. For x_0 to be an approximate zero we must have

$$h(a,r) \leq \frac{1}{2}. \tag{2. 21}$$

which holds if

$$\overline{f}(a,r) \geq 0 \tag{2. 22}$$

which is true for all $r \in [0, \overline{r}^a]$.

We also have that for all $a \in [0,1)$

$$\overline{f}(a,r) > \overline{f}(a,1) \tag{2. 23}$$

which leads as in (2.13) leads to

$$\overline{r}^1 < \overline{r}^a. \tag{2. 24}$$

The implications of (2.25) are the same as (2.12) above. Note that in [5] they found

$$\overline{r}^1 = \frac{3-\sqrt{7}}{2} < r^1. \tag{2. 25}$$

With (2.3) replacing (2.19) in Theorem 2.3 in [5, p. 100] we obtain the improvement:

Proposition 2.4. *Operator F has a unique zero x^* in $\overline{U}(x^*, R_0)$ provided that*

$$R_0 < \frac{1}{2\gamma_0}. \tag{2. 26}$$

Moreover for any other solution $\overline{x}^*$ of F

$$\|x^* - \overline{x}^*\| \geq \frac{1}{2\gamma_0}. \tag{2. 27}$$

Remark 2.5. We already noted that condition (2.3) is weaker than (2.19). If $\gamma_0 = \gamma$ Proposition 2.4 reduces to Theorem 2.3 in [5]. Otherwise it is an improvement. Estimate (2.28) further improves the corresponding result by J.P. Dedieu [4, Ch. 8]

$$\|x^* - \overline{x}^*\| \geq \frac{5-\sqrt{17}}{4\gamma}. \tag{2. 28}$$

We provide an example to show how we choose γ_0, γ so that (2.2), (2.3) are satisfied, and γ_0 can be smaller than γ.

Example 2.5. Let $X = Y = \mathbf{R}$, $R = 1$, $x^* = 0$ and define function F on $\overline{U}(0,1)$ by

$$F(x) = e^x - 1. \tag{2. 29}$$

Using (2.30) we get that

$$|F'(x^*)^{-1}[F'(x) - F'(x^*)]| = \ell_0\|x - x^*\|, \quad \ell_0 = e - 1 \tag{2. 30}$$

and

$$\|F'(x^*)^{-1}[F'(x^t) - F'(x)]\| \leq \ell\|x^t - x\|, \quad \ell = e. \tag{2. 31}$$

Set:

$$\gamma_0 = \frac{e-1}{2}, \text{ and } \gamma = \frac{e}{2}. \tag{2. 32}$$

It can then easily be seen that with the above choices of γ_0 and γ conditions (2.2), (2.3) are both satisfied, and $\gamma_0 < \gamma$.

In order for us to compare parameters γ_0, γ appearing in conditions (2.2) and (2.3) with the Lipschitz constants appearing in the Kantorovich theory [2], [3], [6], let us assume:

there exist parameters $\ell_0 > 0$, $\ell > 0$ such that for all $x \in \overline{U}(x^*, \frac{1}{\ell_0})$, $t \in [0,1]$

$$\|F'(x^*)^{-1}[F'(x) - F'(x^*)]\| \leq \ell_0\|x - x^*\|, \tag{2. 33}$$

and

$$\|F'(x^*)^{-1}[F'(x^t) - F'(x)]\| \leq \ell\|x^t - x\|. \tag{2. 34}$$

Then we will find out how to choose γ_0 and γ so that (2.34) and (2.35) imply (2.2) and (2.3). Set $x - x^* = r$. It follows from (2.1), (2.2), (2.34) and (2.35) that

$$\ell_0 r \leq \frac{1}{(1-\gamma_0 r)^2} - 1, \tag{2. 35}$$

and

$$\ell \leq \frac{2\gamma}{(1-\gamma r)^3} \tag{2. 36}$$

should hold true for all $r \in \left[0, \min\{\frac{2-\sqrt{2}}{2\gamma_0}, \frac{1}{\ell_0}\}\right]$. By solving system (2.36)–(2.37) we conclude

$$\ell \leq 2\gamma, \tag{2. 37}$$

$$\ell_0 \leq 2\gamma_0, \tag{2. 38}$$

and

$$0 \leq r \leq \frac{2-\sqrt{2}}{2\gamma_0}. \tag{2. 39}$$

Therefore parameters γ_0 and γ should be chosen so they will satisfy (2.38) and (2.39). A possible choice is obviously

$$\gamma_0 = \frac{\ell_0}{2} \tag{2. 40}$$

and

$$\gamma = \frac{\ell}{2}. \tag{2. 41}$$

Note that in this case

$$\gamma_0 \leq \gamma, \tag{2. 42}$$

since

$$\ell_0 \leq \ell \tag{2. 43}$$

holds in general. The ratio $\frac{\ell}{\ell_0}$ (i.e. the ratio $\frac{\gamma}{\gamma_0}$) can also be arbitrarily large [2], [3]. Clearly, if strict inequality holds in (2.44) so does in (2.43) for the choices of γ_0 and γ given by (2.41), (2.42), respectively.

Let us also consider the converse problem. In this case (2.2) and (2.3) hold true and we would like to know how to choose ℓ_0, ℓ in terms of γ_0 and γ. This time we solve "complementary" inequalities with "$\leq$" replaced by "$\geq$" to obtain

$$\ell_0 \geq 2\gamma_0, \tag{2. 44}$$

$$\ell \geq 2\gamma, \tag{2. 45}$$

and

$$0 \leq r \leq \min\left\{\frac{2-\sqrt{2}}{2\gamma_0}, \frac{1}{\gamma}\left(1 - \sqrt[3]{\frac{2\gamma}{\ell}}\right)\right\}. \tag{2. 46}$$

Note that $\frac{2-\sqrt{2}}{2\gamma_0}$ is used to secure that 1 is a strict upper bound in (2.2). In view of (2.36) (with $\leq$ replaced by $\geq$ 1) $\frac{2-\sqrt{2}}{2\gamma_0}$ can be replaced by $\frac{1}{\ell_0}$. That is (2.47) can be replaced by

$$0 \leq r \leq \min\left\{\frac{1}{\ell_0}, \frac{1}{\gamma}\left(1 - \sqrt[3]{\frac{2\gamma}{\ell}}\right)\right\}, \tag{2. 47}$$

or by

$$0 \leq r \leq \min\left\{\min\left\{\frac{1}{\ell_0}, \frac{2-\sqrt{2}}{2\gamma_0}\right\}, \frac{1}{\gamma}\left(1 - \sqrt[3]{\frac{2\gamma}{\ell}}\right)\right\}. \tag{2. 48}$$

Rheinboldt's radius of convergence [2], [3] under condition (2.35) is given by

$$r_R = \frac{2}{3\ell}. \tag{2. 49}$$

The radius of convergence given by us [2], [3] using (2.34) and (2.35) is

$$r_A = \frac{2}{2\ell_0 + \ell}. \tag{2. 50}$$

Note that

$$r_R < r_A \tag{2. 51}$$

unless if $\ell_0 = \ell$.

Returning back to Example 2.5 we obtain

$$r_{HW} = .161295704, \quad \text{(see (2.20))} \tag{2. 52}$$

$$r_R = .245252961 \tag{2. 53}$$

and

$$r_A = .324947231. \tag{2. 54}$$

That is our radius of convergence r_A is larger than Han-Wang's r_{HW}, and Rheinboldt's r_R. Using the most favorable (see (2.38)) choice $\gamma = \frac{\ell}{2}$ for the enlargement of radius r_{HW}, we get

$$r_{HW} = \frac{5-\sqrt{17}}{2\ell} = \frac{.438447187}{\ell} < r_R. \tag{2. 55}$$

Hence, under Kantorovich's conditions (2.33) and (2.34), we have:

$$r_{HW} < r_R \leq r_A, \tag{2. 56}$$

with strict double inequality holding in (2.56) if $\ell_0 < \ell$.

Note that even in the case when (2.40) must hold still (2.56) is true for

$$r_R = r_A = \frac{2-\sqrt{2}}{2\gamma_0}, \tag{2. 57}$$

since $\frac{2-\sqrt{2}}{2\gamma_0} > r_{HW}$.

In the case of Example 2.5 we must choose r_R, r_A not given by (2.55) and (2.56) but instead set

$$r_R = r_A = .170457031. \tag{2. 58}$$

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Some Notes On An Integral Inequality Related To G.H. Hardy'S Integral Inequality

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Abstract. We obtain an extension of Hardy inequality for convex functions, which is a special case of Boas's version of the Hardy's integral inequality.

1. Introduction

Hardy in an attempt to simplify Hilbert's integral inequality ([5], Theorem 316) discovered the following result:

Theorem A *If* $p > 1, f(x) \geq 0$ *and* $F(x) = \int_0^x f(t)d(t)$*, then*

$$\int_0^\infty (\frac{F}{x})^p dx < (\frac{p}{p-1})^p \int_0^\infty f(x)^p dx \qquad (1.\ 1)$$

holds,unless $f \equiv 0$. The constant $(\frac{p}{p-1})^p$ is the best possible.

This result is called the Hardy's inequality (see, for example [4] and [5], Theorem 327).

Another inequality due to Hardy ([5], Theorem 328) was given by using the converse of Holder's inequality as follows:

Theorem B *If* $p > 1, f(x) \geq 0$ *and* $F(x) = \int_x^\infty f(t)d(t)$*, then*

$$\int_0^\infty F^p dx < p^p \int_0^\infty (xf)^p dx \qquad (1.\ 2)$$

holds, unless $f \equiv 0$. The constant p^p is the best possible. Since, the inequality

has wide applications in analysis. A number of researchers have developed interest in the results and a lot of effort and time have been expended in the study and extension of the inequality in various directions (see for example [5] ,chapter ix). Of particular interest is the work due to [2]. The main purpose of this paper, therefore, is to study and establish some new inequalities from the Jensen inequality for convex functions. Indeed, our main result is an extension of Boas's version of the Hardy's inequality where the variable x additionally is a linear function of two further variables, says u and v, and then obtains an estimate of the integral of the double integral of f with respect to the variables u and v. Special cases of our result yield some of the earlier, as well as some recent generalization of Hardy's inequality given in [[1], [6], [7], and [9] - [12]] and on convex function [[3] and [8]]. The result is as follows:

Theorem 1. *Let $g(x)$ be continuous and non-decreasing on$[0,\infty]$ with $g(0)=0$, $g(x)>0$, $g(\infty)=\infty.\sum_{i=1}^{n}u_i=u$ and $\sum_{i=1}^{n}v_i=v$ where x,u and v are all positive. Also, let $f:[a,b]\to\Re(a<b)$, be continuous and convex on the real interval $[a,b]$.Assume $\prod_{i=1}^{n}\alpha_i\beta_i\geq 0$ with $\sum_{i=1}^{n}(\alpha+\beta)>0$ for all $i\in\aleph$.*
Then, the following inequality holds:

$$\int_0^\infty g(x)^{-1}[\int_a^b\int_a^b f(\sum_{i=1}^{n}(\alpha_i u_i+\beta_i v_i))u^{\alpha-1}dudv]^p dg(x)\leq L^p\int_0^\infty G(x)dg(x) \tag{1. 3}$$

where, $L=(\alpha^{-1}\beta)(b-a)(b^\alpha-a^\alpha)(k+1)$ and $G(x)=f(x)^p g(x)^{-1}$.

We note, however, that the left sides of (1.1), (1.2) and (1.3) exist when the right sides do.

2. PRELIMINARY LEMMAS

We shall prove the following lemma which is a simple consequence of the Jensen inequality proved by standard methods.

Lemma 2.1 *If Φ is convex and continuous, f is a non-negative and λ is non-decreasing on $[0,\infty]$. Then,*

$$\int_0^\infty g(x)^{-1}\Phi[L^{-1}\int_0^\infty f(v)d\lambda(v)]dg(x)\leq\int_0^\infty g(x)^{-1}\Phi(f(x))dg(x) \tag{2. 1}$$

where, $L=\int_0^\infty d\lambda(v)$ with the inequality reversed when Φ is concave.

Proof. Let Φ be convex, then Jensen's inequality says

$$\Phi(\int_0^\infty f(v)d\lambda(v))\leq\int_0^\infty\Phi f(v)d\lambda(v)$$

Hence,

$$\begin{aligned}\int_0^\infty g(x)^{-1}\Phi[L^{-1}\int_0^\infty f(v)d\lambda(v)]dg(x) &\leq L^{-1}\int_0^\infty g(x)^{-1}dg(x)\int_0^\infty\Phi(f(v))d\lambda(v)\\ &= L^{-1}\int_0^\infty d\lambda(v)\int_0^\infty g(x)^{-1}\Phi(f(x))dg(x)\\ &= L^{-1}L\int_0^\infty g(x)^{-1}\Phi(f(x))dg(x)\\ &= \int_0^\infty g(x)^{-1}\Phi(f(x))dg(x)\end{aligned}$$

Similar result was obtained by replacing $g(x)$ by x in [2] and this proves the lemma when Φ is convex. □

In order to obtain the next result, we shall make use of lemma 2.1 and our method of proof is simply one step of integration by parts.

Corollary 2.1 *If $p > 1$, $f \geq 0$, g is continuous, non-decreasing on $[0, \infty)$. Let Φ be continuous and convex and suppose Φ has a continuous inverse (which is necessarily concave) on $[0, \infty)$ and $d\lambda(v)$ be defined as $v^{\alpha-1}dv$ on $[0,1]$ and 0 for $v > 1, \alpha > 0$. Then,*

$$\int_0^\infty g(x)^{-1}\Phi[\int_0^1 f(v)v^{\alpha-1}dv]^p dg(x) \leq \alpha^{-p}\int_0^\infty g(x)^{-1}\Phi f(x)^p dg(x) \qquad (2.\ 2)$$

Proof. Let

$$I = \int_0^\infty g(x)^{-1}\Phi[\int_0^1 f(v)v^{\alpha-1}dv]^p dg(x)$$

Then, integrating by part of the inner integral yields

$$\begin{aligned} & \int_0^\infty g(x)^{-1}\Phi[\left[\frac{f(v)v^\alpha}{\alpha}\right]_0^1 - \int_0^1 \frac{v^\alpha}{\alpha} f^{'}(v)dv]^p dg(x) \\ = & \int_0^\infty g(x)^{-1}\Phi(\alpha^{-1}f(v))^p dg(x) - [Non-negativeterm] \\ \leq & \alpha^{-p}\int_0^\infty g(x)^{-1}\Phi f(v)^p dg(x). \end{aligned} \qquad (2.\ 3)$$

this completes the proof of the corollary. □

Remark 2. If $v^{\alpha-1} = L^{-1}, \alpha = p = 1$ and dv be defined as $d\lambda(v)$ on $[0, x]$, then we get lemma 2.1.
Also, if $\alpha = 1 - \frac{1}{p}$ replace $g(x)$ by x on $[0, \infty]$ and $\Phi = v = 1$, then we get (1.1).
Similarly, we get (1.2) by letting $\alpha = p, g(x) = x, (x^{-1}\Phi) = x^p$ and $v^{\alpha-1} = x^{-1}$ on $[x, \infty]$.

3. Proof of Theorem 1

The method of proof of this theorem is induction by means of partial integration. From inequality (3), we obtain

$$I = \int_0^\infty g(x)^{-1}[\int_a^b \int_a^b f(\sum_{i=1}^n(\alpha_i u_i + \beta_i v_i))u^{\alpha-1}dudv]^p dg(x)$$

Also, we obtain on using integration by part of the inner integral

$$I \leq \alpha^{-p}(b^\alpha - a^\alpha)^p \int_0^\infty g(x)^{-1}[\int_a^b f(\sum_{i=1}^n(\alpha_i(b_i - a_i) + \beta_i v_i))dv]^p dg(x)$$

Let $i = 1$ then,

$$I \leq \alpha^{-p}(b^\alpha - a^\alpha)^p \int_0^\infty g(x)^{-1}[\int_a^b f(\alpha_1(b_1 - a_1) + \beta_1 v_1)dv]^p dg(x)$$

Assume the theorem for $i = k > 1$, we have

$$I \leq \int_0^\infty g(x)^{-1}[\alpha^{-1}(b^\alpha - a^\alpha)\int_a^b f(\sum_{i=1}^{k}(\alpha_i(b_i - a_i) + \beta_i v_i))dv]^p dg(x)$$

Then, for $i = k+1$, we have

$$I \leq \int_0^\infty g(x)^{-1}[\alpha^{-1}(b^\alpha - a^\alpha)\int_a^b f(\sum_{i=1}^{k+1}(\alpha_i(b_i - a_i) + \beta_i v_i))dv]^p dg(x)$$

To see this, we note that f is a continuous and convex function on a real interval $[a, b]$.
Then,

$$\begin{aligned} f(\sum_{i=1}^{k+1}(\alpha_i(b_i - a_i) + \beta_i v_i)) &= f(\sum_{i=1}^{k}(\alpha_i(b_i - a_i) + \beta_i v_i) + \alpha_{k+1}(b_{k+1} - a_{k+1}) \\ &\quad + \beta_{k+1}v_{k+1}) \\ &\leq (\sum_{i=1}^{k}(\alpha_i(b_i - a_i) + \beta_i f(v_i)) + \alpha_{k+1}(b_{k+1} - a_{k+1}) \\ &\quad + \beta_{k+1} f(v_{k+1})) \qquad (3.\ 1) \end{aligned}$$

Integrating both sides of (3.1) on the $(k+1)$ rectangles $[a,b]X[a,b]X...X[a,b]$ from a to b
We then have,

$$\begin{aligned} \int_a^b [f(\sum_{i=1}^{k+1}(\alpha_i(b_i - a_i) + \beta_i v_i))]dv &\leq \int_a^b [(\sum_{i=1}^{k}(\alpha_i(b_i - a_i) + \beta_i f(v_i) \\ &\quad + \alpha_{k+1}(b_{k+1} - a_{k+1}) + \beta_{k+1} f(v_{k+1})))]dv \\ &= \int_a^b [\sum_{i=1}^{k}(\alpha_i(b_i - a_i) + \beta_i f(v_i))]dv \\ &\quad + \int_a^b [(\alpha_{k+1}(b_{k+1} - a_{k+1}) + \beta_{k+1} f(v_{k+1}))]dv \\ &= k(b-a)\beta f + (b-a)\beta f \\ &= (b-a)(k+1)\beta f \end{aligned}$$

Therefore,

$$\begin{aligned} I &\leq [(\alpha^{-1}\beta)(b-a)(b^\alpha - a^\alpha)(k+1)]^p \int_0^\infty g(x)^{-1} f(x)^p dg(x) \\ &= L^p \int_0^\infty G(x)dg(x). \end{aligned}$$

This completes the proof of the theorem.

Remark 3. If we take $\alpha = 1, v = 0, L = (1 - \frac{1}{p})^{-1}$ and $g(x) = x$ and replace $f(x)^p x^{-1}$ by $f(x)^p$. We then obtain a useful version of Hardy's inequality (1.1). Similarly, we get (1.2) by letting $L = p$.

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A Note On BCI-Algebras Of Order Five

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Abstract. In [2], it was shown that the number of proper BCI-algebras of order five regarding isomorphic BCI-algebras as equal are 70. In this note we investigated that the number of proper BCI-algebras of order five is 31 instead of 70.

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Keywords and Phrases: BCI-algebras, BCI-algebras of order five.

1. Introduction

K. Iseki [7] introduced the concept of BCI-algebras and established certain properties. Unlike, finite order groups the problem of characterizing finite order BCI-algebras has not been investigated so far. S.K. Goel in [5], as a first step, characterized BCI-algebras of order three and partially BCI-algebras of order four. In [2], it was shown that the number of proper BCI-algebras of order five is 70, regarding isomorphic BCI-algebras as equal to each other. It is pointed out that some impossible cases were also included as feasible ones. In this note, we have removed the impossible cases and show that the number of proper BCI-algebras of order five is 31 instead of 70. The mistakes lie in lemma 5, 7, 9, 10 and 12 of [2]. According to Lemmas 5, 7, 9, 10 and 12 of [2], the numbers of distinct proper BCI-algebras of order five are 16, 4, 4, 16 and 18 respectively. However the numbers of proper BCI-algebras of order five are found 5, 3, 2, 3 and 6 respectively by eliminating the impossible cases.

2. Preliminaries

1.Definition [7]
A BCI-algebra X is an abstract algebra $(X, *, o)$ of type (2, 0), where * is a binary

operation, o is a constant which is the smallest element in X, satisfying the following conditions; for all $x, y, z \in X$,

1.1 $((x * y) * (x * z)) * (z * y) = o$
1.2 $(x * (x * y)) * y = o$
1.3 $x * x = o$
1.4 $x * y = o = y * x \Rightarrow x = y$
1.5 $x * o = o \Rightarrow x = o$

where $x * y = o \Leftrightarrow x \leq y$
In a BCI-algebra X, the set $M = \{x \in X : o * x = o\}$ is a subalgebra and is called the BCK-part of X. A BCI-algebra X is called proper if $X - M \neq \phi$.
Moreover, the following properties hold in every BCK/BCI-algebra ([7, 8]):

1.6 $x * o = x$
1.7 $(x * y) * z = (x * z) * y$
1.8 $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$
1.9 Let X be a BCK-algebra. For $x, y \in M$, $x * y \leq x$. [6]
1.10 Let X be a BCI-algebra with M as its BCK-part. For $x \in M$, $y \in X - M$, x*y and $y * x \in X - M$.[7].
1.11 Let X be a BCI-algebra. If $M = \{o\}$, then X is called a p-semisimple BCI-algebra.[9]

1.2. Definition [8]
Let X be a BCK-algebra. An element x_o in X is said to be a Semi-neutral element in X if and only if for all $x \neq x_o$, $x * x_o = x$ and $x_o * x = x_o$.

Note that any nonzero element x of a BCK-algebra X such that $x \leq y$ for some $y \in X (or, y \leq x for some y (\neq o) \in X)$ cannot be a semi-neutral element of X.

1.3. Definition [1]
Let X be a BCI-algebra, for $x, y \in X$, x, y are said to be comparable if $x \leq y$ or $y \leq x$.

1.4 Definition [8]
Let X be a BCI-algebra. We choose an element $x_o \in X$ such that there does not exist any $y \neq x_o$, satisfying $y * x_o = o$ and define

$$A(x_o) = \{x \in X : x_o * x = o\}$$

We note that $A(x_o)$ consists of all those elements of X which succeed x_o. The element x_o is known as the initial element of $A(x_o)$ as well as X. Let I_x denote the set of all initial elements of X. We call it the center of X.

Note that the BCK-part M of X is equal to A(o) because $M = \{x \in X : o * x = o\} = A(o)$.

1.12 The center I_x of a BCI-algebra X is p-semisimple.[3]
1.13 Every p-semisimple BCI-algebra X is an abelian group under the binary operation defined as $x + y = x * (o * y)$.[4, 9]
1.14 Let X be a BCI-algebra and $A(x_o) \subseteq X$, for $x_o \in I_x$. Then $x, y \in A(x_o) \Rightarrow x * y, y * x \in M$ ([1]).

1.5 Definition
Let X be a BCK-algebra. An element $x \neq o$, $x \in X$ is said to be a Neutral element in X if $y \leq x \Rightarrow y = o$ or $y = x$ and $x \leq y \Rightarrow x = y$.

3. Some results on BCI-algebras

Proposition: Let x be a neutral element in a BCK-algebra X. Then for all $y \neq x$ in X, $x * y = x$ and $y * x = y$.
Proof: Let x be any neutral element in a BCK-algebra X. Take any $y \neq x$ in X. Because of (1.9),

$$x * y \leq x$$

As x is a neutral element in X, therefore either $x * y = x$ or $x * y = o$. But $x * y = o$ is not possible because $x * y = o \Rightarrow x \leq y \Rightarrow x = y$, a contradiction. So $x * y = x$. Because of (1.2), $y*(y*x) \leq x$ As x is a neutral element, therefore either $y*(y*x) = x$ or $y * (y * x) = o$. But $y * (y * x) = x$ is not possible otherwise

$$\begin{aligned} y * (y * x) = x \Rightarrow (y * (y * x)) * y &= x * y \\ \Rightarrow (y * y) * (y * x) &= x * y \quad (using\ (1.7)) \\ \Rightarrow o * (y * x) &= x * y \quad (using\ (1.3)) \\ \Rightarrow o &= x * y \quad (using\ (1.5)) \\ \Rightarrow x \leq y \Rightarrow x &= y \quad (Since\ x\ is\ neutral) \end{aligned}$$

a contradiction. So, $y * (y * x) = o$.
Also because of (1.9), $(y*x)*y = o$. Using (1.4), $y*(y*x) = o = (y*x)*y \Rightarrow y*x = y$, which completes the proof.
Corollary:
A neutral element is a semi-neutral element.
Proof:
Let x be a neutral element in X. Then by above proposition, for all $y \neq x \in X$, $x * y = x$ and $y * x = y$. Because of definition 1.2 it follows that x is a semi-neutral element.
Lemma 1
Let X be a BCI-algebra with I_x as its center. If $x \in A(x_o)$, $y \in A(y_o)$, then $x * y \in A(x_o * y_o)$, for x_o, $y_o \in I_x$.
Proof
Let $x \in A(x_o)$, $y \in A(y_o)$, for $x_o \neq y_o \in I_x$. Then

$$x_o \leq x \tag{1}$$

and

$$y_o \leq y \tag{2}$$

Because of (1.8), $y_o \leq y \Rightarrow x_o * y \leq x_o * y_o$. Because of (1.12), the center I_x is p-semisimple and by (1.13), I_x is an abelian group, so I_x is closed i.e for $x_o, y_o \in I_x$, $x_o * y_o \in Ix_o$.
Hence

$$x_o * y = x_o * y_o \tag{3}$$

Further, because of(1.8) and equation (3), $x_o \leq x \Rightarrow x_o*y \leq x*y \Rightarrow x_o*y_o \leq x*y$, which implies that $x * y \in A(x_o * y_o)$.

Lemma 2
If $X = \{o, a, b\}$ is a BCK-algebra of order three, then there exist three such BCK-algebras.

Proof

Let $X = \{o, a, b\}$ be a BCK-algebra of order three, then for the configuration of X, we have following two possibilities:

(i) $o \leq a \leq b$

(ii) $o \leq a$, $o \leq b$ and a, b are incomparable.

Case (i): $o \leq a \leq b$

Routine calculation show that $o * a = o * b = o = a * b = o$

Computation of b*a

Because of (1.8) and (1.3), $o \leq a \Rightarrow b * a \leq b * o \Rightarrow b * a \leq b \Rightarrow b * a \in \{o, a, b\}$. But $b * a \neq o$, otherwise by (1.4), $a * b = o = b * a \Rightarrow a = b$, a contradiction. So, $b * a \in \{a, b\}$. Thus, there exist two distinct BCK-algebras, given by following Multiplication tables:

Table M_1

*	o	a	b
o	o	o	o
a	a	o	o
b	b	a	o

Table M_2

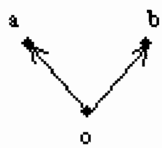
*	o	a	b
o	o	o	o
a	a	o	o
b	b	b	o

Case (ii)

Routine calculation show that $o * a = o * b = o$. From given condition, it follows that a and b are neutral elements. Thus by proposition, $a * b = a$ and $b * a = b$. Hence by using the properties (1.3), (1.6), $a * b = a$ and $b * a = b$ the Multiplication table representing such BCK-algebra is given as follows:

Table M_3

*	o	a	b
o	o	o	o
a	a	o	a
b	b	b	o



From case (i) and Case (ii), it follows that there exist 3 distinct BCK-algebras of order 3.

Lemma 3
Let X be a BCI-algebra with I_x as its center. Let x_o, $y_o \in I_x$. Then for all $y \in A(y_o)$, $x_o * y = x_o * y_o$.
Proof
Let $y \in A(y_o)$. Then $y_o \leq y$. Because of (1.8), it implies that $x_o * y \leq x_o * y_o$. Because of (1.12), the center I_x is p-semisimple and by (1.13), I_x is an abelian group, so I_x is closed i.e for x_o, $y_o \in I_x$, $x_o * y_o \in I_x$. So that $x_o * y = x_o * y_o$.

4. On BCI-algebras of order five

Now we reproduce the results proved in [2], point out the mistakes there in and remove them. We will only discuss the results in which there are mistakes and remaining results will be represented as it is:
Let X be a BCI-algebra of order five with M as its BCK-part. Then we have the following possibilities about the configuration of M:

(i) o(M)=1 (ii) o(M)=2 (iii) o(M)=3 (iv) o(M)=4

1. BCI-ALGEBRA OF ORDER 5 WITH o(M)=1

Theorem 1([2]):
Let X be a BCI-algebra with M as its BCK-part. Let o(X)=5 and o(M)=1, then there is one such proper BCI-algebra.
Proof: The proof is given in [2].

2. BCI-ALGEBRAS OF ORDER 5 WITH o(M)=2

Theorem 4([2]):
Let X be a BCI-algebra with M as its BCK-part. Let o(X)=5 and o(M)=2, then the number of all such BCI-algebras is 5.
Proof: The proof is given in [2].

3. BCI-ALGEBRAS OF ORDER 5 WITH o(M)=3

Theorem 5([2])
Let X be a BCI-algebra with o(X)=5. Let M be its BCK-part with o(M)=3, then the number of all such BCI-algebras is 23.
This theorem 5([2]) depends upon Lemma 5[2], Lemma 6[2] and Lemma 7[2], which are stated as follows:
Lemma 5[2]:
Let X be a BCI-algebra with o(X)=5. Let M be its BCK-part with o(M)=3 and for $o, a, b \in M$, $o \leq a \leq b$, and for $c, d \in X - M$, c, d are comparable, then there are 16 such BCI-algebras.
Lemma 6[2]:
Let X be a BCI algebra with o(X)=5. Let M be its BCK part. If o(M)=3 and $o(X - M) = 2$ such that $c, d \in X - M$, are incomparable. Then the number of all such BCI-algebras is 3.
Lemma 7[2]:
Let X be a BCI algebra with o(X)=5. Let M be its BCK part. Let o(M)=3 and

$o, a, b \in M$ are incomparable and $c, d \in X - M$, are comparable. Then, there exist 4 such BCI-algebras.

According to theorem 5([2]), the author claims that there are 23 distinct BCI-algebras. But we find out that out of these 23 cases, 13 BCI-algebras do not exist, because there are impossible cases as proved in **Lemma 4** and **Lemma 5**, in the sequel:

We have noted that there does not exist any impossible case in Lemma 6[2]. The impossible cases exist in Lemmas 5[2] and 7[2], which are removed and the total number of BCI-algebras of order 5 with o(M)=3 reduces to 11 instead of 23.

In lemma 5[2] and 7[2], the number of BCI-algebras of order five are 16 and 4. However, the number of such proper BCI-algebras of order five are reduced to 5 and 3, respectively, by eliminating impossible cases, as shown in following **Lemma 4** and **Lemma 5**.

Lemma 4

Let X be a BCI-algebra with o(X)=5. Let M be its BCK-part with o(M)=3 and for $o, a, b \in M$, $o \leq a \leq b$, and for $c, d \in X - M$, c, d are comparable, then there are 5 such BCI-algebras.

Proof Let $X = \{o, a, b, c, d\}$ be a BCI-algebra with $M = A(o) = \{o, a, b\}$ as its BCK-part. Then $X - M = \{c, d\}$. Since $c \leq d$ therefore $A(c) = X - M = \{c, d\}$. Note that $I_x = \{o, c\}$. The following table defines the binary operation* in I_x as follows:

Table 1

*	o	c
o	o	c
c	c	o

Since $o \leq a \leq b$, therefore by lemma 2 there are two BCK-algebras of order three. The multiplication tables representing these BCK-algebras are given in lemma 2, case (i), labeled as table M_1 and table M_2. Thus, we have following two cases for the configuration of X:

Case 1: BCK-part M is given by Table M_1 and $A(c) = X - M = \{c, d\}$

Case 2: BCK-part M is given by Table M_2 and $A(c) = X - M = \{c, d\}$

Routine calculations show that c*d=o

Case 1: By using the properties (1.3), (1.6), the Table M_1 and $c * d = o$, the multiplication table representing the BCI-algebra is given as follows:

Table 2

*	o	a	b	c	d
o	o	o	o	c	
a	a	o	o		
b	b	a	o		
c	c			o	o
d	d				

The entries for the blank cells in Table 2 are computed as follows:

Computation of o*d:

By lemma 3, $o * d = o * c = c$. So, c will fill the blank cell of the 2nd row.

Table I

No	Possibility	Valid	No	Possibility	Valid
1	$a*c=c,\ a*d=c$	(C_1)	3	$a*c=d,\ a*d=c$	(C_2)
2	$a*c=c,\ a*d=d$		4	$a*c=d,\ a*d=d$	

Computation of a*c, a*d:
Because of lemma 1, $a*c \in A(o*c) = A(c) = \{c, d\}$ (1)
Also because of (1.8), $c \le d \Rightarrow a*d \le a*c \Rightarrow a*d \in \{c, d\}$ (2)
Combining (1) and (2) simultaneously, we have following 4 possibilities to fill the blank cells of the 3rd row:

Some impossible cases are discussed as follows:

Case 2 is not possible because by (1.8), $c \le d \Rightarrow a*d \le a*c \Rightarrow a*d \le c$. But $c \le d$, therefore $c \le d \le c \Rightarrow c = d$, a contradiction.
Case 4 is not possible because by (1.4), $d = a*d = a*(a*c) \le c \Rightarrow d \le c$. But $c \le d$, therefore $c \le d \le c \Rightarrow c = d$, a contradiction.

Remaining cases 1 and 3 do not conclude in contradictions and denoted as (C_1) and (C_2).

Computation for b*c, b*d are same as above. Thus, to fill the blank cell of the 4th row, we have following two possibilities:

C_3 $b*c = c, b*d = c$
C_4 $b*c = d, b*d = c$

Computation of c*a, c*b
By Lemma 3, $c*a = c*b = c*o = c$. So, c will fill the blank cells of the 5^{th} row.
Computation of d*a, d*b, d*c
Because of (1.8), $o \le a \Rightarrow d*a \le d*o \Rightarrow d*a \le d \Rightarrow d*a \in \{c, d\}$ (3)
Further by (1.8), $a \le b \Rightarrow d*b \le d*a \Rightarrow d*b \in \{c, d\}$ (4)
Because of (1.14), $d*c \in M = A(o) = \{o, a, b\}$. But $d*c \ne o$, otherwise because of (1.4), $c*d = o = d*c \Rightarrow c = d$, a contradiction. So, $d*c \in \{a, b\}$ (5)
Combining (3), (4) and (5) simultaneously, we have following 8 possibilities to fill the 3rd , 4th and 5th blank cells of the 6th row.:

Table II

No	Possibility	Valid
1	$d*a=c,\ d*b=c,\ d*c=a$	(E)
2	$d*a=c,\ d*b=c,\ d*c=b$	
3	$d*a=c,\ d*b=d,\ d*c=a$	
4	$d*a=c,\ d*b=d,\ d*c=b$	
5	$d*a=d,\ d*b=c,\ d*c=a$	
6	$d*a=d,\ d*b=c,\ d*c=b$	(F)
7	$d*a=d,\ d*b=d,\ d*c=a$	
8	$d*a=d,\ d*b=d,\ d*c=b$	

Some impossible cases are discussed as follows:

Case 2 is not possible because by (1.2), $b = d*c = d*(d*a) \leq a \Rightarrow b \leq a$. But $a \leq b$, therefore $a \leq b \leq a$. By (1.4) $a \leq b \leq a \Rightarrow a = b$, a contradiction.
Case 3 is not possible because by (1.8), $a \leq b \Rightarrow d*b \leq d*a \Rightarrow d \leq c$. But $c \leq d$, therefore $c \leq d \leq c$. By (1.4), $c \leq d \leq c \Rightarrow c = d$, a contradiction.
Case 4 and 8 are not possible because $d = d*b = d*(d*c) \leq c \Rightarrow d \leq c$. But $c \leq d$, therefore $c \leq d \leq c$. By (1.4), $c \leq d \leq c \Rightarrow c = d$, a contradiction.
Case 5 and 7 are not possible because by (1.2), $d = d*a = d*(d*c) \leq c \Rightarrow d \leq c$. But $c \leq d$, therefore $c \leq d \leq c$. By (1.4), $c \leq d \leq c \Rightarrow c = d$, a contradiction.
Remaining Cases 1 and 6 do not conclude in contradictions and denoted as (E) and (F) and all the computations are fixed in each case. Now combining these with $(C_1) - (C_4)$, simultaneously, we have the following 8 cases such that each case may represent a distinct BCI-algebra. However, these include impossible cases, too, which are to be pointed out in the sequel:

Table III

No	Possibility
1	$a*c=c,\ a*d=c,\ b*c=c,\ b*d=c,\ d*a=c,\ d*b=c,\ d*c=a$
2	$a*c=c,\ a*d=c,\ b*c=d,\ b*d=c,\ d*a=c,\ d*b=c,\ d*c=a$
3	$a*c=d,\ a*d=c,\ b*c=c,\ b*d=c,\ d*a=c,\ d*b=c,\ d*c=a$
4	$a*c=d,\ a*d=c,\ b*c=d,\ b*d=c,\ d*a=c,\ d*b=c,\ d*c=a$
5	$a*c=c,\ a*d=c,\ b*c=c,\ b*d=c,\ d*a=d,\ d*b=c,\ d*c=b$
6	$a*c=c,\ a*d=c,\ b*c=d,\ b*d=c,\ d*a=d,\ d*b=c,\ d*c=b$
7	$a*c=d,\ a*d=c,\ b*c=c,\ b*d=c,\ d*a=d,\ d*b=c,\ d*c=b$
8	$a*c=d,\ a*d=c,\ b*c=d,\ b*d=c,\ d*a=d,\ d*b=c,\ d*c=b$

The impossible cases are discussed as follows:

Cases 3 is not possible because $a \leq b \Rightarrow a*c \leq b*c \Rightarrow d \leq c$, but $c \leq d$, therefore $c \leq d \leq c \Rightarrow c = d$, a contradiction. From table M_1 it follows $b*a = a$ and because of (i), $b*d = c$. **case 4** is not possible because by (1.1), $(b*a)*(b*d) \leq d*a \Rightarrow a*c \leq d*a \Rightarrow d \leq c$, but $c \leq d$, therefore $c \leq d \leq c \Rightarrow c = d$, a contradiction. From table M_o it follows $b*a = a$, so, **Case 5-Case 8** are not possible because $b = d*c = (d*a)*c = (d*c)*a = b*a = a$, a contradiction.

Remaining Cases 1 and 2 do not conclude in contradictions. Hence, by filling the corresponding blank cells with the remaining cases namely 1 and 2 respectively, the resulting two BCI-algebras are given as follows:

Table 3

*	o	a	b	c	d
o	o	o	o	c	c
a	a	o	o	c	c
b	b	a	o	c	c
c	c	c	c	o	o
d	d	c	c	a	o

Table 4

*	o	a	b	c	d
o	o	o	o	c	c
a	a	o	o	c	c
b	b	a	o	d	c
c	c	c	c	o	o
d	d	c	c	a	o

Case 2: By using the properties (1.3), (1.6), the Table M_2 and the values computed above, the multiplication table representing the BCI-algebra is given as follows:

Table 5

*	o	a	b	c	d
o	o	o	o	c	
a	a	o	o		
b	b	b	o		
c	c			o	o
d	d				

The entries for the blank cells in Table 5 are computed as follows:

Computations of o*d, a*c, a*d, b*c, b*d, c*a, c*b, d*a, d*b and d*c are same as in case (1). So, $o * d = c$, $a * d = c$, $b * d = c$ and $c * a = c * b = c$. Thus, c will fill the blank cells of the 6^{th} column and 3^{rd} and 4^{th} blank cell of the 5^{th} row. To fill the remaining blank cells of the 5^{th} column and 3^{rd} and 4^{th} blank cell of the 6^{th} row, we have 8 possibilities, given in table III of case (1). However, some of these are not possible and discussed below:

From table M_2 it follows $b * a = b$, so, **case 2 and 4** are not possible because $d = b * c = (b * a) * c = (b * c) * a = d * a = c$, a contradiction.

Case 3 is not possible because $a \leq b \Rightarrow a * c \leq b * c \Rightarrow d \leq c$, but $c \leq d$, therefore $c \leq d \leq c \Rightarrow c = d$, a contradiction.
Cases 7 and 8 are not possible because $d = d*a = (a*c)*a = (a*a)*c = o*c = c$, a contradiction.
Remaining Cases 1, 5 and 6 do not conclude in contradictions and all the computations are fixed in each case. Hence, by filling the corresponding blank cells with these cases respectively, the resulting three BCI-algebras are given as follows:

Table 6

*	o	a	b	c	d
o	o	o	o	c	c
a	a	o	o	c	c
b	b	b	o	c	c
c	c	c	c	o	o
d	d	c	c	a	o

Table 7

*	o	a	b	c	d
o	o	o	o	c	c
a	a	o	o	c	c
b	b	b	o	c	c
c	c	c	c	o	o
d	d	c	c	b	o

Table 8

*	o	a	b	c	d
o	o	o	o	c	c
a	a	o	o	c	c
b	b	b	o	d	c
c	c	c	c	o	o
d	d	c	c	b	o

From, cases 1 and 2, it follows that the total number of such BCI-algebras is 5.

Lemma 5

Let X be a BCI algebra with o(X)=5. Let M be its BCK part. Let o(M)=3 and $o \neq a, b \in M$ are incomparable and $c, d \in X - M$, are comparable. Then, there exist 3 such BCI-algebras.

Proof

Let $X = \{o, a, b, c, d\}$ be a BCI-algebra with $M = A(o) = \{o, a, b\}$ as its BCK-part. Then $X - M = \{c, d\}$. Since $c \leq d$ therefore $A(c) = X - M = \{c, d\}$. Note that $I_o = \{o, c\}$. The binary operation * in I_x is defined as in table 1. Since $o \neq a, b \in M$ are incomparable, therefore $o \leq a$ and $o \leq b$. Thus it follows from lemma 2, case (ii), there is only one BCK-algebra of order three. The multiplication table representing this BCK-algebra is given in lemma 3, labeled as table M_3. Routine calculations show that $c * d = o$ By using the properties (1.3), (1.6), the Table M_3, table 1 and $c * d = o$, the multiplication table representing the proper BCI-algebra is given as follows:

Table 9

*	o	a	b	c	d
o	o	o	o	c	
a	a	o	a		
b	b	b	o		
c	c			o	o
d	d				o

The entries for the blank cells in Table 9 are computed as follows:

Computations of o*d and c*a, c*b are same as in case (1), lemma 4. So, **o*d=c** and **c*a=c*b=c**. So, c will fill the blank cells of the 2^{nd} and 5^{th} rows.
Computation of a*c and a*d are same as done in case 1, lemma 4. Thus to fill the blank cells of the 3rd row, we have following two possibilities:

$$C_1 \quad a * c = c, a * d = c \quad C_2 \quad a * c = d, a * d = c$$

Computation of b*c and b*d are same as done for a*c and a*d in case (1), lemma 4.. Thus to fill the blank cells of the 4_{th} row, we have following two possibilities:

$$C_3 \quad b*c=c, b*d=c \quad C_4 \quad b*c=d, b*d=c$$

Computation of d*a, d*b, d*c

Because of (1.8), $o \leq a \Rightarrow d*a \leq d*o \Rightarrow d*a \leq d \Rightarrow d*a \in \{c,d\}$ (3)
Likewise, $d*b \in \{c,d\}$ (4)

Because of (1.14), $d*c \in M = A(o) = \{o,a,b\}$. But $d*c \neq o$, otherwise because of (1.4), $c*d = o = d*c \Rightarrow c = d$, a contradiction. So, $d*c \in \{a,b\}$ (5)

Combining (3), (4) and (5) simultaneously, we have 8 possibilities, given in table II, case 1, lemma 4. Out of these cases 2, 4, 5, 7 and 8 are not possible as shown in case 1, lemma 4. Remaining three cases 1, 3 and 6 do not conclude in contradiction. Combining these with $C_1 - C_4$ simultaneously, we have following 12 possibilities. However these include some impossible cases, too, which are to be pointed out in sequel:

Table IV

No	Possibility
1	$a*c=c,\ a*d=c,\ b*c=c,\ b*d=c,\ d*a=c,\ d*b=c,\ d*c=a$
2	$a*c=c,\ a*d=c,\ b*c=d,\ b*d=c,\ d*a=c,\ d*b=c,\ d*c=a$
3	$a*c=d,\ a*d=c,\ b*c=c,\ b*d=c,\ d*a=c,\ d*b=c,\ d*c=a$
4	$a*c=d,\ a*d=c,\ b*c=d,\ b*d=c,\ d*a=c,\ d*b=c,\ d*c=a$
5	$a*c=c,\ a*d=c,\ b*c=c,\ b*d=c,\ d*a=c,\ d*b=d,\ d*c=a$
6	$a*c=c,\ a*d=c,\ b*c=d,\ b*d=c,\ d*a=c,\ d*b=d,\ d*c=a$
7	$a*c=d,\ a*d=c,\ b*c=c,\ b*d=c,\ d*a=c,\ d*b=d,\ d*c=a$
8	$a*c=d,\ a*d=c,\ b*c=d,\ b*d=c,\ d*a=c,\ d*b=d,\ d*c=a$
9	$a*c=c,\ a*d=c,\ b*c=c,\ b*d=c,\ d*a=d,\ d*b=c,\ d*c=b$
1o	$a*c=c,\ a*d=c,\ b*c=d,\ b*d=c,\ d*a=d,\ d*b=c,\ d*c=b$
11	$a*c=d,\ a*d=c,\ b*c=c,\ b*d=c,\ d*a=d,\ d*b=c,\ d*c=b$
12	$a*c=d,\ a*d=c,\ b*c=d,\ b*d=c,\ d*a=d,\ d*b=c,\ d*c=b$

The impossible cases are discussed as follows:

Case 1-Case 4 are not possible because $a = d*c = d*(d*b) \leq b$, a contradiction.
Cases 6 and 8 are not possible because $d = d*b = (b*c)*b = (b*b)*c = o*c = c$, a contradiction.
Cases 11 and 12 are not possible because $d = d*a = (a*c)*a = (a*a)*c = o*c = c$, a contradiction.
Remaining 4 Cases 5, 7, 9 and 10 do not conclude in contradictions and all the computations are fixed in each case. Hence, by filling the corresponding blank cells with the remaining four cases respectively, the resulting four proper BCI-algebras are given as follows:

Table 10

*	o	a	b	c	d
o	o	o	o	c	c
a	a	o	a	c	c
b	b	b	o	c	c
c	c	c	c	o	o
d	d	c	d	a	o

Table 11

*	o	a	b	c	d
o	o	o	o	c	c
a	a	o	a	d	c
b	b	b	o	c	c
c	c	c	c	o	o
d	d	c	d	a	o

Table 12

*	o	a	b	c	d
o	o	o	o	c	c
a	a	o	a	c	c
b	b	b	o	c	c
c	c	c	c	o	o
d	d	d	c	b	o

Table 13

*	o	a	b	c	d
o	o	o	o	c	c
a	a	o	a	c	c
b	b	b	o	d	c
c	c	c	c	o	o
d	d	d	c	b	o

Note that proper BCI-algebra represented by Table 10 is isomorphic to the proper BCI-algebra represented by Table 12 under the mapping $f : Table10 \rightarrow Table12$ defined as f (o) = o, f (a) = b, f (b) = a, f (c) = c and f (d) = d. Hence in this case regarding isomorphic BCI-algebras as equal there are three such proper BCI-algebras represented by the multiplication **table 10, table 11 and table 13.**

So, Theorem 5 is restated as follows:

Theorem 5

Let X be a BCI-algebra with o(X)=5. Let M be its BCK-part with o(M)=3, then the number of all such BCI-algebras is 11.

Proof

It follows from Lemma 4, Lemma 5 and Lemma 6[2].

4. BCI-ALGEBRAS OF ORDER 5 WITH o(M)=4

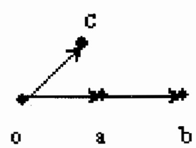
Theorem 6([2]): Let X be a BCI algebra with o(X)=5. Let M be its BCK part. If o(M)=4, then there exist 41 such distinct BCI-algebras.
This theorem 6([2]) depends upon Lemma 8, Lemma 9, Lemma 10, Lemma 11 and Lemma 12, which are stated as follows:
Lemma 8 [2]: Let X be a BCK-algebra with o(X)=4, and for each pair $a, b \in X$ is incomparable, then X is unique.
Lemma 9 [2]: Let X be a BCK algebra with o(X)=4, and for $o \neq a, b, c \in X$, $a \leq b$ and c is not comparable with a and b, then number of such BCK- algebras is 4.
Lemma 10 [2]: Let X be a BCK algebra with o(X)=4, and for $o \neq a, b, c \in X$, $a \leq b$ and $a \leq c$, but b, c are incomparable, then there are 16 such BCK-algebras.
Lemma 11[2]: Let X be a BCK algebra with o(X)=4, and for $o \neq a, b, c \in X$, $o \leq a \leq b$ and $o \leq b \leq c$, where a, b are not comparable. Then, there exist two such BCK-algebras.
Lemma 12 [2]: Let X be a BCK algebra with o(X)=4, and for $o \neq a, b, c \in X$, $o \leq a \leq b \leq c$, then there exist 18 such BCK-algebras.
According to theorem 6([2]), the author claims that there are 41 distinct BCI-algebras. But we find out that out of these 41 cases, 27 BCI-algebras do not exist, because there are impossible cases as proved in Lemma 6, Lemma 7 and Lemma 8, in the sequel.
We have noted that there does not exist any impossible case in Lemma 8[2] and Lemma 11[2]. The impossible cases only exist in Lemma 9[2], Lemma 10[2] and Lemma 12[2] which are removed and the total number of BCK-algebras of order four reduces to 14 instead of 41. It is shown that in Lemma 9[2], Lemma 10[2] and Lemma 12[2], the numbers of BCK-algebras of order four are 4, 16, and 18 respectively. However, the number of BCK-algebras of order four are reduced to 2, 3 and 6 respectively by eliminating the impossible cases, as follows from **Lemma 6, Lemma 7 and Lemma 8** in the sequel:

Lemma 6

Let X be a BCK algebra with o(X) = 4, and for $o \neq a, b, c \in$ X, $a \leq b$ and c is not comparable with a and b, then number of such BCK- algebras is 2.

Proof

Let $X = \{o, a, b, c\}$ be a BCK-algebra and for $o \neq a, b, c \in$ X, $a \leq b$ and c is not comparable with a and b i.e. $o \leq a \leq b$, $o \leq c$.



Routine calculations show that $o * a = o * b = a * b = o * c = o$. By definition 1.5, c is a Neutral element. Thus by proposition, $c * a = c * b = c$, $a * c = a$, $b * c = b$.
By using the properties (1.3), (1.6) and the values computed above, the Multiplication table representing the BCK- algebra of order 4 with the given conditions, will be shown as follows:

Table 14

*	o	a	b	c
o	o	o	o	o
a	a	o	o	a
b	b		o	b
c	c	c	c	o

The computations for b*a is same as given in lemma 2. So, $b * a \in \{a, b\}$. Thus we have two possibilities to fill the 3^{rd} blank cell of 4^{th} row. Thus, there exist two distinct BCK-algebras, given as follows:

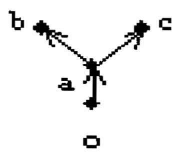
Table 15

*	o	a	b	c
o	o	o	o	o
a	a	o	o	a
b	b	a	o	b
c	c	c	c	o

Table 16

*	o	a	b	c
o	o	o	o	o
a	a	o	o	a
b	b	b	o	b
c	c	c	c	o

Lemma 7
Let X be a BCK algebra with o(X) = 4, and for $o \neq a, b, c \in X$, $o \leq a \leq b$, $o \leq a \leq c$. Then there are 3 such BCK-algebras.
Proof
Let $X = \{o, a, b, c\}$ be a BCK-algebra and for $o \neq a, b, c \in X$, $o \leq a \leq b$, $o \leq a \leq c$. Routine calculations show that $o * a = o * b = a * b = o$ and $o * a = o * c = a * c = o$.



By using the properties (1.3), (1.6) and the values computed above, the Multiplication table representing the BCK-algebra of order four with the given conditions, will be shown as follows:

Table 17

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b		o	
c	c			o

The entries for the blank cells in Multiplication table 17 are computed as follows:

Computation of b*a, b*c
Because of (1.8), $o \leq a \Rightarrow b * a \leq b * o \Rightarrow b * a \leq b \Rightarrow b * a \in \{o, a, b\}$. But $b * a \neq o$,

otherwise by (1.4), $a * b = o = b * a \Rightarrow a = b$, a contradiction. So $b * a \in \{a, b\}$ (1)
Further, $a \leq c \Rightarrow b * c \leq b * a \Rightarrow b * c \in \{a, b\}$ (2)
Combining (1) and (2) simultaneously, we have the following four possibilities to fill the blank cells of the 4th row:

Table I

No	Possibility	Valid	No.	Possibility	Valid
1	$b * a = a, b * c = a$	(A)	3	$b * a = b, b * c = a$	
2	$b * a = a, b * c = b$		4	$b * a = b, b * c = b$	(B)

Some impossible cases are discussed as follows:
Case 2 is not possible because by (1.8), $a \leq c \Rightarrow b * c \leq b * a \Rightarrow b \leq a$. But $a \leq b$, $a \leq b \leq a \Rightarrow a = b$, a contradiction
Case 3 is not possible because by (1.2), $b = b * a = b * (b * c) \leq c$, a contradiction. Remaining two Cases 1 and 4 do not conclude in contradiction, thus we have two possibilities to fill the blank cells in 4th row, denoted by (A) and (B).

Computation of c*a, c*b
In the same way as computed above, we have two possibilities to fill the blank cells of the 5th row given as follows:

$$\text{(C).}\quad c * a = a,\ c * b = a \quad \text{(D)} \quad c * a = c,\ c * b = c$$

Now depending upon (A), (B), (C) and (D) we have the following four cases, which may represent four BCK-algebras:

Set A

1.	$b * a = a,\ b * c = a,\ c * a = a,\ c * b = a$
2.	$b * a = a,\ b * c = a,\ c * a = c,\ c * b = c$
3.	$b * a = b,\ b * c = b,\ c * a = a,\ c * b = a$
4.	$b * a = b,\ b * c = b,\ c * a = c,\ c * b = c$

None of the above cases conclude in contradiction and all the computations are fixed in each case. Hence, by filling the corresponding blank cells with the above cases namely 1, 2, 3 and 4 respectively, the resulting four BCK-algebras are given as follows:

Table 18

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b	a	o	a
c	c	a	a	o

Table 19

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b	a	o	a
c	c	c	c	o

Table 20

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b	b	o	b
c	c	a	a	o

Table 21

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b	b	o	b
c	c	c	c	o

Note that BCK-algebra represented by Table 19 is isomorphic to the BCK-algebra represented by Table 20 under the mapping f: Table 19 Table 20 defined as f (o) = o, f (a) = a, f (b) = c and f (c) = b. Hence in this case regarding isomorphic BCK-algebras as equal there are three such BCK-algebras represented by the multiplication **table 18**, **table 19** and **table 21**.

Lemma 8
Let X be a BCK algebra with o(X) = 4, and for $o \neq a, b, c \in X$, $o \leq a \leq b \leq c$. Then there exist 6 such BCK-algebras.
Proof
Let $X = \{o, a, b, c\}$ be a BCK-algebra and for $o \neq a, b, c \in X$, $o \leq a \leq b \leq c$.

Routine calculations show that $o * a = o * b = o * c = a * b = a * c = b * c = o$

By using the properties (1.3), (1.6) and the values computed above, the Multiplication table representing the BCK- algebra of order 4 with the given conditions, will be shown as follows:

Table 22

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b		o	o
c	c			o

The entries for the blank cells in Multiplication table 22 are computed as follows:

Computation of b*a

The computations for b*a is same as given in lemma 3. So, $b * a \in \{a, b\}$ (D) Thus, we have two possibilities to fill the 3rd blank cell of the 4^{th} row.

Computation of c*a, c*b

Because of (1.8), $o \leq a \Rightarrow c{*}a \leq c{*}o \Rightarrow c{*}a \leq c \Rightarrow c{*}a \in \{o, a, b, c\}$. Now $c{*}a \neq o$, otherwise by (1.4), $a{*}c = o = c{*}a \Rightarrow a = c$, a contradiction, so $c{*}a \in \{a, b, c\}$ (1). Further, $a \leq b \Rightarrow c * b \leq c * a \Rightarrow c * b \in \{a, b, c\}$ (2)

Combining (1) and (2) simultaneously, we have nine possibilities given in the following Table I, to fill the 3rd and 4th blank cells of 5th row:

Table I

No	Possibility	Valid	No.	Possibility	Valid
1	$c * a = a,\ c * b = a$	(i)	6	$c * a = b,\ c * b = c$	
2	$c * a = a,\ c * b = b$		7	$c * a = c,\ c * b = a$	
3	$c * a = a,\ c * b = c$		8	$c * a = c,\ c * b = b$	(iii)
4	$c * a = b,\ c * b = a$	(ii)	9	$c * a = c,\ c * b = c$	(iv)
5	$c * a = b,\ c * b = b$				

Some impossible cases are discussed as follows:

Case 2 is not possible because by (1.8), $a \leq b \Rightarrow c * b \leq c * a \Rightarrow b \leq a$. But $a \leq b$, therefore by (1.4), $a \leq b \leq a \Rightarrow a = b$, a contradiction. Similarly, cases 3 and 6 are not possible.

Case 5 is not possible because by (1.7), $b = c * b = c * (c * a) \leq a \Rightarrow b \leq a$. But $a \leq b$, therefore by (1.4), $a \leq b \leq a \Rightarrow a = b$, a contradiction.

Case 7 is not possible because by (1.4), $c = c * a = c * (c * b) \leq b$, a contradiction.

Remaining Cases 1, 4, 8 and 9 do not conclude in contradiction and denoted by (i), (ii), iii) and (iv). Combining these with (D) simultaneously, we have the following eight cases, which may represent eight BCK-algebras. However, these include some impossible cases, too, which are to be pointed out in the sequel:

Set B

1. $b * a = a,\ c * a = a,\ c * b = a$ (A)
2. $b * a = a,\ c * a = b,\ c * b = a$ (B)
3. $b * a = a,\ c * a = c,\ c * b = b$
4. $b * a = a,\ c * a = c,\ c * b = c$ (C)
5. $b * a = b,\ c * a = a,\ c * b = a$
6. $b * a = b,\ c * a = b,\ c * b = a$ (D)
7. $b * a = b,\ c * a = c,\ c * b = b$ (E)
8. $b * a = b,\ c * a = c,\ c * b = c$ (F)

Some impossible cases are discussed as follows:
Case 3 is not possible because by (1.1), $(c*a)*(c*b) \leq b*a \Rightarrow c*b \leq a \Rightarrow b \leq a$. But $a \leq b$, therefore (1.4), $a \leq b \leq a \Rightarrow a = b$, a contradiction.
Case 5 is not possible because by (1.8), b c b*a c*a b a. But a b, therefore by (1.4), a b a a = b, a contradiction.
Remaining 6 Cases 1, 2, 4, 6, 7 and 8 do not conclude in contradictions and all the computations are fixed in each case. Hence, by filling the corresponding blank cells with the remaining cases namely 1, 2, 4, 6, 7 and 8 respectively, the resulting 6 BCK-algebras are given as follows:

Table 23

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b	a	o	o
c	c	a	a	o

Table 24

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b	a	o	o
c	c	b	a	o

Table 25

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b	a	o	o
c	c	b	c	o

Table 26

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b	b	o	o
c	c	b	a	o

Table 27

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b	b	o	o
c	c	c	b	o

Table 28

*	o	a	b	c
o	o	o	o	o
a	a	o	o	o
b	b	b	o	o
c	c	b	c	o

Hence, Theorem 6 is restated as follows:

Theorem 6

Let X be a BCI-algebra with o(X)=5 and M be its BCK-part. If o(M)=4, then there exist 14 such BCI-algebras.

Proof

Let $X = \{o, a, b, c, d\}$. Without any loss of generality, we take $M = A(o) = \{o, a, b, c\}$. and $X - M = A(d) = \{d\}$. We have the following possibilities about the configuration of X:

(i) For each pair $o \neq x,\ y \in M$, x, y are incomparable i.e. $o \leq a$, $o \leq b$, $o \leq c$ and $X - M = A(d) = \{d\}$.
(ii) For each pair $o \neq x, y \in M$,$o \leq a \leq b$, $o \leq c$ and $X - M = A(d) = \{d\}$.
(iii For each pair $o \neq x, y \in M$,$o \leq a \leq b$, $o \leq a \leq c$ and $X - M = A(d) = \{d\}$.
(iv) For each pair $o \neq x, y \in M$,$o \leq a \leq b$, $o \leq b \leq c$ and $X - M = A(d) = \{d\}$.
(v) For each pair $o \neq x, y \in M$,$o \leq a \leq b \leq c$ and $X - M = A(d) = \{d\}$.

Note, that $I_x = \{o, d\}$. The binary operation* in I_o is defined as follows :

Table 29

*	o	d
o	o	d
d	d	o

By Lemma 1, a*d, b*d, $c * d \in A(o * d) = A(d) = \{d\}$. So, a*d=b*d=c*d=d. Also by Lemma 3, d*a=d*b=d*c=d*o=d. By using the properties (1.3), (1.8), Table 29 and the values computed above, the multiplication table representing the BCI-algebra of order 5 with the given conditions, will be shown as follows:

Table 30

*	o	a	b	c	d
o	o				d
a	a	o			d
b	b		o		d
c	c				d
d	d	d	d	d	o

The entries for the blank cells in Table 30 are computed as follows:

In Case (i) by Lemma 8[2], there exist a unique BCK-algebra of order four. So, using the entries in the corresponding cells, given in table 15[2], will fill the blank cells, so the resulting BCI-algebra is shown as follows:

Table 31

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	a	a	d
b	b	b	o	b	d
c	c	c	c	o	d
d	d	d	d	d	o

In Case (ii) by Lemma 6, there exist 2 BCK-algebra of order four. So, using the entries in the corresponding cells, given in Table 15 and 16 respectively, will fill the blank cells, so the resulting BCI-algebra are shown as follows:

Table 32

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	o	a	d
b	b	a	o	b	d
c	c	c	c	o	d
d	d	d	d	d	o

Table 33

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	o	a	d
b	b	b	o	b	d
c	c	c	c	o	d
d	d	d	d	d	o

In Case (iii) by Lemma 7, there exist 3 BCK-algebra of order four. So, using the entries in the corresponding cells, given in Tables 18, 19 and 21 respectively, will fill the blank cells, so the resulting BCI-algebra are shown as follows:

Table 34

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	o	o	d
b	b	a	o	a	d
c	c	a	a	o	d
d	d	d	d	d	o

Table 35

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	o	o	d
b	b	a	o	a	d
c	c	c	c	o	d
d	d	d	d	d	o

Table 36

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	o	o	d
b	b	b	o	b	d
c	c	c	c	o	d
d	d	d	d	d	o

In Case (iv) by Lemma 11 [2], there exist 2 BCK-algebra of order four. So, using the entries in the corresponding cells, given in Table18[2] and 19[2] respectively, will fill the blank cells, so the resulting BCI-algebra are shown as follows:

Table 37

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	a	o	d
b	b	b	o	o	d
c	c	b	c	o	d
d	d	d	d	d	o

Table 38

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	a	o	d
b	b	b	o	o	d
c	c	c	c	o	d
d	d	d	d	d	o

In Case (v) by Lemma 8, there exist 6 BCK-algebra of order four. So, using the entries in the corresponding cells, given in Table 23, 24, 25, 26, 27 and 28 respectively, will fill the blank cells, so the resulting BCI-algebra are shown as follows:

Table 39

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	o	o	d
b	b	a	o	o	d
c	c	a	a	o	d
d	d	d	d	d	o

Table 40

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	o	o	d
b	b	a	o	o	d
c	c	b	a	o	d
d	d	d	d	d	o

Table 41

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	o	o	d
b	b	a	o	o	d
c	c	b	c	o	d
d	d	d	d	d	o

Table 42

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	o	o	d
b	b	b	o	o	d
c	c	b	a		d
d	d	d	d	d	o

Table 43

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	o	o	d
b	b	b	o	o	d
c	c	c	b	o	d
d	d	d	d	d	o

Hence, the total number of proper BCI-algebras of order five with o(M)=4 is1+2+3+6+2=14.

Theorem 5

Let X be a proper BCI-algebra with o(X)=5 and M be its BCK-part M. Then, there exist 30 such BCI-algebras.

Proof

Let X be a proper BCI-algebra with o(X)=5 and M be its BCK-part. Then there are following possibilities for the BCK-part M:

(i) o(M)=1 (ii) o(M)=2 (iii) co(M)=3 (iv) o(M)=4

We have seen in Theorem 1[2], Theorem 4[2], Theorem 5, and Theorem 6 that there exist 1, 5, 11, and 14 proper BCI-algebras in each case respectively. Hence, the total number of proper BCI-algebras of order fives is 1+5+11+14=31.

Table 44

*	o	a	b	c	d
o	o	o	o	o	d
a	a	o	o	o	d
b	b	b	o	o	d
c	c	b	c	o	d
d	d	d	d	d	o

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On The Convergence Of Fixed Slope Iterations

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Abstract. We provide a local as well as a semilocal convergence analysis for a certain class of fixed slope iterations in a Banach space setting. Using a weaker Hölder condition on the operator involved and more precise estimates than in [1], [2] we provide in the semilocal case: finer error estimates on the distances involved and an at least as precise information on the location of the solution; in the local case: a larger radius of convergence. Finally numerical examples are used to compare favorably our results with earlier ones [1], [2].

AMS (MOS) Subject Classification: 65H10, 65G99, 47H17, 49M15.
Keywords: Banach space, Hölder/center Hölder continuity, contraction mapping principle, fixed slope iterations, radius of convergence.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of a nonlinear equation

$$F(x) = 0, \tag{1. 1}$$

where F is a Fréchet-differentiable operator, defined on an open convex subset D of a Banach space X with values in a Banach space Y.

Recently for $A^{-1} \in L(Y, X)$, M. Ahues [1], [2] used the fixed slope iteration

$$x_{n+1} = x_n - A^{-1}F(x_n) \qquad (x_n \in D)(n \geq 0) \tag{1. 2}$$

to approximate x^*. Ahues provided a local as well as a semilocal convergence for method (1. 2) under the Hölder continuity condition

$$\|F'(x) - F'(y)\| \leq \bar{l}\, \|x - y\|^{\lambda} \tag{1. 3}$$

for all $x, y \in D$, some $\bar{l} > 0$ and $\lambda \in (0, 1]$.
Iteration (1. 2) is a special case of a Newton-like method of the form

$$x_{n+1} = x_n - A(x_n)^{-1}F(x_n) \quad (x_0 \in D), (n \geq 0). \tag{1. 4}$$

Here $A(x)^{-1} \in L(Y, X)$ and approximates $F'(x)^{-1}$ in some sense. Many authors have provided convergence results for method (1. 4) under various assumptions.

Therefore we can refer the reader to [1]–[5] and the references there for convergence results concerning iteration (1. 2). However we decided to employ a direct approach and using center-Hölder condition (which is actually needed)

$$\left\|A^{-1}(F'(x) - F'(x_0))\right\| \leq l_0 \left\|x - x_0\right\|^{\lambda} \tag{1. 5}$$

for all $x \in D$, some $l_0 > 0$ instead of stronger (1. 3) we provided a convergence analysis with the advantages over the corresponding as stated above in the abstract of the paper. Note that even if the affine variant form of (1. 3)

$$\left\|A^{-1}(F'(x) - F'(y)\right\| \leq l \left\|x - y\right\|^{\lambda} \tag{1. 6}$$

was used in [2] instead of (1. 3) still since

$$l_0 \leq l \tag{1. 7}$$

holds in general (and $\frac{l}{l_0}$ can be arbitrarily large) [3],[4], the advantages of our approach still hold true.

Finally we complete this study with numerical examples where our results compare favorably with the corresponding ones in [1], [2].

2. Semilocal convergence analysis of method (1. 2).

We can show the following semilocal convergence theorem for fixed slope method (1. 2).

Theorem 1. *Under condition (1. 5) further assume:*
there exist parameters $\delta \geq 0$, $\eta > 0$ such that

$$\left\|A^{-1}(F'(x_0) - A)\right\| \leq \delta \tag{2. 1}$$

$$\left\|A^{-1}F(x_0)\right\| \leq \eta \tag{2. 2}$$

$$\delta < 1 \tag{2. 3}$$

$$h_A = l_0 \eta^{\lambda} \leq \lambda^{\lambda} \left(\frac{1-\delta}{1+\lambda}\right)^{1+\lambda} = g(\lambda, \delta) \tag{2. 4}$$

and

$$\bar{U}(x_0, r_0) = \{x \in X \,|\, \|x - x_0\| \leq r_0| \subseteq D, \tag{2. 5}$$

where r_0 is the smallest positive zero of function f defined on $[0, +\infty)$, and given by

$$f(r) = l_0 r^{1+\lambda} + (\delta - 1)r + \eta = 0. \tag{2. 6}$$

Then sequence $\{x_n\}$ $(n \geq 0)$ generated by method (1. 2) remains in $\bar{U}(x_0, r_0)$ for all $n \geq 0$ and converges to a unique solution x^* of equation $F(x) = 0$. Moreover the following estimates hold for all $n \geq 0$:

$$\|x_n - x^*\| \leq \frac{q^n}{1-q}\eta, \tag{2. 7}$$

where the geometric ratio q is given by

$$0 \leq q = l_0 r_0^{\lambda} + \delta < 1. \tag{2. 8}$$

Proof. Function f is strictly convex, and vanishes at the unique

$$r_1 = \left[\frac{1-\delta}{(1+\lambda)l_0}\right]^{\frac{1}{\lambda}} > 0$$

at which

$$f(r_1) = \eta - \frac{\lambda r_1(1-\delta)}{1+\lambda} \leq 0.$$

Moreover, we have

$$f(0) = \eta > 0.$$

That is

$$f(0)f(r_1) \leq 0.$$

It follows by the intermediate value theorem that function f has a smallest zero $r_0 \in (0, r_1]$.

We shall apply the contraction mapping principle [4] on the operator $G : \bar{U}(x_0, r_0) \to Y$ given by

$$G(x) = x - A^{-1}F(x). \tag{2. 9}$$

We also need to define the auxiliary operator

$$H : \bar{U}(x_0, r_0) \to Y$$

by

$$H(x) = F(x_0) + A(x - x_0) - F(x). \tag{2. 10}$$

Using (1. 5), (2. 1) and (2. 10)we get

$$\begin{aligned} \|A^{-1}H'(x)\| &= \|A^{-1}[A - F'(x_0) + F'(x_0) - F'(x)]\| \\ &\leq \|A^{-1}[A - F'(x_0)]\| + \|A^{-1}[F'(x_0) - F'(x)]\| \\ &\leq l_0 r_0^{\lambda} + \delta. \end{aligned} \tag{2. 11}$$

Moreover by (2. 2), (2. 9) and (2. 11) we obtain

$$\begin{aligned} \|G(x) - x_0\| &= \|A^{-1}(H(x) - F(x_0))\| \\ &\leq \|A^{-1}H(x)\| + \|A^{-1}F(x_0)\| \\ &\leq \|A^{-1}((H(x) - H(x_0))\| + \eta \\ &\leq (l_0 r_0^{\lambda} + \delta) r_0 + \eta = r_0, \end{aligned} \tag{2. 12}$$

by the choice of r_0.

That is we showed $G(\bar{U}(x_0, r_0)) \subseteq \bar{U}(x_0, u_0)$. Furthermore for all $x \in \bar{U}(x_0, r_0)$ using (2. 9) we get in turn

$$\begin{aligned} \|G'(x)\| &= \|A^{-1}(A - F'(x))\| = \|A^{-1}[A - F'(x_0) + F'(x_0) - F'(x)]\| \\ &= \|A^{-1}H'(x)\| \leq q := l_0 r_0^{\lambda} + \delta \\ &\leq l_0 r_1^{\lambda} + \delta = \frac{1+\lambda\delta}{1+\lambda} < 1, \end{aligned} \tag{2. 13}$$

which shows that operator G is a contraction on $\bar{U}(x_0, r_0)$.

The result now follows from the contraction mapping principle.

That completes the proof of the theorem. □

Remark 2. If $l_0 = l$ our Theorem 1 reduces to Theorem 5 in [2, p.385]. Otherwise it is an improvement over it. Indeed, let us denote by $\bar{h}_A, \bar{r}_0, \bar{f}$ and $\bar{q}$ the quantities corresponding to h_A, r_0, f and q given by

$$\bar{h}_A = l\eta^\lambda \leq \lambda^\lambda \left[\frac{1-\delta}{1+\lambda}\right]^{1+\lambda} = g(\lambda, \delta) \tag{2. 14}$$

$$\bar{f}(r) = lr^{1+\lambda} + (\delta - 1)r + \eta, \tag{2. 15}$$

$$\bar{q} = l\bar{r}_0^\lambda + \delta, \tag{2. 16}$$

and $\bar{r}_0$ denoting the smallest positive zero of function $\bar{f}$. It follows that

$$\bar{h}_A \leq g(\lambda, \delta) \Longrightarrow h_A \leq g(\lambda, \delta) \tag{2. 17}$$

but not vice versa (only if $l_0 = l$)

$$r_0 < \bar{r}_0 \tag{2. 18}$$

and

$$q < \bar{q} \tag{2. 19}$$

Estimates (2. 17)-(2. 19) justify the claims made in the abstract for the semilocal case.

Remark 3. If $\lambda = 1$ (Lipschitz case) and $A = F'(x_0)$ (modified Newton method) parameters r_0 and q can be given in closed form by

$$r_0 = \frac{1 - \sqrt{1 - 4l_0\eta}}{2l_0}, \tag{2. 20}$$

and

$$q = \frac{1}{2}(1 - \sqrt{1 - 4l_0\eta}), \tag{2. 21}$$

provided that

$$4l_0\eta \leq 1 \tag{2. 22}$$

whereas the corresponding ones in [2] are given by

$$\bar{r}_0 = \frac{1 - \sqrt{1 - 4l\eta}}{2l}, \tag{2. 23}$$

and

$$\bar{q} = \frac{1}{2}(1 - \sqrt{1 - 4l\eta}), \tag{2. 24}$$

provided that

$$4l\eta \leq 1. \tag{2. 25}$$

Let us provide a numerical example for the choices of A and λ as in Remark 2.

Example 1:

Let $X = Y = \mathbf{R}$, $x_0 = 1, D = [p, 2-p], p \in [0, .7]$, and define function F on D by

$$F(x) = x^3 - p. \tag{2. 26}$$

Using (1. 5),(1. 6),(2. 2) we obtain

$$l_0 = 3 - p < 2(2-p) = l, \eta = \frac{1}{3}(1-p). \tag{2. 27}$$

Condition (2. 25) [2] is violated since

$$4l\eta = \frac{8}{3}(2-p)(1-p) > 1, \text{ for all } p \in [0, .7]. \tag{2. 28}$$

That is there is no guarantee that method (1. 2) starting at $x_0 = 1$ converges to $x^* = \sqrt[3]{p}$. However, our condition (2. 22) holds for all

$$p \in \left[\frac{4-\sqrt{7}}{2}, .7\right],$$

since

$$4l_0\eta = \frac{4}{3}(3-p)(1-p) \leq 1. \tag{2. 29}$$

3. Local convergence for method (1. 2).

In order for us to study the local convergence of method (1. 2) we assume: there exist a solution x^* of equation (1. 1), and a positive parameter L such that for all $x \in D$

$$\left\|A^{-1}(F'(x) - F'(x^*))\right\| \leq L \left\|x - x^*\right\|^{\lambda}. \tag{3. 1}$$

Then we can show the main local convergence theorem:

Theorem 4. *Assume:*

$$s = \left\|A^{-1}(A - F'(x^*))\right\| + \sup_{x \in D} \left\|F'(x) - F'(x^*)\right\| < 1. \tag{3. 2}$$

Then sequence $\{x_n\} (n \geq 0)$ generated by method (1. 2) converges to x^* provided $x_0 \in U(x^*, R)$ for some $R > 0$ such that $\bar{U}(x^*, R) \subset D$ with

$$\left\|x_{n+1} - x^*\right\| \leq s \left\|x_n - x^*\right\|. \tag{3. 3}$$

If conditions (3. 1) and

$$\left\|A^{-1}(A - F'(x^*))\right\| = s_1 < 1 \tag{3. 4}$$

hold true then R is given by

$$R = \left(\frac{1 - s_1}{L}\right)^{\frac{1}{\lambda}}. \tag{3. 5}$$

Proof. We shall show that sequence $\{x_n\}$ remains in $\bar{U}(x^*, R)$ and (3. 3) holds for all $n \geq 0$. By hypothesis $x_0 \in \bar{U}(x^*, R)$. Then for $x_k \in \bar{U}(x^*, R)$, the point

$$x_k(t) = (1-t)x^* + tx_k \in \bar{U}(x^*, R) \tag{3. 6}$$

for all $t \in [0, 1]$.Using (1. 2) we obtain the identity.

$$x_{k+1} - x^* = x_k - x^* - A^{-1} \int_0^1 F'(x_k(t))(x_k - x^*)dt \tag{3. 7}$$

$$= A^{-1} \int_0^1 [(A - F'(x^*)) + (F'(x^*) - F'(x_k(t))](x_k - x^*)dt$$

In view of (3. 2) and (3. 7) we obtain

$$\left\|x_{k+1} - x^*\right\| \leq s \left\|x_k - x^*\right\| < R, \tag{3. 8}$$

which shows $x_{k+1} \in \bar{U}(x^*, R)$, (3. 3) and $\lim x_k = x^*$.In case (3. 1) holds, we get

$$s \leq s_1 + L \left\|x_k - x^*\right\|^{\lambda} < s_1 + LR^{\lambda} = 1, \tag{3. 9}$$

by the choice of R, which shows (3. 2).That completes the proof of the Theorem. □

Remark 5. Our Theorem 4 provides an actual radius of convergence R under marker hypotheses than in the corresponding Theorem 4 in [2].

We will now finish this study by providing numerical examples. For simplicity we choose

$$A = F'(x^*). \tag{3. 10}$$

Example 2:

Let $X = Y = \mathbf{R}$, $x^* = 0$, $D = \bar{U}(0,1)$ and define function F on D by

$$F(x) = e^x - 1. \tag{3. 11}$$

Using (3. 1),(3. 4), (3. 10), and (3. 11) we obtain

$$s_1 = 0, L = e - 1, \quad \text{and } \lambda = 1. \tag{3. 12}$$

In view of (3. 5) we obtain

$$R = \frac{1}{e-1} = .581976706. \tag{3. 13}$$

Example 3:

Let $X = Y = \mathbf{R}$,

$$x^* = \frac{9}{4}, \quad D = [.81, 6.25]$$

and define function F on D by

$$F(x) = \frac{2}{3}x^{\frac{3}{2}} - x \tag{3. 14}$$

This time we obtain

$$s_1 = 0, L = 1, \text{and } \lambda = \frac{1}{2}, \tag{3. 15}$$

since

$$F'(x^*) = \frac{1}{2}, \text{ and}$$

$$\begin{aligned} \|F'(x^*)^{-1}[F'(x) - F'(x^*)]\| &= 2\left|\sqrt{x} - \frac{3}{2}\right| \\ &= 2\left|\sqrt{x} - \frac{3}{2}\right|^{\frac{1}{2}}\left|\sqrt{x} - \frac{3}{2}\right|^{\frac{1}{2}} \\ &\leq |x - x^*|^{\frac{1}{2}} \end{aligned} \tag{3. 16}$$

holds for all $x \in D$. In view of (3. 5) and (3. 15), we obtain

$$R = 1. \tag{3. 17}$$

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A Note On BCI-Algebras Of Order Six

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Abstract. In [4], author has characterized the BCI-algebras of order 6 by considering BCK-parts of order 5, 4, 3, 2 and 1 respectively vide his theorem on page 34. The total of all these cases is 88+14+6+1+1=110. According to the authors these are known as simpler cases. Again they claimed that there are some complicated BCI-algebras with BCK-parts of order 4, 3 and 2 respectively. He computed 69 such complicated BCI-algebras. Adding simpler and complicated BCI-algebras of order 6, the number springs out to be 110+69=179. In this note it is worked out and shown that the number of proper BCI-algebras of order six up to isomorphism is 197 instead of 179.

AMS (MOS) Subject Classification: 03G25, 06F35
Keywords and Phrases: BCI-algebras, BCI-algebras of order Six.

1. Introduction

K. Iseki [3] introduced the theory of BCI-algebras and established some of its properties. On wards, so many eminent researchers have contributed to the discipline. S.K. Goel in [2], as a first step, characterized BCI-algebras of order three and partially BCI-algebras of order four. In [4], author has characterized the BCI-algebras of order 6 by considering BCK-parts of order 5, 4, 3, 2 and 1 respectively vide his theorem on page 34. The total of all these cases is 88+14+6+1+1=110. According to the author these are known as simpler cases. Again he claimed that there are some complicated BCI-algebras with BCK-parts of order 4, 3 and 2 respectively. He computed 69 such complicated BCI-algebras. Adding simpler and complicated BCI-algebras of order 6, the number springs out to be 110+69=179. In this note it is worked out and shown that the number of proper BCI-algebras of order six up to isomorphism is 197 instead of 179. In [4], in case of complicated BCI-algebras, it was shown that there are 69 complicated proper BCI-algebras of order 6. However, it is worked out and found that regarding isomorphic BCI-algebras as equal, these are 68 instead of 69.

2. Preliminaries

1.Definition [3]
A BCI-algebra X is an abstract algebra $(X, *, o)$ of type (2, 0), satisfying the following conditions; for all $x, y, z \in X$,

1 $((x * y) * (x * z)) * (z * y) = o$
2 $(x * (x * y)) * y = o$
3 $x * x = o$
4 $x * y = o = y * x \Rightarrow x = y$

where $x * y = o \Leftrightarrow x \leq y$
In a BCI-algebra X, the set $M = \{x \in X : o * x = o\}$ is a subalgebra and is called the BCK-part of X. A BCI-algebra X is called proper if $X - M \neq \phi$.

1.2. Definition [1]
Let X be a BCI-algebra and $x, y \in X$. Then x, y are said to be **comparable** if and only if x*y = o or y*x = o. Further, we shall say that x precedes y and y succeeds x if and only if x*y = o and denote it by $x \to y$ or $x \leq y$.

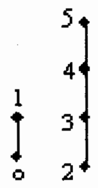
If x and y are not comparable, then they are said to be **incomparable**.

3. On BCI-algebras of order six

In [4], the author defined the partial order on a proper BCI-algebra X by making Hasse diagram.

Note that the Hasse diagram associated with table no. 56 on page 41 in [4] is given as follows:

Fig. (1)



This Hasse diagram implies that $4 \leq 5 \Rightarrow 4 * 5 = o$, but in table 56 in [4] it is given that 4*5 = 1, a contradiction to the partial order defined by figure (1).

Let us consider the figure 2 and figure 3, given below:

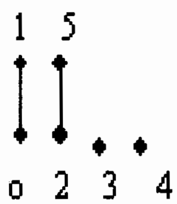


Fig. 2

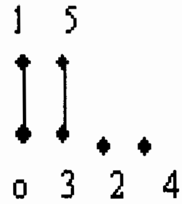


Fig. 3

Note that in [4], it was shown that the number of BCI-algebras of order 6 with the partial order defined by Fig. 2 is 2 for which the binary operation is given by tables 62 and 63 on page 42. It is pointed out that entry in (3, 4)th cell of table 62 should be 2 instead of 5 according to bonafide properties of BCI-algebras; otherwise tables 62 and 63 coincide. Further, number of BCI-algebras of order 6 with the partial order defined by Fig. 3 is 3 for which the operation is given by tables 64, 65 and 66 on page 42 in [4].

Now consider the multiplication table 62 (after correction) and table 64 along with their associated diagrams on page 42 in [4], which are shown below:

Table 62

*	o	1	2	3	4	5
o	o	o	2	3	4	2
1	1	o	2	3	4	2
2	2	2	o	4	3	o
3	3	3	4	o	2	4
4	4	4	3	2	o	3
5	5	2	1	4	3	o

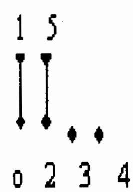


Fig. 2

Table 64

*	o	1	2	3	4	5
o	o	o	2	3	4	3
1	1	o	2	3	4	3
2	2	2	o	4	3	4
3	3	3	4	o	2	o
4	4	4	3	2	o	2
5	5	3	4	1	2	o

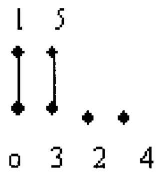


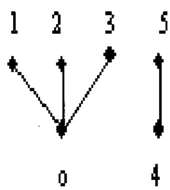
Fig. 3

Since the choice of ordering of elements contained in any BCI-branch of order 2 is arbitrary, therefore **Fig. 2** and **Fig. 3** coincide. Also note that table 62 is isomorphic to table 64 under the mapping $f : Table62 \rightarrow Table64$ defined as f(o)=o, f(1)=1, f(2)=3, f(3)=2, f(4)=4 and f(5)=5.

Thus it follows that the number of complicated BCI-algebras of order 6, given in [4], is 68 instead of 69.

Moreover in [4], in case of complicated BCI-algebras of order 6, all the authors did not consider the following five cases:

Case (i) : Let $X = \{o, 1, 2, 3, 4, 5\}$ be a BCI-algebra with $M = A(o) = \{o, 1, 2, 3\}$ as its BCK-part. Then $X - M = \{4, 5\}$. The partial order on X is defined as $o \leq 1$, $o \leq 2$, $o \leq 3$, $4 \leq 5$. Geometrically we represent it as follows:



In this case regarding isomorphic BCI-algebras as equal, there are two such BCI-algebras of order 6. The multiplication tables representing these BCI-algebras are given below:

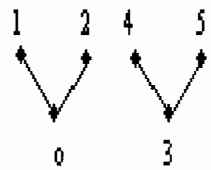
Table 1

*	o	1	2	3	4	5
o	o	o	o	o	4	4
1	1	o	1	1	4	4
2	2	2	o	2	4	4
3	3	3	3	o	4	4
4	4	4	4	4	o	o
5	5	4	5	5	1	o

Table 2

*	o	1	2	3	4	5
o	o	o	o	o	4	4
1	1	o	1	1	5	4
2	2	2	o	2	4	4
3	3	3	3	o	4	4
4	4	4	4	4	o	o
5	5	4	5	5	1	o

Case (ii) : Let $X = \{o, 1, 2, 3, 4, 5\}$ be a BCI-algebra with $M = A(o) = \{o, 1, 2\}$ as its BCK-part. Then $X - M = \{3, 4, 5\}$. The partial order on X is defined as $o \leq 1$, $o \leq 2$, $3 \leq 4$, $3 \leq 5$. Geometrically we represent it as follows:



In this case regarding isomorphic BCI-algebras as equal, there are 5 such BCI-algebras of order 6. The multiplication tables representing these BCI-algebras are given below:

Table 3

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	1	3	3	3
2	2	2	o	3	3	3
3	3	3	3	o	o	o
4	4	3	4	1	o	1
5	5	3	5	1	1	o

Table 4

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	1	3	3	3
2	2	2	o	3	3	3
3	3	3	3	o	o	o
4	4	4	3	2	o	2
5	5	3	5	1	1	o

Table 5

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	1	5	5	3
2	2	2	o	3	3	3
3	3	3	3	o	o	o
4	4	4	3	2	o	2
5	5	3	5	1	1	o

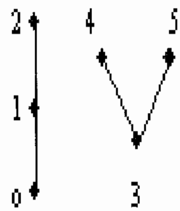
Table 6

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	1	3	3	3
2	2	2	o	4	3	4
3	3	3	3	o	o	o
4	4	4	3	2	o	2
5	5	3	5	1	1	o

Table 7

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	1	5	5	3
2	2	2	o	4	3	4
3	3	3	3	o	o	o
4	4	4	3	2	o	2
5	5	3	5	1	1	o

Case (iii) : Let $X = \{o, 1, 2, 3, 4, 5\}$ be a BCI-algebra with $M = A(o) = \{o, 1, 2\}$ as its BCK-part. Then $X - M = \{3, 4, 5\}$. The partial order on X is defined as $o \leq 1 \leq 2$, $3 \leq 4$, $3 \leq 5$. Geometrically we represent it as follows:



In this case there are 10 such BCI-algebras of order 6. The multiplication tables representing these BCI-algebras are given below:

Table 8

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	o	3	3	3
2	2	1	o	3	3	3
3	3	3	3	o	o	o
4	4	3	3	1	o	1
5	5	3	3	1	1	o

Table 9

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	o	3	3	3
2	2	1	o	5	5	3
3	3	3	3	o	o	o
4	4	3	3	1	o	1
5	5	3	3	1	1	o

Table 10

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	o	3	3	3
2	2	1	o	4	3	4
3	3	3	3	o	o	o
4	4	3	3	1	o	1
5	5	3	3	1	1	o

Table 11

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	o	3	3	3
2	2	1	o	3	3	3
3	3	3	3	o	o	o
4	4	3	3	1	o	1
5	5	5	3	2	2	o

Table 12

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	o	3	3	3
2	2	2	o	3	3	3
3	3	3	3	o	o	o
4	4	3	3	1	o	1
5	5	3	3	1	1	o

Table 13

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	o	3	3	3
2	2	2	o	3	3	3
3	3	3	3	o	o	o
4	4	4	3	2	o	2
5	5	3	3	1	1	o

Table 14

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	o	3	3	3
2	2	2	o	4	3	4
3	3	3	3	o	o	o
4	4	4	3	2	o	2
5	5	3	3	1	1	o

Table 15

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	o	3	3	3
2	2	2	o	3	3	3
3	3	3	3	o	o	o
4	4	3	3	1	o	1
5	5	5	3	2	2	o

Table 16

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	o	3	3	3
2	2	2	o	5	5	3
3	3	3	3	o	o	o
4	4	3	3	1	o	1
5	5	5	3	2	2	o

Table 17

*	o	1	2	3	4	5
o	o	o	o	3	3	3
1	1	o	o	3	3	3
2	2	2	o	3	3	3
3	3	3	3	o	o	o
4	4	4	3	2	o	2
5	5	5	3	2	2	o

Case (iv) : Let $X = \{o, 1, 2, 3, 4, 5\}$ be a BCI-algebra with $M = A(o) = \{o, 1\}$ as its BCK-part. Then $X - M = \{2, 3, 4, 5\}$. The partial order on X is defined as $o \le 1$, $2 \le 3 \le 4$, $2 \le 3 \le 5$. The Cayley table and the Hasse diagram representing such BCI-algebra is given as follows:

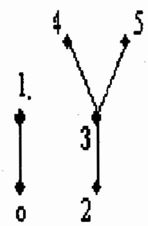


Table 18

*	o	1	2	3	4	5
o	o	o	2	2	2	2
1	1	o	2	2	2	2
2	2	2	o	o	o	o
3	3	2	1	o	o	o
4	4	2	1	1	o	1
5	5	2	1	1	1	o

Case (v) : Let $X = \{o, 1, 2, 3, 4, 5\}$ be a BCI-algebra with $M = A(o) = \{o, 1\}$ as its BCK-part. Then $X - M = \{2, 3, 4, 5\}$. The partial order on X is defined as $o \leq 1$, $2 \leq 3$, $2 \leq 4$, $2 \leq 5$. The Cayley table and the Hasse diagram representing such BCI-algebra is given as follows:

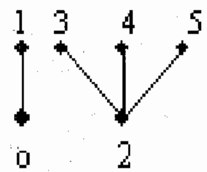


Table 19

*	o	1	2	3	4	5
o	o	o	2	2	2	2
1	1	o	2	2	2	2
2	2	2	o	o	o	o
3	3	2	1	o	1	1
4	4	2	1	1	o	1
5	5	2	1	1	1	o

Thus it follows that there are 19 more proper BCI-algebras of order 6 represented by table 1-table 19 in case of complicated BCI-algebras.

Further from cases (i)-(v), it follows that the total number of proper BCI-algebras of order 6 is 197 instead of 179.

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On The Singular LQ Control Problem For Nonregular Implicit System

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Abstract. This paper concerns with the LQ control problem for nonregular implicit system. Main objective of this paper is to solve LQ control problem for nonregular implicit system by utilizing the equivalent relationship principle between two optimal control problems. By constructing a LQ control problem subject to the standard state space system which is equivalent to the original problem, we point out that solvability of the new one is the sufficient condition to guarantees existence and uniqueness of solution for the original LQ control problem.

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Keywords: Nonregular implicit systems, Singular LQ problem.

1. Introduction

The linear quadratic (LQ) control problem is important in control and optimization theory and has been used in practice widely. In another hand, the implicit system has received considerable interest over the last decade because it has the specificity in the structure of its solution.

Thus the LQ control problem subject to the implicit system has a great potential for the system modeling, because they can preserve the structure of physical systems and can include non dynamic constraint and impulsive element.
To the best of the author's knowledge, these issues have been investigated in depth for the cases where the implicit system is regular and the control weighting matrix in objective function being positive definite (see [2], [3], [5]). However, not much work has been reported for the cases in which the constraint is nonregular implicit system and the control weighting matrix in objective function being positive semidefinite. Geerts [4] discussed the LQ control problem for nonregular implicit system via linear matrix inequalities, but the finding of this work involves impulse over optimal control-state. The LQ control problem for implicit system with the output free of control is considered in [7] in which the optimal control-state pair that free impulse is obtained. Nonetheless, this finding is inadequate to handle cases in which the output equation depends on control vector.

In this paper, we combine the control vector terms into the output equation, thus the control-state weighting matrix term in objective function appear, and we obtain some new results which significantly disparate from the finding in [7].

To solve the problem, we utilize the equivalent relationship principle between two optimal control problems, so that we can construct a new LQ control problem, i.e., LQ control problem subject to standard state space system, which is equivalent to the original LQ control problem. By applying the existing theories over this new LQ control problem, we may obtain the solution of the original LQ control problem.

Notation: Throughout this paper, the superscript "T" stands for the transpose, I is the identity matrix with appropriate dimension, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices, $C_p^+[\mathbb{R}^n]$denotes the n-dimensional piecewise continuous functions space with domain in $[0, \infty]$, and $\mathbb{C}$ denotes set of complex number.

2. Preliminaries and Problem Statement

Consider the following continuous time implicit system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t),\ t \geq 0,\ Ex(0) = x_0 \\ y &= Cx(t) + Du(t), \end{aligned} \tag{2. 1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}^r$ denotes the control (input) vector and $y(t) \in \mathbb{R}^m$ denotes the output vector. The matrices $E, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times r}$ are constant, with rank $E \equiv p < n$. We often write (E, A, B, C, D) as a shorthand notation for system (2.1). The system (E, A, B, C, D) is said to be *regular* if $\det(sE - A) \neq 0$ for some $s \in \mathbb{C}$, and it is called as *nonregular* if $\det(sE - A) = 0$ for each $s \in \mathbb{C}$. It is well known that the solution of (2.1) exists and unique if it is regular for an admissible initial state $x_0 \in \mathbb{R}^n$. However, it is possible to have many solutions, or no solution at all, if it is nonregular.

Next, for a given admissible initial state $x_0 \in \mathbb{R}^n$, we consider the associated objective function (cost functional) as follows:

$$J(u(.), x_0) = \int_0^\infty y^T(t)y(t)dt. \tag{2. 2}$$

In general, the problem of determining the control $u(t) \in \mathbb{R}^r$ which minimizes the cost functional (2.2) and satisfies the dynamic system (2.1) for an admissible initial state $x_0 \in \mathbb{R}^n$, is often called as LQ *control problem for implicit system.* If $D^T D$ is positive semidefinite, it is called as *singular* LQ control problem for implicit system. We denote, for simplicity, this LQ control optimal as Ω. Next, we define the set of admissible control-state pairs of problem Ω by:

$$A_{\rm ad} \equiv \{(u(.), x(.)) \mid u(.) \in C_p^+[\mathbb{R}^r] \text{ and } x(.) \in C_p^+[\mathbb{R}^n] \text{ satisfy (2.1) and } J(u(.), x_0) < \infty\}.$$

The optimization problem under consideration is to find the pair $(u^*, x^*) \in A_{\rm ad}$ for a given admissible initial condition $x_0 \in \mathbb{R}^n$, such that

$$J(u^*, x_0) = \underset{(u(.),x(.))\in A_{\rm ad}}{\text{minimize}} \; J(u(.), x_0), \tag{2. 3}$$

under the assumption that (2.1) is nonregular and $D^T D$ is positive semidefinite.

Assumption 1 The implicit system (2.1) is solvable and impulse controllable.

Note that the Assumption 1 implies that there exists an impulse free control $u(t)$ so that $J(u(.), x_0)$ exist and finite [2], and it follows that $A_{\rm ad}$ is not an empty set. The following definition has actually been mentioned in [8], and it is restated in an equivalent form in [7]. It is useful in the sequel.

Definition 1. Two systems (E, A, B, C, D) and $(\bar{E}, \bar{A}, \bar{B}, \bar{C}, D)$ are termed restricted system equivalent (r.s.e.), denote as $(E, A, B, C, D) \sim (\bar{E}, \bar{A}, \bar{B}, \bar{C}, D)$, if there exists nonsingular matrices $M, N \in \mathbb{R}^{n\times n}$such that their associated system matrices are related by $MEN = \bar{E}$, $MAN = \bar{A}$, $MB = \bar{B}$ and $CN = \bar{C}$.

Remark 2. The operations of r.s.e. correspond to the constant nonsingular transformations of (2.1) itself and of the basis in the space of internal variables x. The behavior of x in the original system may thus be simply recovered from the behavior of any system r.s.e. to it. These operations therefore constitute an eminently safe set of transformations that are unlikely to destroy any important properties of the system. Furthermore, such operations suffice to display the detailed structure of the original system.

Definition 3. Two optimal control problems are said to be equivalent if there exist a bijection between the two sets of admissible control - state pairs of the problems, and the quadratic cost value of any image is equal to that of corresponding preimage.

Obviously, definition 3 conforms to the reflexivity, symmetry, and transitivity of an equivalent relation, thus the two equivalent optimal control problems will have the same solvability, uniqueness of solution and optimal cost. Thus solving one can be replaced by solving the other.

3. Constructing into Equivalent LQ Control Problem

Firstly, we will construct a LQ control problem which is equivalent to the original LQ control problem. Since rank $E = p < n$, then by applying the Singular Value Decomposition (SVD) theorem[6] to the matrix E, one can obtain a nonsingular matrices $M, N \in \mathbb{R}^{n\times n}$, such that

$$MEN = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}.$$

Accordingly, let

$$MAN = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \; MB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \; CN = \begin{pmatrix} C_1 & C_2 \end{pmatrix},$$

and

$$N^{-1}x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{3. 1}$$

where $A_{11} \in \mathbb{R}^{p\times p}$, $A_{12} \in \mathbb{R}^{n\times(n-p)}$, $A_{21} \in \mathbb{R}^{(n-p)\times p}$, $A_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$, $B_1 \in \mathbb{R}^{p\times r}$, $B_2 \in \mathbb{R}^{(n-p)\times r}$, $C_1 \in \mathbb{R}^{m\times p}$, $C_2 \in \mathbb{R}^{m\times(n-p)}$, $x_1 \in \mathbb{R}^p$ and $x_2 \in \mathbb{R}^{n-p}$. Therefore, for a given admissible initial state $x_0 \in \mathbb{R}$, the system (2.1) is $r.s.e$ to the system

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), \; x_1(0) = x_{10} \in \mathbb{R}^p \\ 0 &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \\ y(t) &= C_1x_1(t) + C_2x_2(t) + Du(t) \end{aligned} \tag{3. 2}$$

where $x_{10} = \begin{pmatrix} I_p & 0 \end{pmatrix} Mx_0$.

Using the expression (3.2), the objective function (2.2) can be changed into

$$J_1(u(.), x_{10}) = \int_0^\infty \begin{pmatrix} x_1(t) \\ x_2(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} C_1^TC_1 & C_1^TC_2 & C_1^TD \\ C_2^TC_1 & C_2^TC_2 & C_2^TD \\ D^TC_1 & D^TC_2 & D^TD \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ u(t) \end{pmatrix} dt. \tag{3. 3}$$

Likewise, we have a new LQ problem which minimize $J_1(u(.), x_{10})$ subject to the dynamic system (3.2). Denote this LQ problem as Ω_1 and define the set of admissible control-state pairs of problem Ω_1 by :

$A^1_{\text{ad}} = \{(u(.), x_1(.), x_2(.)) \mid u(.) \in C_p^+[\mathbb{R}^r], x_1(.) \in C_p^+[\mathbb{R}^p]$ and $x_2(.) \in C_p^+[\mathbb{R}^{n-p}]$ satisfy (3.2)and $J_1(u(.), x_{10}) < \infty\}$.

By virtue of Definition 3, it is easily seen that the LQ control problem Ω_1 is equivalent to Ω.

Lemma 4. [2] *The implicit system* (2.1) *is impulse controllable if and only if*

$$rank\begin{pmatrix} A_{22} & B_2 \end{pmatrix} = n - p.$$

Furthermore, since the matrix $\begin{pmatrix} A_{22} & B_2 \end{pmatrix}$has full row rank, then the solution of the second equation of (3.2) can be stated as

$$\begin{pmatrix} x_2(t) \\ u(t) \end{pmatrix} = -\hat{A}^+A_{21}x_1(t) + Wv(t) \tag{3. 4}$$

for some $v(t) \in \mathbb{R}^r$ and for some full column rank matrix $W \in \mathbb{R}^{(n-p+r)\times r}$ satisfying $\begin{pmatrix} A_{22} & B_2 \end{pmatrix} W = 0$, where

$$\hat{A}^+ = \begin{pmatrix} A_{22} & B_2 \end{pmatrix}^T \left[\begin{pmatrix} A_{22} & B_2 \end{pmatrix}\begin{pmatrix} A_{22} & B_2 \end{pmatrix}^T\right]^{-1}$$

is the generalized inverse of the matrix $\begin{pmatrix} A_{22} & B_2 \end{pmatrix}$.

Remark 5. Note that the matrix W is not necessary unique. In the next section, we will show that although W is not unique, it preserves the uniqueness of the optimal control-state pair.

Using expression (3.4), we can create the following transformation

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ -\hat{A}^+ A_{21} & W \end{pmatrix} \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix}. \tag{3. 5}$$

Substituting (3.5) into the Ω_1, we obtain a new LQ control problem as follows

$$\Omega_2 : \quad \begin{array}{ll} \underset{(v,x_1)}{\text{minimize}} & J_2(v(.), x_{10}) \\ \text{subject to} & \begin{array}{l} \dot{x}_1(t) = \bar{A}x_1(t) + \bar{B}v(t), \;\; x_1(0) = x_{10} \\ y(t) \;= \bar{C}x_1(t) + \bar{D}v(t), \end{array} \end{array} \tag{3. 6}$$

where

$$J_2(v(.), x_{10}) = \int_0^\infty \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix} dt,$$

$$\begin{aligned} \bar{A} &= A_{11} - \begin{pmatrix} A_{12} & B_1 \end{pmatrix} \hat{A}^+ A_{21}, \; \bar{B} = \begin{pmatrix} A_{12} & B_1 \end{pmatrix} W, \\ \bar{C} &= C_1 - \begin{pmatrix} C_2 & D \end{pmatrix} \hat{A}^+ A_{21}, \; \bar{D} = \begin{pmatrix} C_2 & D \end{pmatrix} W, \\ Q_{11} &= \left[C_1 - \begin{pmatrix} C_2 & D \end{pmatrix} \hat{A}^+ A_{21}\right]^T \left[C_1 - \begin{pmatrix} C_2 & D \end{pmatrix} \hat{A}^+ A_{21}\right], \\ Q_{12} &= \left[C_1 - \begin{pmatrix} C_2 & D \end{pmatrix} \hat{A}^+ A_{21}\right]^T \begin{pmatrix} C_2 & D \end{pmatrix} W, \end{aligned}$$

and

$$Q_{22} = \left[\begin{pmatrix} C_2 & D \end{pmatrix} W\right]^T \begin{pmatrix} C_2 & D \end{pmatrix} W. \tag{3. 7}$$

Define the set of admissible control-state pairs of problem Ω_2 by

$$A^2_{\text{ad}} = \{(v(.), x_1(.)) \mid v(.) \in C_p^+[\mathbb{R}^r] \text{ and } x_1(.) \in C_p^+[\mathbb{R}^p] \text{ satisfy (3.6) and } J_2(v(.), x_{10}) < \infty\}.$$

It is obvious that the system (3.6) is a standard state space system with the state x_1, the control v and the output y, so Ω_2 is a LQ control problem for the standard state space system. Next, we need to show that Ω_1 is equivalent to Ω_2.

Theorem 6. *The transformation defined by* (3.5) *is a bijection from* A^2_{ad} *to* A^1_{ad}, *and thus the problem* Ω_2 *is equivalent to the problem* Ω_1.

Proof. It suffices to prove that the transformation (3.5) is a mapping, injection and surjection from A^2_{ad} to A^1_{ad}. For any $(v, x_1) \in A^2_{\text{ad}}$, $(u, (x_1, x_2))$ determined by (3.5) satisfies

$$\begin{aligned}
&\begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} \\
&= \left(\begin{array}{c|cc} A_{11} & A_{12} & B_1 \\ \hline A_{21} & A_{22} & B_2 \end{array}\right)\begin{pmatrix} I_p & 0 \\ -\hat{A}^+A_{21} & W \end{pmatrix}\begin{pmatrix} x_1 \\ v \end{pmatrix} \\
&= \left(\begin{array}{c|c} A_{11} - \begin{pmatrix} A_{12} & B_1 \end{pmatrix}\hat{A}^+A_{21} & \begin{pmatrix} A_{12} & B_1 \end{pmatrix}W \\ \hline A_{21} - \begin{pmatrix} A_{22} & B_2 \end{pmatrix}\hat{A}^+A_{21} & \begin{pmatrix} A_{22} & B_2 \end{pmatrix}W \end{array}\right)\begin{pmatrix} x_1 \\ v \end{pmatrix} \\
&= \begin{pmatrix} \bar{A} & \bar{B} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ v \end{pmatrix} \\
&= \begin{pmatrix} \dot{x}_1 \\ 0 \end{pmatrix}.
\end{aligned}$$

Hence $(u,(x_1,x_2)) \in A^1_{\text{ad}}$. Therefore, the transformation (3.5) is a mapping from A^2_{ad} to A^1_{ad}. Note that this transformation matrix has full column rank, and it follows that it is an injection from A^2_{ad} to A^1_{ad}. Next we will show that the transformation (3.5) is a surjection. Since the matrix $\begin{pmatrix} \begin{pmatrix} A_{22} & B_2 \end{pmatrix} \\ I_{n-p+r} \end{pmatrix}$ has full column rank, then

$$\begin{aligned}
\operatorname{rank}\begin{pmatrix} \hat{A}^+ & W \end{pmatrix} &= \operatorname{rank}\left[\begin{pmatrix} \begin{pmatrix} A_{22} & B_2 \end{pmatrix} \\ I_{n-p+r} \end{pmatrix}\begin{pmatrix} \hat{A}^+ & W \end{pmatrix}\right] \\
&= \operatorname{rank}\begin{pmatrix} I_{n-p} & 0 \\ \hat{A}^+ & W \end{pmatrix} \\
&= \operatorname{rank}\begin{pmatrix} I_{n-p} & 0 \\ 0 & W \end{pmatrix} \\
&= n-p+r,
\end{aligned}$$

and it follows that the matrix $\begin{pmatrix} \hat{A}^+ & W \end{pmatrix} \in \mathbb{R}^{(n-p+r)\times(n-p+r)}$ is nonsingular. Therefore for any $(u^0,(x_1^0,x_2^0)) \in A^1_{\text{ad}}$ we can take

$$\begin{pmatrix} \bar{x}_2^0 \\ v^0 \end{pmatrix} = \begin{pmatrix} \hat{A}^+ & W \end{pmatrix}^{-1}\begin{pmatrix} x_2^0 \\ u^0 \end{pmatrix},$$

and from the equation of (3.4) we have $\bar{x}_2^0 = -A_{21}x_1^0$. It follows that

$$\begin{aligned}
\begin{pmatrix} x_1^0 \\ x_2^0 \\ u^0 \end{pmatrix} &= \begin{pmatrix} I_p & 0 & 0 \\ 0 & \hat{A}^+ & W \end{pmatrix}\begin{pmatrix} x_1^0 \\ \bar{x}_2^0 \\ v^0 \end{pmatrix} \\
&= \begin{pmatrix} I_p & 0 & 0 \\ 0 & \hat{A}^+ & W \end{pmatrix}\begin{pmatrix} x_1^0 \\ -A_{21}x_1^0 \\ v^0 \end{pmatrix} \\
&= \begin{pmatrix} I_p & 0 & 0 \\ 0 & \hat{A}^+ & W \end{pmatrix}\begin{pmatrix} I_p & 0 \\ -A_{21} & 0 \\ 0 & I_r \end{pmatrix}\begin{pmatrix} x_1^0 \\ v^0 \end{pmatrix} \\
&= \begin{pmatrix} I_p & 0 \\ -\hat{A}^+A_{21} & W \end{pmatrix}\begin{pmatrix} x_1^0 \\ v^0 \end{pmatrix}
\end{aligned}$$

Thus (v^0, x_1^0) is the preimage of $(u^0, (x_1^0, x_2^0))$ under (3.5). Further, we know that (v^0, x_1^0) satisfies the first equation of the system (3.6), i.e.

$$\dot{x}_1^0 = \bar{A}x_1^0 + \bar{B}\,v^0, \;\; x_1^0(0) = x_{10},$$

so $(v^0, x_1^0) \in A_{\text{ad}}^2$. Therefore the transformation (3.5) is surjection from A_{ad}^2 to A_{ad}^1 and therefore problem Ω_2 is equivalent to Ω_1 by definition 3. □

4. Main Result

In the previous section, we have constructed a LQ problem which is equivalent to the original LQ problem. We have showed that Ω is equivalent to Ω_1, and Ω_1 is equivalent to Ω_2, hence Ω is equivalent to Ω_2. Therefore we suffice to utilize the existence and uniqueness conditions of optimal control-state pair for Ω_2.

In accordance to the linear optimal control theory for the standard state-space system [1], if the matrix Q_{22} is positive definite , the pair $(\bar{A}, \bar{B})$ is stabilizable and the pair $(\bar{A} - \bar{B}Q_{22}^{-1}Q_{12}^T, Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T)$ is detectable, then Ω_2 has a unique optimal control-state pair (v^*, x_1^*), where v^* is given by

$$v^* = -Lx_1^*, \tag{4. 1}$$

the state x_1^* is the solution of differential equation

$$\dot{x}_1(t) = (\bar{A} - \bar{B}L)x_1(t), \qquad x_1(0) = x_{10} \tag{4. 2}$$

with $L = Q_{22}^{-1}(Q_{12}^T + \bar{B}^T P)$, P is the unique positive semidefinite solution of the following algebraic Riccati equation:

$$0 = \bar{A}^T P + P\bar{A} + Q_{11} - (P\bar{B} + Q_{12})Q_{22}^{-1}(P\bar{B} + Q_{12})^T \tag{4. 3}$$

and every eigenvalue λ of $\bar{A} - \bar{B}L$ satisfies $\text{Re}(\lambda) < 0$.However, Q_{22} can be positive semidefinite, so we are now in a position to give the necessary and sufficient conditions in order that the matrix Q_{22} is positive definite. Moreover, we also need to establish the necessary and sufficient condition in order the pair $(\bar{A}, \bar{B})$ is stabilizable and $(\bar{A} - \bar{B}Q_{22}^{-1}Q_{12}^T, Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T)$ is detectable.

Theorem 7. *The following statements are equivalent :*

(*i*) Q_{22} *is positive definite*

(*ii*) $\text{rank}\begin{pmatrix} A_{22} & B_2 \\ C_2 & D \end{pmatrix} = n - p + r$

(*iii*) $\text{rank}\begin{pmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & D \end{pmatrix} = n + p + r.$

Proof. (*i*) ⇔ (*ii*) From (3.6) we have

$$Q_{22} = \left[\begin{pmatrix} C_2 & D \end{pmatrix} W\right]^T \begin{pmatrix} C_2 & D \end{pmatrix} W.$$

Since $Q_{22}\in \mathbb{R}^{r\times r}$ then $Q_{22}>0 \Leftrightarrow \text{rank}\left[\begin{pmatrix} C_2 & D \end{pmatrix} W\right] = r$. Since the matrix $\begin{pmatrix} \hat{A}^+ & W \end{pmatrix}$ is nonsingular then

$$\begin{aligned}\text{rank}\begin{pmatrix} A_{22} & B_2 \\ C_2 & D \end{pmatrix} &= \text{rank}\left[\begin{pmatrix} A_{22} & B_2 \\ \hline C_2 & D \end{pmatrix}\begin{pmatrix} \hat{A}^+ & W \end{pmatrix}\right] \\ &= \text{rank}\begin{pmatrix} I_{n-p} & 0 \\ \begin{pmatrix} C_2 & D \end{pmatrix}\hat{A}^+ & \begin{pmatrix} C_2 & D \end{pmatrix}W \end{pmatrix} \\ &= \text{rank}\begin{pmatrix} I_{n-p} & 0 \\ 0 & \begin{pmatrix} C_2 & D \end{pmatrix}W \end{pmatrix} \\ &= n-p+r.\end{aligned}$$

(*ii*) $\Leftrightarrow$ (*iii*) It is easy to see that

$$\begin{aligned}&\text{rank}\begin{pmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & D \end{pmatrix} \\ &\quad = \text{rank}\left[\begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & I_q \end{pmatrix}\begin{pmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & D \end{pmatrix}\begin{pmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & I_r \end{pmatrix}\right] \\ &\quad = \text{rank}\begin{pmatrix} 0 & I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \\ 0 & 0 & A_{22} & B_2 \\ 0 & 0 & C_2 & D \end{pmatrix} \\ &\quad = p+p+\text{rank}\begin{pmatrix} A_{22} & B_2 \\ C_2 & D \end{pmatrix} \\ &\quad = n+p+r.\end{aligned}$$

□

Theorem 8. *The pair $(\bar{A},\bar{B})$ is stabilizable if and only if*

$$\textit{rank}\begin{pmatrix} A-\lambda E & B \end{pmatrix} = n, \tag{4. 4}$$

for each λ with $Re(\lambda)\geq 0$.

Proof. According to the stability theory for linear time invariant system [1], the pair $(\bar{A},\bar{B})$ is stabilizable if and only if $\text{rank}\begin{pmatrix} \bar{A}-\lambda I_p & \bar{B} \end{pmatrix} = p$. It follows that

$$\begin{aligned}\text{rank}\begin{pmatrix} A-\lambda E & B \end{pmatrix} &= \text{rank}\left[M\begin{pmatrix} A-\lambda E & B \end{pmatrix}\begin{pmatrix} N & 0 \\ 0 & I_r \end{pmatrix}\right] \\ &= \text{rank}\begin{pmatrix} MAN-\lambda MEN & MB \end{pmatrix} \\ &= \text{rank}\left(\begin{array}{c|cc} A_{11}-\lambda I_p & A_{12} & B_1 \\ \hline A_{21} & A_{22} & B_2 \end{array}\right) \\ &= \text{rank}\left(\begin{array}{c|cc} A_{11}-\begin{pmatrix} A_{12} & B_1 \end{pmatrix}\hat{A}^+A_{21}-\lambda I_p & A_{12} & B_1 \\ \hline A_{21}-\begin{pmatrix} A_{22} & B_2 \end{pmatrix}\hat{A}^+A_{21} & A_{22} & B_2 \end{array}\right) \\ &= \text{rank}\left(\begin{array}{c|cc} \bar{A}-\lambda I_p & A_{12} & B_1 \\ \hline 0 & A_{22} & B_2 \end{array}\right)\begin{pmatrix} I_p & 0 & 0 \\ 0 & W & I_{n-p+r} \end{pmatrix} \\ &= \text{rank}\left(\begin{array}{c|c|cc} \bar{A}-\lambda I_p & \bar{B} & A_{12} & B_1 \\ \hline 0 & 0 & A_{22} & B_2 \end{array}\right) \\ &= p+(n-p)=n.\end{aligned}$$

□

Theorem 9. *The pair* $(\bar{A}-\bar{B}Q_{22}^{-1}Q_{12}^T, Q_{11}-Q_{12}Q_{22}^{-1}Q_{12}^T)$ *is detectable if and only if*

$$\text{rank}\begin{pmatrix} A-\lambda E & B \\ C & D \end{pmatrix} = n+r, \tag{4. 5}$$

for each λ, $Re(\lambda) \geq 0$.

Proof. It is well known that the pair $(\bar{A}-\bar{B}Q_{22}^{-1}Q_{12}^T, Q_{11}-Q_{12}Q_{22}^{-1}Q_{12}^T)$ is detectable if and only if

$$\text{rank}\begin{pmatrix} \bar{A}-\bar{B}Q_{22}^{-1}Q_{12}^T-\lambda I_p \\ Q_{11}-Q_{12}Q_{22}^{-1}Q_{12}^T \end{pmatrix} = p \text{ (full column rank)}$$

for all λ, $Re(\lambda) \geq 0$. It follows that

$$\begin{aligned}
\text{rank}\begin{pmatrix} A-\lambda E & B \\ C & D \end{pmatrix} &= \text{rank}\left[\begin{pmatrix} M & 0 \\ 0 & I_q \end{pmatrix}\begin{pmatrix} A-\lambda E & B \\ C & D \end{pmatrix}\begin{pmatrix} N & 0 \\ 0 & I_r \end{pmatrix}\right] \\
&= \text{rank}\left(\begin{array}{c|cc} A_{11}-\lambda I_p & A_{12} & B_1 \\ \hline A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right) \\
&= \text{rank}\left(\begin{array}{c|cc} A_{11}-\begin{pmatrix} A_{12} & B_1 \end{pmatrix}\hat{A}^{+}A_{21}-\lambda I_p & A_{12} & B_1 \\ \hline A_{21}-\begin{pmatrix} A_{22} & B_2 \end{pmatrix}\hat{A}^{+}A_{21} & A_{22} & B_2 \\ \hline C_1-\begin{pmatrix} C_2 & D \end{pmatrix}\hat{A}^{+}A_{21} & C_2 & D \end{array}\right) \\
&= \text{rank}\left\{\left(\begin{array}{c|cc} \bar{A}-\lambda I_p & A_{12} & B_1 \\ \hline 0 & A_{22} & B_2 \\ \hline \bar{C} & C_2 & D \end{array}\right)\begin{pmatrix} I_p & 0 & 0 \\ 0 & W & I_{n-p+r} \end{pmatrix}\right\} \\
&= \text{rank}\begin{pmatrix} \bar{A}-\lambda I_p & \bar{B} & \begin{pmatrix} A_{12} & B_1 \end{pmatrix} \\ \bar{C} & \bar{D} & \begin{pmatrix} C_2 & D \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} A_{22} & B_2 \end{pmatrix} \end{pmatrix} \\
&= \text{rank}\begin{pmatrix} \bar{A}-\bar{B}Q_{22}^{-1}Q_{12}^T-\lambda I_p & \bar{B} & \begin{pmatrix} A_{12} & B_1 \end{pmatrix} \\ \bar{C}-\bar{D}Q_{22}^{-1}Q_{12} & \bar{D} & \begin{pmatrix} C_2 & D \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} A_{22} & B_2 \end{pmatrix} \end{pmatrix} \\
&= \text{rank}\left[\begin{pmatrix} I_p & 0 & 0 \\ 0 & \bar{C}^T & 0 \\ 0 & \bar{D}^T & 0 \\ 0 & 0 & I_{n-p} \end{pmatrix}\begin{pmatrix} \bar{A}-\bar{B}Q_{22}^{-1}Q_{12}^T-\lambda I_p & \bar{B} & \begin{pmatrix} A_{12} & B_1 \end{pmatrix} \\ \bar{C}-\bar{D}Q_{22}^{-1}Q_{12} & \bar{D} & \begin{pmatrix} C_2 & D \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} A_{22} & B_2 \end{pmatrix} \end{pmatrix}\right] \\
&= \text{rank}\begin{pmatrix} \bar{A}-\bar{B}Q_{22}^{-1}Q_{12}^T-\lambda I_p & \bar{B} & \begin{pmatrix} A_{12} & B_1 \end{pmatrix} \\ Q_{11}-Q_{12}Q_{22}^{-1}Q_{12}^T & Q_{12} & \bar{C}^T\begin{pmatrix} C_2 & D \end{pmatrix} \\ 0 & Q_{22} & \bar{D}^T\begin{pmatrix} C_2 & D \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} A_{22} & B_2 \end{pmatrix} \end{pmatrix} \\
&= \text{rank}\begin{pmatrix} \bar{A}-\bar{B}Q_{22}^{-1}Q_{12}^T-\lambda I_p \\ Q_{11}-Q_{12}Q_{22}^{-1}Q_{12}^T \end{pmatrix} + \text{rank}\, Q_{22} + \text{rank}\begin{pmatrix} A_{22} & B_2 \end{pmatrix} \\
&= p+r+(n-p) = n+r.
\end{aligned}$$

□

Theorem 10. *If the implicit system* (2.1) *is impulse controllable and satisfies part (iii) of theorem 7, relations* (4.4) *and* (4.5), *then* Ω *has a unique optimal control-state pair and the optimal control can be synthesized as state feedback.*

Proof. Let the hypothesis holds, then in accordance to the linear optimal control theory for the standard state space system [1], Ω_2 has a unique optimal control-state pair (v^*, x^*), where the optimal control v^* is given by (4.1), the state x_1^* is the solution of differential equation (4.2), P is the unique positive semidefinite solution of the algebraic Riccati equation (4.3) and every eigenvalue λ of the closed loop system $\bar{A}-\bar{B}L$ satisfies $\mathrm{Re}(\lambda)<0$. From (3.5) and (3.7), we have

$$\begin{aligned}\begin{pmatrix} x_1^* \\ x_2^* \\ u^* \end{pmatrix} &= \begin{pmatrix} I_p & 0 \\ -\hat{A}^+A_{21} & W \end{pmatrix}\begin{pmatrix} I_p \\ -L \end{pmatrix} x_1^* \\ &= \begin{pmatrix} I_p & 0 \\ \Lambda_1 & W_1 \\ \Lambda_2 & W_2 \end{pmatrix}\begin{pmatrix} I_p \\ -L \end{pmatrix} x_1^* \\ &= \begin{pmatrix} I_p \\ \Lambda_1 - W_1L \\ \Lambda_2 - W_2L \end{pmatrix} x_1^* \end{aligned} \tag{4. 6}$$

where $\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} \equiv -\hat{A}^+A_{21}, W \equiv \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \Lambda_1 \in \mathbb{R}^{(n-p)\times p}, \Lambda_2 \in \mathbb{R}^{r\times p}$, $W_1 \in \mathbb{R}^{(n-p)\times r}$ and $W_2 \in \mathbb{R}^{r\times r}$. Finally, by using (3.1) we get the unique optimal control-state pair (u^*, x^*) of Ω as follows:

$$\begin{pmatrix} x^* \\ u^* \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & I_r \end{pmatrix}\begin{pmatrix} x_1^* \\ x_2^* \\ u^* \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & I_r \end{pmatrix}\begin{pmatrix} I_p \\ \Lambda_1 - W_1L \\ \Lambda_2 - W_2L \end{pmatrix} x_1^*. \tag{4. 7}$$

Now, we are going to synthesize u^* as state feedback, i.e.

$$u^* = Kx^* \equiv \begin{pmatrix} K_1 & K_2 \end{pmatrix}\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = (K_1 + K_2(\Lambda_1 - W_1L))x_1^*, \tag{4. 8}$$

where $K \in \mathbb{R}^{r\times n}$ is a feedback matrix. Recall that $u^* = (\Lambda_2 - W_2L)x_1^*$, one can see that (4.6) holds if and only if K_1 and K_2 satisfy $K_1 + K_2(\Lambda_1 - W_1L) = \Lambda_2 - W_2L$. But, since the matrix $\begin{pmatrix} A_{22} & B_2 \end{pmatrix}$ has full row rank then there exists a matrix $K_2 \in \mathbb{R}^{r\times(n-p)}$ such that $A_{22} + B_2K_2$ is non singular. It follows that if we choose $K_1 = (\Lambda_2 - W_2L) - K_2(\Lambda_1 - W_1L)$ then (4.6) holds and the proof is completed. □

To end this section, we would like to point out that for every full column rank matrix $W \in \mathbb{R}^{(n-p+r)\times r}$ which satisfies $\begin{pmatrix} A_{22} & B_2 \end{pmatrix} W = 0$ will preserve the uniqueness optimal state-control pair of the LQ optimal control problem Ω. At this end, let $\widehat{W}, \bar{W} \in \mathbb{R}^{(n-p+r)\times r}$, with $\widehat{W} \neq \bar{W}$, have full column rank and satisfy $\begin{pmatrix} A_{22} & B_2 \end{pmatrix}\widehat{W} = \begin{pmatrix} A_{22} & B_2 \end{pmatrix}\bar{W} = 0$. Next, let $\widehat{L}$ and $\bar{L}$, with $\widehat{L} \neq \bar{L}$, be the feedback matrices corresponding to the choosing $\widehat{W}$ and $\bar{W}$, respectively. It will suffice to show that the solution of differential equation (4.2) is unique for a given admissible initial state $x_{10} \in \mathbb{R}^p$. For this reason, suppose that $\widehat{x}_1$ and $\bar{x}_1$, with $\widehat{x}_1 \neq \bar{x}_1$, are two solutions of the differential equation (4.2) corresponding to choosing $\widehat{W}$ and $\bar{W}$, respectively. It follows that there exists $\widehat{x}_2, \bar{x}_2 \in \mathbb{R}^{n-p}$,such that (3.5) becomes

$$\begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{u} \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ -\widehat{A}^+A_{21} & \widehat{W} \end{pmatrix}\begin{pmatrix} \widehat{x}_1 \\ \widehat{v} \end{pmatrix}$$

and

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{u} \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ -\hat{A}^+ A_{21} & \widehat{W} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{v} \end{pmatrix},$$

where $\widehat{v}$ and $\bar{v}$ are the control vectors corresponding to $\widehat{x}_1$ and $\bar{x}_1$, respectively. Subtracting both the above, we have

$$\begin{aligned} \begin{pmatrix} \widehat{x}_1 - \bar{x}_1 \\ \widehat{x}_2 - \bar{x}_2 \\ \widehat{u} - \bar{u} \end{pmatrix} &= \begin{pmatrix} I_p & 0 \\ -\hat{A}^+ A_{21} & \widehat{W} \end{pmatrix} \begin{pmatrix} I_p \\ -\hat{L} \end{pmatrix} \widehat{x}_1 \\ &\quad - \begin{pmatrix} I_p & 0 \\ -\hat{A}^+ A_{21} & \bar{W} \end{pmatrix} \begin{pmatrix} I_p \\ -\bar{L} \end{pmatrix} \bar{x}_1 \\ &= \begin{pmatrix} I_p \\ -\hat{A}^+ A_{21} - \widehat{W}\hat{L} \end{pmatrix} \widehat{x}_1 - \begin{pmatrix} I_p \\ -\hat{A}^+ A_{21} - \bar{W}\bar{L} \end{pmatrix} \bar{x}_1 . \end{aligned} \quad (4.\ 9)$$

Premultipling both sides of (4.9) by $\begin{pmatrix} I_p & 0 \\ 0 & \widehat{W}^T \end{pmatrix}$, we have

$$\begin{pmatrix} \widehat{x}_1 - \bar{x} \\ \widehat{W}^T \begin{pmatrix} \widehat{x}_2 - \bar{x}_2 \\ \widehat{u} - \bar{u} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} I_p \\ -\widehat{W}^T \widehat{W} \hat{L} \end{pmatrix} \widehat{x}_1 - \begin{pmatrix} I_p \\ -\widehat{W}^T \bar{W} \bar{L} \end{pmatrix} \bar{x}_1, \quad (4.\ 10)$$

and in particular, we have

$$\widehat{W}^T \begin{pmatrix} \widehat{x}_2 - \bar{x}_2 \\ \widehat{u} - \bar{u} \end{pmatrix} = -\widehat{W}^T \widehat{W} \hat{L} \widehat{x}_1 + \widehat{W}^T \bar{W} \bar{L} \bar{x}_1.$$

Since $\widehat{W}^T$ has full row rank then we have

$$\begin{pmatrix} \widehat{x}_2 - \bar{x}_2 \\ \widehat{u} - \bar{u} \end{pmatrix} = -(\widehat{W}^T)^+ \widehat{W}^T \widehat{W} \hat{L} \widehat{x}_1 + (\widehat{W}^T)^+ \widehat{W}^T \bar{W} \bar{L} \bar{x}_1, \quad (4.\ 11)$$

where $(\widehat{W}^T)^+ = \widehat{W}(\widehat{W}^T \widehat{W})^{-1}$ is the generalized inverse of the matrix $\widehat{W}^T$. Premultiply the both sides of (4.11) by $\begin{pmatrix} A_{22} & B_2 \end{pmatrix}$, we have

$$\begin{pmatrix} A_{22} & B_2 \end{pmatrix} \begin{pmatrix} \widehat{x}_2 - \bar{x}_2 \\ \widehat{u} - \bar{u} \end{pmatrix} = 0, \quad (4.\ 12)$$

and it follows that

$$\begin{pmatrix} \widehat{x}_2 \\ \widehat{u} \end{pmatrix} = \begin{pmatrix} \bar{x}_2 \\ \bar{u}_2 \end{pmatrix}. \quad (4.\ 13)$$

which is valid for every $t \in [0, \infty)$. By considering (3.4) then we should have $-A_{21}\widehat{x}_1 = -A_{21}\bar{x}_1$. But, since this is valid to every $t \in [0, \infty)$, then we should have $\widehat{x}_1 = \bar{x}_1$, and this is a contradiction. Therefore $\widehat{x}_1 = \bar{x}_1$. This fact and together with (4.13) show that the optimal control- state pair (u^*, x^*) of the LQ control problem for implicit system is unique.

Remark 11. This finding is also valid for the cases in which the matrix $D^T D$ is positive definite as well as for the regular cases. Moreover, if D is a zero matrix of appropriate dimension, then one can see that theorem 1 and lemma 2 in [7] are a special case of our theorem 7 and theorem 9, respectively.

5. Numerical Examples

In this section, we give some numerical examples.

Example 1 Consider the singular LQ control problem for nonregular implicit system, where

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{pmatrix}, D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, x_0 = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}^T.$$

By taking $M = I_4$ and

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

it is easy to verify that the implicit constraint is impulse controllable. By choosing any full column matrix $W \in \ker \begin{pmatrix} A_{22} & B_2 \end{pmatrix}$, the problem Ω can be equivalently changed into the following LQ control problem subject to standard state space system:

$$\begin{aligned} &\underset{(v,x_1)}{\text{minimize}} \int_0^\infty \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix} dt \\ &\text{subject to} \quad \begin{aligned} \dot{x}_1(t) &= \begin{pmatrix} -1 & 0 \\ -1 & -2 \end{pmatrix} x_1(t) + \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} v(t),\ x_1(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ y(t) &= \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} x_1(t) + \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix} v(t), \end{aligned} \end{aligned}$$

where $x_1, v \in \mathbb{R}^2, Q_{11} = \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}, Q_{12} = \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix}$, and $Q_{22} = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$.

The solution of this problem is $v = -Lx_1^*$, where $L = \begin{pmatrix} 1.5 & 1 \\ 3 & 2 \end{pmatrix}$ and

$$x_1^* = \begin{pmatrix} -2e^{-2t} + 3e^{-0.5t} \\ 5e^{-2t} - 3e^{-0.5t} \end{pmatrix}$$

satisfies the following differential equation:

$$\dot{x}_1 = \begin{pmatrix} 0.5 & 1 \\ -2.5 & -3 \end{pmatrix} x_1,\ x_1(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Moreover, we also have

$$x_2^* = \begin{pmatrix} 2e^{-2t} + 1.5e^{-0.5t} \\ 0 \end{pmatrix}.$$

Thereby, the optimal solution of the LQ control problem Ω is as follows:

$$x^* = \begin{pmatrix} -2e^{-2t} + 3e^{-0.5t} \\ -3e^{-2t} \\ 2e^{-2t} + 1.5e^{-0.5t} \\ 0 \end{pmatrix}$$

and

$$u = \begin{pmatrix} 3e^{-2t} \\ -4e^{-2t} - 3e^{-0.5t} \end{pmatrix},$$

with the optimal cost is

$$J(u^*, x_0) = 0.$$

Moreover, the optimal control can be synthesized as

$$u^* = \begin{pmatrix} -0.5 & 0 \\ -3 & -2 \end{pmatrix} x_1^* + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x_2^*.$$

The trajectories for the optimal control-state pair are given in the figures 1.a and 1.b below.

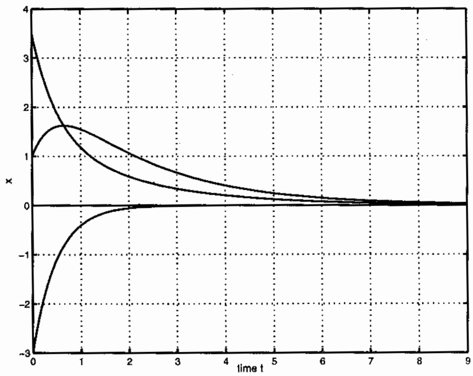

Figure 1.a. State Trajectory

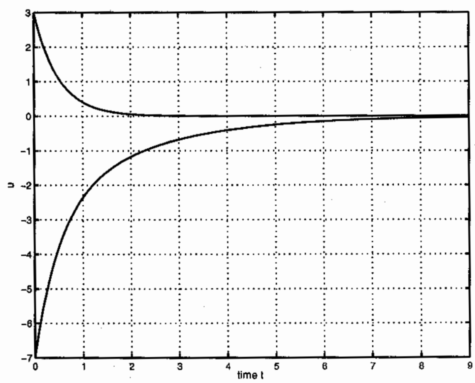

Figure 1.b. Control Trajectory

Example 2 The following is an example of LQ control problem for regular descriptor system, where

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} -1 & 0 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & -2 & 0 & -1 \\ -1 & 2 & -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 4 & -4 \\ 5 & 0 \\ 1 & 0 \\ 3 & 1 \end{pmatrix},$$

$$C=\begin{pmatrix} -1 & 0 & 3 & 0 \\ -1 & -1 & 4 & 0 \end{pmatrix}, D=\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}, x_0=\begin{pmatrix} 2 & 0 & 0 & 0 \end{pmatrix}^T.$$

By taking the matrices $M = I_4$and

$$N=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the implicit constraint is impulse controllable. By choosing any full colomn matrix $W \in \ker\begin{pmatrix} A_{22} & B_2 \end{pmatrix}$, the problem Ω can be equivalently changed into the following LQ problem for standard state space system:

$$\underset{(v,x_1)}{\text{minimize}} \int_0^\infty \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix} dt$$

$$\text{s. t.} \begin{cases} \dot{x}_1(t) = \begin{pmatrix} 4.2308 & -5.5385 \\ 2.7692 & -0.4615 \end{pmatrix} x_1(t) + \begin{pmatrix} -3.4988 & -5.2099 \\ -0.1783 & -1.5339 \end{pmatrix} v(t), \\ x_1(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ y(t) = \begin{pmatrix} 0.9231 & -2.1538 \\ -0.4615 & -1.9231 \end{pmatrix} x_2(t) + \begin{pmatrix} -2.9638 & -0.6082 \\ -3.0444 & 1.8083 \end{pmatrix} v(t) \end{cases}$$

where

$$x_1, v \in \mathbb{R}^2, Q_{11} = \begin{pmatrix} 1.0651 & -1.1006 \\ -1.1006 & 8.3373 \end{pmatrix}, Q_{12} = \begin{pmatrix} -1.3307 & -1.3960 \\ 12.2382 & -2.1675 \end{pmatrix},$$

and

$$Q_{22} = \begin{pmatrix} 18.0524 & -3.7026 \\ -3.7026 & 3.6399 \end{pmatrix}.$$

Thus the optimal control for this LQ control problem is $v^* = -Lx_1^*$, where

$$L = \begin{pmatrix} -0.2270 & 0.6958 \\ -0.6958 & 0.0856 \end{pmatrix},$$

and

$$x_1^* = e^{-0.1923t} \begin{pmatrix} 2\cos(2.1035t) + 0.0132\sin(2.1035t) \\ 1.5826\sin(2.1035t) \end{pmatrix}$$

satisfies the following differential equation:

$$\dot{x}_1(t) = \begin{pmatrix} -0.1784 & -2.6583 \\ 1.6645 & -0.2062 \end{pmatrix} x_1(t), \ x_1(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Moreover, we also have

$$x_2^* = e^{-0.1923t} \begin{pmatrix} 0.6846\cos(2.1035t)\text{-}0.1925\sin(2.1035t) \\ \text{-}4.3973\cos(2.1035t) + 3.6646\sin(2.1035t) \end{pmatrix}.$$

Hence, the optimal solution of the LQ control problem Ω is

$$x^* = e^{-0.1923t} \begin{pmatrix} 2\cos(2.1035t) + 0.0132\sin(2.1035t) \\ 2\cos(2.1035t) - 1.5694\sin(2.1035t) \\ 0.6846\cos(2.1035t) - 0.1925\sin(2.1035t) \\ -4.3973\cos(2.1035t) + 3.6646\sin(2.1035t) \end{pmatrix},$$

and

$$u^* = e^{-0.1923t}\begin{pmatrix} -0.3973\cos(2.1035t) + 0.5257\sin(2.1035t) \\ -0.1236\cos(2.1035t) + 1.3823\sin(2.1035t) \end{pmatrix}$$

with the optimal cost is

$$J(u^*, x_0) = 0.2287.$$

Moreover, the optimal control can be synthesized as

$$u^* = \begin{pmatrix} -0.5409 & 0.4584 \\ 2.1369 & -1.4599 \end{pmatrix} x_1^* + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x_2^*.$$

The curves of trajectory for the optimal control-state pair are given in the figures 2.a and 2.b below.

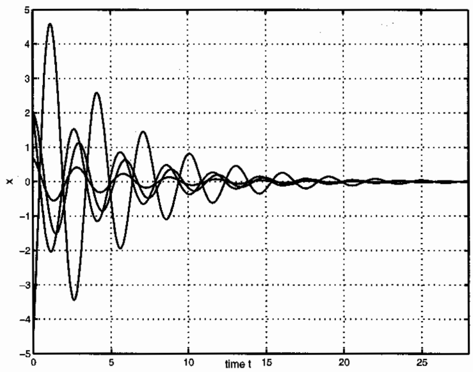

Figure 2.a. State Trajectory

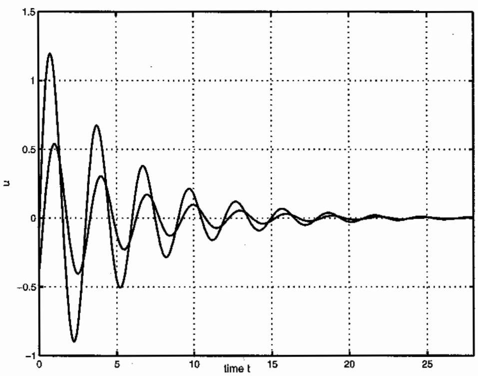

Figure 2.b. Control Trajectory

6. Conclusion

This paper discussed the singular LQ control problem for nonregular implicit system, and has given the relationship between this problem and the LQ control problem for standard state space system. We pointed out that solvability of the LQ control for standard state space system is sufficient condition to guarantee existence and uniqueness of the original problem.

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On Some Finite BCI-Algebras with $o(M) = 2$

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Abstract. In this paper we give some results for the characterization of BCI-algebras of order n with $o(M) = 2$ by means of center and branches.

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Keywords and Phrases: BCK-algebras, BCI-algebras, Initial element, Center and Branch.

1. Introduction

K. Iseki [10] introduced the theory of BCI-algebras and established some of its properties. On wards, so many eminent researchers have contributed to the discipline. S. K. Goel [6], as a first step characterized BCI-algebras of order 3 and partially BCI-algebras of order 4. In [4], S.A. Bhatti, M.A. Chaudhry and A. H. Zaidi posed an open problem stated as follows:

"How many proper BCI-algebras of order n exist?"

In this paper we solve this problem partially and give some results for the characterization of BCI-algebras of order n with $o(M) = 2$ by means of center and branches.

2. Preliminaries

1.Definition [10]

Let X be an abstract algebra of type (2, o) with a binary operation * and a constant o. Then X is a BCI-algebra, if the following conditions are satisfied for all x, $x, y, z \in X$,

1 $((x * y) * (x * z)) * (z * y) = o$

2 $(x * (x * y)) * y = o$

3 $x * x = o$
4 $x * y = o = y * x \Rightarrow x = y$
5 $x * o = o \Rightarrow x = o$

where $x * y = o \Leftrightarrow x \leq y$
If o * x = o, holds for all $x \in X$, then X is a BCK-algebra [9].

Moreover, the following properties hold in every BCI-algebra ([10]):

6 $x * o = x$
7 $(x * y) * z = (x * z) * y$
8 $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$

In a BCI-algebra X, the set $M = \{x \in X : o * x = o\}$ is called the BCK-part of X. A BCI-algebra X is called proper if $X - M \neq \emptyset$. In a BCI-algebra X, $X - M = \{x \in X : o * x \neq o\}$ is known as the BCI-part of X.

9 Let X be a BCI-algebra. If $M = \{o\}$, then X is called a p-semisimple BCI-algebra.[12]
10 Let X be a p-semisimple BCI-algebra. If we define $x + y = x * (o * y)$, then (X, +, o) is an abelian group.[5, 12]

2. Definition [2]
Let X be a BCI-algebra and $x, y \in X$. Then x, y are said to be comparable if and only if $x \leq y$ or $y \leq x$. Further, we shall say that x precedes y and y succeeds x if and only if $x * y = o$ and denote it by $x \to y$ or $x \leq y$.

If x and y are not comparable, then they are said to be **incomparable**.

3. Definition [2]
Let X be a BCI-algebra. An element $x_o \in X$ is said to be an initial element of X, if $x \leq x_o \Rightarrow x = x_o$. Obviously o is an initial element.

4 Definition [2]
Let I_x denote the set of all initial elements of X. We call it the center of X. The reason for calling I_x as the center of X is that each branch (defined below) originates from a unique point of this subset. The cardinality of the center is same as that as the set of branches of X.

5 Definition [2]
Let X be a BCI-algebra with I_x as its center. Let $x_o \in I_x$, then the set $A(x_o) = \{x \in X : x_o \leq x\}$ is known as the branch of X determined by x_o. Each branch $A(x_o)$ is nonempty, because by property (3), $x_o * x_o = o \Rightarrow x_o \in A(x_o)$. We note that $A(x_o)$ consists of all those elements of X which succeed x_o .

If $x_o \neq o$, then $A(x_o)$, the branch of X determined by x_o is called a BCI-branch of X. But if $x_o = o$, then $A(x_o)$, the branch of X determined by x_o is called a BCK-branch of X.

Also note that the BCK-part M of the BCI-algebra X is equal to A(o) because $M = \{x \in X : o * x = o\} = A(o)$. Hence, it follows that a BCK-algebra is a single branch BCI-algebra.

6 Definition [2]
If $A(x_o) = \{x_o\}$, then $A(x_o)$, the branch determined by x_o, is known as a uniary comparable.

11 The center I_x of a BCI-algebra X is p-semisimple.[3]
12 Let X be a BCI-algebra and $A(x_o) \subseteq X$. Then $x, y \in A(x_o) \Rightarrow x * y, y * x \in M$.[2]
13 Let X be a BCI-algebra. If $x \leq y$, then x, y are contained in the same branch of X [2].
14 Let X be a BCI-algebra with I_x as its center. If $x \in A(x_o)$, $y \in A(y_o)$, then $x * y \in A(x_o * y_o)$, for $x_o, y_o \in I_x$. [7]
15 Let X be a BCI-algebra with I_x as its center. Let $x_o, y_o \in I_x$. Then for all $y \in A(y_o)$, $x_o * y = x_o * y_o$.[7]
16 Let X be a p-semisimple algebra.Then X is fully nonassociative if and only if o(X) is odd [4].
17 A BCI-algebra X is said to be associative if $o * x = x$, for all $x \in X$, otherwise it is called non-associative [1, 8].
18 A BCI-algebra X is said to be fully-nonassociative if $o * x \neq x$, for all $x \in X - \{o\}$. [2].
19 A BCI-algebra X is said to be neutral non-associative if $o * x = x$ holds for some $x \in X - M$[2].

3. Some BCI-algebras of order n with $o(M) = 2$

Theorem 1: Let X be a proper BCI-algebra of order $n \lesseqgtr 4$ with $o(M) = 2$. If $o(I_x) = 2$, then X is unique.
Proof: Let $X = \{o = x_1, x_2, x_3, ..., x_n\}$ be a proper BCI-algebra of order $n \lesseqgtr 4$ with BCK-part $M = A(o) = \{o = x_1, x_2\}$ and the BCI-part $X - M = \{x_3, ..., x_n\}$. Take $I_x = \{o, x_3\}$. Since $o(I_x) = 2$, therefore , X has only two branches. As $M = A(o)$ is the BCK-branch, so we take BCI-branch $A(x_3) = \{x \in X - M : x_3 \leq x\} = \{x_3, ..., x_n\}$. By (11), I_x is p-semisimple BCI-algebra. The binary operation * in I_x is defined as follows:

Table P_1

*	o	x_3
o	o	x_3
x_3	x_3	o

Since $x_2 \in M$, so $o * x_2 = o$. Now by using the properties (3), (6) and the corresponding values from table P_1, the multiplication table representing such BCI-algebra is given as follows:

Table 1

*	o	x_2	x_3	x_4	...	x_n
o	o	o	x_3	Δ	Δ	Δ
x_2	x_2	o	Δ	Δ	Δ	Δ
x_3	x_3	Δ	o	Δ	Δ	Δ
x_4	x_4	Δ	Δ	o	Δ	Δ
⋮	⋮	Δ	Δ	Δ	Δ	Δ
x_n	x_n	Δ	Δ	Δ	Δ	o

In the above multiplication table the dotted column represent the missing $6^{th} - n^{th}$ columns and the dotted row represent the missing $6^{th} - n^{th}$ rows. The entries for the blank cells denoted by Δ in Table 1 are computed as follows:

Computation of values for the blank cells of 2^{nd} row
For $o, x_3 \in I_x$ and any $x \in A(x_3)$, by (15), $o * x_3 = o * x \Rightarrow o * x = x_3$ (Since $o * x_3 = x_3$). Thus x_3 will fill all the blank cells of 2^{nd} row.

Computation of values for the blank cells of ^{3rd} row
For any $x \in A(x_3)$, by definition 5, $x_3 \leq x \Rightarrow x_3 * x = o$. By (12), $x * x_3 \in M = A(o) = \{o, x_2\}$. But $x * x_3 \neq o$, otherwise because of property (4), $x * x_3 = o = x_3 * x \Rightarrow x = x_3$, a contradiction. Thus, $x * x_3 = x_2$ **(1)**
Now

$$x_2 * x = (x * x_3) * x \quad (using\ equation\ (1))$$

$$= (x * x) * x_3 \quad (using\ property\ (7))$$

$$= o * x_3 = x_3 \quad (using\ property\ (4)\ and\ o * x_3 = x_3)$$

Thus x_3 will fill all the blank cells of 3^{rd} row.

Computation of values for the blank cells of $4^{th} - (n+1)^{th}$ rows
For $x, x_3 \in A(x_3) \subset X$, by (2) and equation (1), $x * (x * x_3) \leq x_3 \Rightarrow x * x_2 \leq x_3 \Rightarrow x * x_2 = x_3$, as x_3 is an initial element. Thus x_3 will fill all the blank cells of 3^{rd} column.

Now for $x, y \in A(x_3)$ such that $x \neq y \neq x_3$, we have following three possibilities:

Case (i): $x \leq y$ **Case (ii):** $y \leq x$ **Case (iii):** x, y are incomparable.

Case (i): $x \leq y \Rightarrow x * y = o$ (2)

Case (ii): $x \geq y \Rightarrow y * x = o$. Since $x, y \in A(x_3)$, therefore by (12), $x * y \in M = A(o) = \{o, x_2\}$. But $x * y \neq o$, otherwise by (4), $x * y = o = y * x \Rightarrow x = y$, a contradiction. So, $x * y = x_2$ (3)

Case (iii): Since $x, y \in A(x_3)$, therefore by (12) $x * y \in M = A(o) = \{o, x_2\}$. But $x * y \neq o$, otherwise $x * y = o$ implies x, y are comparable, a contradiction. So, $x * y = x_2$ (3)

Thus from equations $(1) - (4)$, it follows that $4^{th} - (n+1)^{th}$ blank cells of $4^{th} - (n+1)^{th}$ rows have fixed entries i. e either o or x_2.
Hence, by filling the corresponding blank cells with the fixed values computed above in table 1, we have a unique BCI-algebra.

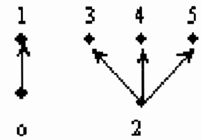
Example 1
Let $X = \{o, 1, 2, 3, 4, 5\}$ be a BCI-algebra with $M = A(o) = \{o, 1\}$ as its BCK-part. Then the BCI-part $X - M = \{2, 3, 4, 5\}$. The partial order on $X - M$ is defined as $2 \leq 3$, $2 \leq 4$, $2 \leq 5$. By (13), $A(2) = \{2, 3, 4, 5\}$. Thus it follows that $I_x = \{o, 2\}$. Note that $o(X) = 6 > 4$ and $o(I_x) = 2$. Hence by theorem 1, X is unique. The

Multiplication table and the Geometric figure representing such BCI-algebra are given below:

Table 2

*	o	1	2	3	4	5
o	o	o	2	2	2	2
1	1	o	2	2	2	2
2	2	2	o	o	o	o
3	3	2	1	o	1	1
4	4	2	1	1	o	1
5	5	2	1	1	1	o

Fig. 1



For more examples see [4] and [11].

Theorem 2:
Let X be a BCI-algebra of order n with $o(M) = 2$ and $o(I_x) = 3$. If for some $x_o \in I_x$, $A(x_o)$ is uniary comparable, then X is unique.
Proof:
Let $X = \{o = x_1, x_2, x_3, \ldots\ldots, x_n\}$ be a proper BCI-algebra of order n with BCK-part $M = A(o) = \{o = x_1, x_2\}$ and the BCI-part $X - M = \{x_3, \ldots\ldots, x_n\}$. Take $I_x = \{o, x_3, x_n\}$. Since $o(I_x) = 3$, therefore , X has three branches. As $M = A(o)$ is the BCK-branch, so we take two BCI-branches $A(x_3) = \{x \in X - M : x_3 \leq x\} = \{x_3, \ldots\ldots, x_{n-1}\}$ and $A(x_n) = \{x_n\}$. By (11), I_x is p-semisimple. Since $o(I_x)$ is odd, therefore by (18), I_x is fully non-associative. The binary operation * in Ix is defined as follows:

Table P_2

*	o	x_3	x_n
o	o	x_n	x_3
x_3	x_3	o	x_n
x_n	x_n	x_3	o

Since $x_2 \in M$, so $o * x_2 = o$. Now by using the properties (3), (6) and the corresponding values from table P_2, the multiplication table representing such BCI-algebra is given as follows:

Table 3

*	o	x_2	x_3	x_4	$\ldots$	x_{n-1}	x_n
o	o	o	x_n	Δ	Δ	Δ	x_3
x_2	x_2	o	Δ	Δ	Δ	Δ	Δ
x_3	x_3	Δ	o	Δ	Δ	Δ	x_n
x_4	x_4	Δ	Δ	o	Δ	Δ	Δ
$\vdots$	$\vdots$	Δ	Δ	Δ	Δ	Δ	Δ
x_n	x_n	Δ	Δ	Δ	Δ	Δ	o

In the above multiplication table the dotted column represent the missing $6^{th} - (n-1)^{th}$ columns and the dotted row represent the missing $6^{th} - (n-1)^{th}$ rows. The entries for the blank cells denoted as Δ in Table 3 are computed as follows:

Computation of values for the blank cells of 2^{nd} row
For $o, x_3 \in I_x$ and any $x \in A(x_3)$, by (15), $o * x_3 = o * x \Rightarrow o * x = x_n$ (Since $o * x_3 = x_n$). Thus x_n will fill the blank cells of 2^{nd} row.

Computation of values for the blank cells of 3^{rd} row
For $x_2 \in A(o)$ and $x \in A(x_3)$, by (14) $x_2 * x \in A(o * x_3) = A(x_n) = \{x_n\}$. So, $x_2 * x = x_n$ (1)
Thus x_n will fill the corresponding blank cells of 3^{rd} row.
Now for $x_2 \in A(o)$ and $x_3 \in A(x_3)$, because by equation (1) it follows that $x_2 * x_3 = x_n$. So using (2), $x_2 * (x_2 * x_3) \leq x_3 \Rightarrow x_2 * x_n \leq x_3 \Rightarrow x_2 * x_n = x_3$, as x_3 is an initial element. Thus x_3 will fill the $(n+1)^{th}$ blank cell of 3^{rd} row

Computation of values for the blank cells of $4^{th} - n^{th}$ rows
Computation of $x*x_2$ for any $x \in A(x_3)$ and $x_2 \in A(o)$, are the same as done in theorem 1. So, $x*x_2 = x_3$, Thus x_3 will fill the corresponding blank cells of 3^{rd} column.

Computation of $x * x_3$ and $x * y$ for $x \neq y \in A(x_3)$ are the same as done in theorem 1. Thus $4th - n^{th}$ blank cells of $4th - n^{th}$ rows have fixed entries i.e o or x_2.

Also for $x \in A(x_3)$ and $x_n \in A(x_n)$, by (14) $x * x_n \in A(x_3 * x_n) = A(x_n) = x_n$. So $x * x_n = x_n$. Thus x_n will fill all the blank cells of $(n+1)^{th}$ column.

Computation of values for the blank cells of $(n+1)^{th}$ row
For $x_n \in A(x_n)$ and $x_2 \in A(o)$ by (15) $x_n * o = x_n * x_2 = x_n$. Thus x_n will fill the 3^{rd} blank cell of $(n+1)^{th}$ row. Again by (15) for $x_n \in A(x_n)$ and any $x \in A(x_3)$, $x_n * x_3 = x_n * x = x_3$ (Since $x_n * x_3 = x_3$). Thus x_3 will fill remaining blank cells of $(n+1)^{th}$ row.

Hence, by filling the corresponding blank cells with the fixed computed values in table 3, we have a unique BCI-algebra.

Example 2:
Let $X = \{o, a, b, c, d, e, f\}$ be a BCI-algebra with $M = A(o) = \{o, a\}$ as its BCK-part. Then BCI-part $X - M = \{b, c, d, e, f\}$. The partial order on $X - M$ is defined as $b \leq c \leq e$, $b \leq d \leq e$ and $f \leq f$. Therefore by (13) $A(b) = \{b, c, d, e\}$ and $A(f) = \{f\}$. Note that $I_x = \{o, b, f\}$. Since $o(I_x) = 3$ and for $f \in I_x$, $A(f)$ is uniary comparable, So by theorem 2, X is unique. The Multiplication table and the Geometric figure representing such BCI-algebra are given below:

Table 4

*	o	a	b	c	d	e	f
o	o	o	f	f	f	f	b
a	a	o	f	f	f	f	b
b	b	b	o	o	o	o	f
c	c	b	a	o	a	o	f
d	d	b	a	a	o	o	f
e	e	b	a	a	a	o	f
f	f	b	b	b	b	b	o

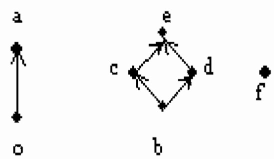


Fig. 2

For more examples see [4] and [11].

Theorem 3:
Let X be a BCI-algebra of order $n \leqslant 6$ with $o(M) = 2$ and $o(I_x) = 4$. If for some $x_o \neq y_o \in I_x$, $A(x_o)$ and $A(y_o)$ are uniary comparable, then there exist two such BCI-algebras.
Proof:
Let $X = \{o = x_1, x_2, ..., x_n\}$ be a proper BCI-algebra of order $n \leqslant 6$ with BCK-part $M = A(o) = \{o = x_1, x_2\}$ and BCI-part as $X - M = \{x_3, x_4, ..., x_n\}$. Take $I_x = \{o, x_3, x_{n-1}, x_n\}$. Since $o(I_x) = 4$, therefore X has 4 branches. As $M = A(o)$ is a BCK-branch, so we take three BCI-branches $A(x_3) = \{x \in X : x_3 \leq x\} = \{x_3, ..., x_{n-2}\}$, $A(x_{n-1}) = \{x_{n-1}\}$ and $A(x_n) = \{x_n\}$. By (11) I_x is a p-semisimple BCI-algebra. Since p-semisimple BCI-algebras are precisely abelian groups (see [5, 12]), therefore isomorphism classes of p-semisimple BCI-algebras are the isomorphism classes of abelian groups. Since there are two abelian groups of order 4, therefore there are two p-semisimple BCI-algebras of order four. Keeping in view properties (17) and (19), the multiplication tables representing such p-semisimple BCI-algebras are given as follows:

Table P_3

*	o	x_3	x_{n-1}	x_n
o	o	x_3	x_{n-1}	x_n
x_3	x_3	o	x_n	x_{n-1}
x_{n-1}	x_{n-1}	x_n	o	x_3
x_n	x_n	x_{n-1}	x_3	o

Table P_4

*	o	x_3	x_{n-1}	x_n
o	o	x_3	x_n	x_{n-1}
x_3	x_3	o	x_{n-1}	x_n
x_{n-1}	x_{n-1}	x_n	o	x_3
x_n	x_n	x_{n-1}	x_3	o

Thus construction of proper BCI-algebras of order 7, which depend upon $o(I_x) = 4$ can be obtained by using Table P_3 and Table P_4 respectively.

Case (1): (Using **Table** P_3)
By using the properties (3), (6) and the corresponding values from Table P3, the multiplication table representing such BCI-algebra is given as follows:

Table 5

$*$	o	x_2	x_3	$\dots$	x_{n-2}	x_{n-1}	x_n
o	o	o	x_3	Δ	Δ	x_{n-1}	x_n
x_2	x_2	o	Δ	Δ	Δ	Δ	Δ
x_3	x_3	Δ	o	Δ	Δ	x_n	x_{n-1}
$\vdots$	$\vdots$	Δ	Δ	Δ	Δ	Δ	Δ
x_{n-1}	x_{n-1}	Δ	x_n	Δ	Δ	o	x_3
x_n	x_n	Δ	x_{n-1}	Δ	Δ	x_3	o

In the above multiplication table, the dotted column represent the missing $5^{th} - (n-2)^{th}$ columns and the dotted row represent the missing $5^{th} - (n-2)^{th}$ rows. The entries for the blank cells denoted as Δ in Table 5 are computed as follows:

Computation of values for the blank cells of 2^{nd} row
For $o, x_3 \in I_x$ and any $x \in A(x_3)$, by (15), $o * x_3 = o * x \Rightarrow o * x = x_3$ (since $o * x_3 = x_3$). Thus x_3 will fill the blank cells of 2^{nd} row.

Computation of values for the blank cells of 3^{rd} row
Computation of $x_2 * x$ for $x_2 \in A(o)$ and any $x \in A(x_3)$ are the same as done in theorem 1. Thus x_3 will fill $4^{th} - (n-1)^{th}$ blank cells of 3^{rd} row.

Now for $x_2 \in A(o)$ and $x_{n-1} \in A(x_{n-1})$, by (14) $x_2 * x_{n-1} \in A(o * x_{n-1}) = A(x_{n-1}) = \{x_{n-1}\}$ (since, $o * x_{n-1} = x_{n-1}$). So, $x_2 * x_{n-1} = x_{n-1}$. Thus x_{n-1} will fill the n^{th} blank cell of 3^{rd} row.

Also for $x_2 \in A(o)$ and $x_n \in A(x_n)$, by (14) $x_2 * x_n \in A(o * x_n) = A(x_n) = \{x_n\}$(since $o * x_n = x_n$). So, $x_2 * x_n = x_n$. Thus x_n will fill the $(n+1)^{th}$ blank cell of 3^{rd} row.

Computation of values for the blank cells of $4^{th} - (n-1)^{th}$ row
Computation of $x * x_2$ for any $x \in A(x_3)$ and $x_2 \in A(o)$ are the same as done in theorem 1. Thus x_3 will fill $4^{th} - (n-1)^{th}$ the blank cells of 3^{rd} column. For $x, y \in A(x_3)$, such that $x \neq y$, computation of $x * x_3$ and $x * y$ are the same as done in theorem 1. Thus $4^{th} - (n-1)_{th}$ blank cells of $4^{th} - (n-1)^{th}$ rows have fixed entries i.e o or x_2.

Now for any $x \in A(x_3)$ and $x_{n-1} \in A(x_{n-1})$ by (14) $x * x_{n-1} \in A(x_3 * x_{n-1}) = A(x_n) = \{x_n\}$. So $x * x_{n-1} = x_n$. Thusx_n will fill all the blank cells of n^{th} column. Also for any $x \in A(x_3)$ and $x_n \in A(x_n)$ by (14) $x * x_n \in A(x_3 * x_n) = A(x_{n-1}) = \{x_{n-1}\}$. So $x * x_n = x_{n-1}$. Thus x_{n-1} will fill all the blank cells of $(n+1)^{th}$ column.

Computation of values for the blank cells of n^{th} row

For $x_{n-1} \in A(x_{n-1})$ and $x_2 \in A(o)$ by (15) $x_{n-1} * o = x_{n-1} * x_2 = x_{n-1}$. Thus x_{n-1} will fill the 3^{rd} blank cell of n^{th} row. Also for $x_{n-1} \in A(x_{n-1})$ and $x \in A(x_3)$ by (15), $x_{n-1} * x_3 = x_{n-1} * x = x_n$. Thus x_n will fill the $5^{th} - (n-1)^{th}$ blank cells of n^{th} row.

Computation of values for the blank cells of $(n+1)^{th}$ row

For $x_n \in A(x_n)$ and $x_2 \in A(o)$ by (15), $x_n * o = x_n * x_2 = x_n$. Thus x_n will fill the 3^{rd} blank cell of $(n+1)^{th}$ row. Also for $x_n \in A(x_n)$ and $x \in A(x_3)$ by (15), $x_n * x_3 = x_n * x = x_{n-1}$. Thus x_{n-1} will fill the $5^{th} - (n-1)^{th}$ blank cells of $(n+1)^{th}$ row.

Hence, by filling the corresponding blank cells with the fixed values computed above in table 5, we have a unique BCI-algebra.

Case (2): (Using Table P_4).
By using the properties (3), (6) and the corresponding values from Table P_4, the multiplication table representing the BCI-algebra is given as follows:

Table 6

$*$	o	x_2	x_3	$\cdots$	x_{n-2}	x_{n-1}	x_n
o	o	o	x_3	Δ	Δ	x_n	x_{n-1}
x_2	x_2	o	Δ	Δ	Δ	Δ	Δ
x_3	x_3	Δ	o	Δ	Δ	x_{n-1}	x_n
$\vdots$	$\vdots$	Δ	Δ	Δ	Δ	Δ	Δ
x_{n-1}	x_{n-1}	Δ	x_n	Δ	Δ	o	x_3
x_n	x_n	Δ	x_{n-1}	Δ	Δ	x_3	o

In the above multiplication table, the dotted column represent the missing $5^{th} - (n-2)^{th}$ columns and the dotted row represent the missing $5^{th} - (n-2)^{th}$ rows. The entries for the blank cells denoted as Δ in Table 6 are computed as follows:

Computation of values for the blank cells of 2^{nd} row

For $o, x_3 \in I_x$ and any $x \in A(x_3)$ by (15), $o * x_3 = o * x \Rightarrow o * x = x_3$ (since $o * x_3 = x_3$). Thus x_3 will fill the blank cells of 2^{nd} row.

Computation of values for the blank cells of 3^{rd} row

Computation of $x_2 * x$ for $x_2 \in A(o)$ and any $x \in A(x_3)$ are the same as done in theorem 1. Thus x_3 will fill $4^{th} - (n-1)^{th}$ the blank cells of 3^{rd} column.

Now for $x_2 \in A(o)$ and $x_{n-1} \in A(x_{n-1})$ by (14), $x_2 * x_{n-1} \in A(o * x_{n-1}) = A(x_n) = \{x_n\}$. So, $x_2 * x_{n-1} = x_n$. Thus x_n will fill the n^{th} blank cell of 3^{rd} row. Also for $x_2 \in A(o)$ and $x_n \in A(x_n)$ by (14), $x_2 * x_n \in A(o * x_n) = A(x_{n-1}) = \{x_{n-1}\}$. So, $x_2 * x_n = x_{n-1}$. Thus x_{n-1} will fill the $(n+1)^{th}$ blank cell of 3^{rd} row

Computation of values for the blank cells of $4^{th} - (n-1)^{th}$ rows

Computation of $x * x_2$ for any $x \in A(x_3)$ and $x_2 \in A(o)$ are the same as done in theorem 1. Thus x_3 will fill $4^{th} - (n-1)^{th}$ blank cells of 3^{rd} column. For $x, y \in A(x_3)$

such that $x \neq y$, computation of $x * x_3$ and $x * y$ are the same as done in theorem 1. Thus $4^{th} - (n-1)^{th}$ blank cells of $4^{th} - (n-1)^{th}$ rows have fixed entries i.e o or x_2.

Now for any $x \in A(x_3)$ and $x_{n-1} \in A(x_{n-1})$ by (14), $x * x_{n-1} \in A(x_3 * x_{n-1}) = A(x_{n-1}) = \{x_{n-1}\}$. So $x * x_{n-1} = x_{n-1}$. Thus x_{n-1} will fill all the blank cells of n^{th} column.

Also for any $x \in A(x_3)$ and $x_n \in A(x_n)$ by (14), $x*x_n \in A(x_2*x_n) = A(x_n) = \{x_n\}$. So $x * x_n = x_n$. Thus x_n will fill all the blank cells of $(n+1)^{th}$ column.

Computation of values for the blank cells of n^{th} row
For $x_{n-1} \in A(x_{n-1})$ and $x_2 \in A(o)$ by (15) $x_{n-1}*o = x_{n-1}*x_2 = x_{n-1}$. Thus x_{n-1} will fill the 3^{rd} blank cell of n^{th} row. Also for $x_{n-1} \in A(x_{n-1})$ and any $x \in A(x_3)$ by (15) $x_{n-1} * x_3 = x_{n-1} * x = x_n$. Thus x_n will fill the $5^{th} - (n-1)^{th}$ blank cells of n^{th} row.

Computation of values for the blank cells of $(n+1)^{th}$ row
For $x_n \in A(x_n)$ and $x_2 \in A(o)$ by (15), $x_n * o = x_n * x_2 = x_n$. Thus x_n will fill the 3^{rd} blank cell of $(n+1)^{th}$ row. Also for $x_n \in A(x_n)$ and any $x \in A(x_3)$ by (15), $x_n * x_3 = x_n * x = x_{n-1}$. So, x_{n-1} will fill the $5^{th} - (n-1)^{th}$ blank cells of $(n+1)^{th}$ row.

Thus, by filling the corresponding blank cells with the fixed values computed above in table 6, we have a unique BCI-algebra.

Hence from case (1) and case (2) it follows that there are two such BCI-algebras of order n.

Example 3:
Let $X = \{o, a, b, c, d, e, f\}$ be a BCI-algebra with $M = A(o) = \{o, a\}$ as its BCK-part. Then BCI-part $X - M = \{b, c, d, e, f\}$. The partial order on $X - M$ is defined as $b \leq c \leq d$, $e \leq e$ and $f \leq f$. Therefore by (13), $A(b) = \{b, c, d\}$, $A(e) = \{e\}$ and $A(f) = \{f\}$. Note that $I_x = \{o, b, e, f\}$.. Since $o(Ix) = 4$ and for $e, f \in I_x$, A(e) and A (f) are uniary comparable, So by theorem 3, there exist 2 such BCI-algebras of order 7. The Multiplication tables and the Geometric figure representing such BCI-algebras are given as follows:

Table 7

*	o	a	b	c	d	e	f
o	o	o	b	b	b	e	f
a	a	o	b	b	b	e	f
b	b	b	o	o	o	f	e
c	c	b	a	o	o	f	e
d	d	b	a	a	o	f	e
e	e	e	f	f	f	o	b
f	f	f	e	e	e	b	o

Table 8

*	o	a	b	c	d	e	f
o	o	o	b	b	b	f	e
a	a	o	b	b	b	f	e
b	b	b	o	o	o	e	f
c	c	b	a	o	o	e	f
d	d	b	a	a	o	e	f
e	e	e	f	f	f	o	b
f	f	f	e	e	e	b	o

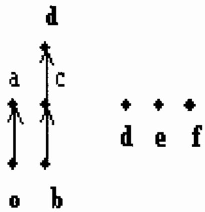


Fig. 3

For more examples see [4] and [11].

Theorem 4.
Let X be a BCI-algebra of order n with $o(M) = 2$ and $o(I_x) = 5$. If for some $x_o \neq y_o \neq z_o \in I_x$, $A(x_o)$, $A(y_o)$ and $A(z_o)$ are uniary comparable, then X is a unique BCI-algebra

Proof:
Let $X = \{o = x_1, x_2, ., x_n\}$ be a proper BCI-algebra of order n with BCK-part $M = A(o) = \{o = x_1, x_2\}$. Then BCI-part $X - M = \{x_3, ., x_n\}$. Take $I_x = \{o, x_3, x_{n-2}, x_{n-1}, x_n\}$. Since $o(I_x) = 5$, therefore X has 5 branches. As $M = A(o)$ is a BCK-branch, so we take four BCI-branches $A(x_3) = \{x \in X : x_3 \leq x\} = \{x_3, , x_{n-3}\}$, $A(x_{n-2}) = \{x_{n-2}\}$, $A(x_{n-1}) = \{x_{n-1}\}$ and $A(x_n) = \{x_n\}$. By (11), I_x is p-semisimple. Since $o(I_x)$ is odd, therefore by (16), I_x is fully non-associative. The binary operation * in Ix is defined as follows:

Table P_5

*	o	x_3	x_{n-2}	x_{n-1}	x_n
o	o	x_n	x_{n-1}	x_{n-2}	x_3
x_3	x_3	o	x_n	x_{n-1}	x_{n-2}
x_{n-2}	x_{n-2}	x_3	o	x_n	x_{n-1}
x_{n-1}	x_{n-1}	x_{n-2}	x_3	o	x_n
x_n	x_n	x_{n-1}	x_{n-2}	x_3	o

Any other p-semisimple BCI-algebra of order 5 is isomorphic to the above table. By using the properties (3), (6), the corresponding values from table P_5, the multiplication table representing such BCI-algebra is given below:

Table 9

$*$	o	x_2	x_3	$\dots$	x_{n-3}	x_{n-2}	x_{n-1}	x_n
o	o	o	x_n	Δ	Δ	x_{n-1}	x_{n-2}	x_3
x_2	x_2	o	Δ	Δ	Δ	Δ	Δ	Δ
x_3	x_3	Δ	o	Δ	Δ	x_n	x_{n-1}	x_{n-2}
$\vdots$	$\vdots$	Δ	Δ	Δ	Δ	Δ	Δ	
x_{n-3}	x_{n-3}	Δ	Δ	Δ	Δ	Δ	Δ	
x_{n-2}	x_{n-2}	Δ	x_3	Δ	Δ	o	x_n	x_{n-1}
x_{n-1}	x_{n-1}	Δ	x_{n-2}	Δ	Δ	x_3	o	x_n
x_n	x_n	Δ	x_{n-1}	Δ	Δ	x_{n-3}	x_3	o

In the above multiplication table, the dotted column represent the missing $5^{th} - (n-3)^{th}$ columns and the dotted row represent the missing $5^{th} - (n-3)^{th}$ rows. The entries for the blank cells denoted as Δ in Table 9 are computed as follows:

Computation of values for the blank cells of 2^{nd} row
For $o, x_3 \in I_x$ and any $x \in A(x_3)$ by (15), $o * x_2 = o * x = x_{n-1}$. Thus x_{n-1} will fill the blank cells of 2^{nd} row.

Computation of values for the blank cells of 3^{rd} row
For $x_2 \in A(o)$ and any $x \in A(x_3)$ by (14), $x_2 * x \in A(o * x_3) = A(x_n) = \{x_n\}$ (since $o * x_3 = x_n$) So, $x_2 * x = x_n$. (1)

Thus xn will fill the corresponding blank cells of 3rd row.

Now for $x_2 \in A(o)$ and $x_{n-2} \in A(x_{n-2})$ by (14), $x_2 * x_{n-2} \in A(o * x_{n-2}) = A(x_{n-1}) = \{x_{n-1}\}$. So, $x_2 * x_{n-2} = x_{n-1}$. Thus x_{n-1} will fill the $(n-1)^{th}$ blank cell of 3^{rd} row.

Also, for $x_2 \in A(o)$ and $x_{n-1} \in A(x_{n-1})$ by (14), $x_2 * x_{n-1} \in A(o * x_{n-1}) = A(x_{n-2}) = \{x_{n-2}\}$. So, $x_2 * x_{n-1} = x_{n-2}$. Thus x_{n-2} will fill the n^{th} blank cell of 3^{rd} row.

Further for $x_2 \in A(o)$ and $x_3 \in A(x_3)$, from equation (1), it follows that $x_2 * x_3 = x_n$. Using property (2), $x_2 * (x_2 * x_3) \leq x_3 \Rightarrow x_2 * x_n \leq x_3 \Rightarrow x_2 * x_n = x_3$, as x_3 is an initial element. Thus x_3 will fill the $(n+1)^{th}$ blank cell of 3^{rd} row.

Computation of values for the blank cells of $4^{th} - (n-2)^{th}$ row
For any $x \in A(x_3)$ and $x_2 \in A(o)$ by (14), $x * x_2 \in A(x_3 * o) = A(x_3) = \{x_3\}$. So, $x * x_2 = x_3$, Thus x_3 will fill the corresponding blank cells of 3^{rd} column. For $x, y \in A(x_3)$ such that $x \neq y$, computation of $x * x_3$ and $x * y$ are same as done in theorem 1. Thus $4^{th} - (n-1)^{th}$ blank cells of $4^{th} - (n-1)^{th}$ rows have fixed entries i.e o or x_2

For any $x \in A(x_3)$ and $x_{n-2} \in A(x_{n-2})$ by (14), $x * x_{n-2} \in A(x_3 * x_{n-2}) = A(x_n) = \{x_n\}$ (since, $x_3 * x_{n-2} = x_n$). So $x * x_{n-2} = x_n$. Thus x_n will fill all the blank cells of $(n-1)^{th}$ column.

Also for $x \in A(x_3)$ and $x_{n-1} \in A(x_{n-1})$ by (14), $x * x_{n-1} \in A(x_3 * x_{n-1}) = A(x_{n-1}) = \{x_{n-1}\}$. So $x * x_{n-1} = x_{n-1}$. Thus x_{n-1} will fill all the blank cells of n^{th} column.

Further for $x \in A(x_3)$ and $x_n \in A(x_n)$ by (14), $x * x_n \in A(x_3 * x_n) = A(x_{n-2}) = \{x_{n-2}\}$. So $x * x_{n-2} = x_{n-2}$. Thus x_{n-2} will fill all the blank cells of $(n+1)^{th}$ column.

Computation of values for the blank cells of $(n-1)^{th}$ row
For $x_{n-2} \in A(x_{n-2})$ and $x_2 \in A(o)$ by (15), $x_{n-2} * o = x_{n-2} * x_2 = x_{n-2}$. Thus x_{n-2} will fill the 3^{rd} blank cell of $(n-1)^{th}$ row. Also for $x_{n-2} \in A(x_{n-2})$ and any $x \in A(x_3)$ by (15), $x_{n-2} * x_3 = x_{n-3} * x = x_3$. Thus x_3 will fill the remaining blank cells of $(n-1)^{th}$ row.

Computation of values for the blank cells of n^{th} row
For $x_{n-1} \in A(x_{n-1})$ and $x_2 \in A(o)$ by (15), $x_{n-1} * o = x_{n-1} * x_2 = x_{n-1}$. Thus x_{n-1} will fill the 3rd blank cell of n^{th} row. Also for $x_{n-1} \in A(x_{n-1})$ and any $x \in A(x_3)$ by (15), $x_{n-1} * x_3 = x_{n-1} * x = x_{n-2}$. Thus x_{n-2} will fill the remaining blank cells of n^{th} row.

Computation of values for the blank cells of $(n+1)^{th}$ row
For $x_n \in A(x_n)$ and $x_2 \in A(o)$, by (15), $x_n * o = x_n * x_2 = x_n$. Thus x_n will fill the 3^{rd} blank cell of $(n+1)^{th}$ row. Also for $x_n \in A(x_n)$ and any $x \in A(x_3)$ by (15), $x_n * x_3 = x_n * x = x_{n-1}$. Thus x_{n-1} will fill the remaining blank cells of $(n+1)^{th}$ row.

Hence, by filling the corresponding blank cells with the fixed values computed above in table 9, we have a unique BCI-algebra.

Example 4:
Let $X = \{o, a, b, c, d, e, f\}$ be a BCI-algebra with $M = A(o) = \{o, a\}$ as its BCK-part. Then $X - M = \{b, c, d, e, f\}$. The partial order on X is defined as $b \leq c$, $d \leq d$, $e \leq e$, $f \leq f$, therefore by (13) $A(b) = \{b, c\}$, $A(d) = \{d\}$, $A(e) = \{e\}$ and $A(f) = \{f\}$. Note that $I_x = \{o, b, d, e, f\}$. Since $o(Ix) = 5$ and for $d, e, f \in I_x$, $A(d)$, $A(e)$ and $A(f)$ are uniary comparable. So, by theorem 4, X is unique. The Multiplication table and the Geometric figure representing such BCI-algebra are given below:

Table 10

*	o	a	b	c	d	e	f
o	o	o	f	f	e	d	b
a	a	o	f	f	e	d	b
b	b	b	o	o	f	e	d
c	c	b	a	o	f	e	d
d	d	d	b	b	o	f	e
e	e	e	d	d	b	o	f
f	f	f	e	e	d	b	o

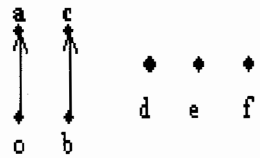


Fig. 4

For more examples see [4] and [11].

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Visualization of Surface Data Using Rational Bicubic Spline

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Abstract. A rational cubic spline [11] is extended to rational bicubic spline. Simple constraints are made on the free parameters in the description of rational bicubic spline to preserve the shape of positive surafac data and to preserve the shape of the data that lie above a plane.

keywords: Visualization, Positive Data, Rational Bicubic Spline.

1. Introduction

Scientific visualization is concerned with the presentation of interactive or animated digital images to users to understand data. The advantages of scientific visualization are: huge amount of data is converted into one picture, it correlates different quantities. One of the problems faced by scientific visualization community is visualization of positive data. Visualization of positive data in the view of positive surface is essential in visualizing the entities that cannot be negative e.g. amount of rainfall, volume, area, density, population etc.

The problem of visualization of positive data has been considered by many authors [1-14]. But most of the authors considered the problem of visualization of positive scattered data [1,3,4,6,8,9,10,14] only a few have considered the problem of visualization of positive data arranged over rectangular grid [2,13]. A brief review is: Asim, Mustafa and Brodlie [1] visualized positive scattered data subject to positivity constraints using modified quadratic Shepard method. Brodlie, Mashwama and Butt [2] addressed the problem of visualization of positive data arranged over a rectangular mesh. Sufficient conditions for positivity are derived in terms of the first partial derivatives and mixed partial derivatives at the grid points. Brodlie, Asim and Unsworth [3] modified the quadratic Shepard method, which interpolates scattered data of any dimensionality to preserve positivity. Chan and Ong

[4] described a local scheme for scattered data range restricted interpolation. Sufficient conditions for the non-negativity of cubic Bzier triangle are derived and these conditions prescribed lower bounds to Bzier ordinates. Non-negativity is achieved by modifying if necessary the first order partial derivatives of data sites and some Bzier ordinates. Kong et al [8] discussed the problem of range restricted scattered data interpolation using cubic Bzier triangles. Nadler [9] have considered non-negative data arranged over a triangular mesh and have interpolated each triangular patch using a bivariate quadratic function. Piah, Goodman and Unsworth [10] constructed the interpolating surfaces comprising cubic Bzier triangular patches. They imposed sufficient conditions on the ordinates of the Bzier control net in each triangle to preserve the positivity. The derivatives at the data points are specified to be consistent with these conditions. Schmidt [13] provided the solution to the problem of shape preserving interpolation of data sets given on rectangular grids. Shepard [14] proposed a basic form that involves an inverse distance-weighted average of data values to visualize the positive data.

In Section 2, the rational cubic function [11] used in this paper is rewritten. In Section 3, the rational cubic function is extended to the rational bicubic function. In Section 4, the problem of visualization of positive data has been considered. In Section 5, the problem of visualization constrained data has been considered. Section 6 applies the results developed in Section 4 and 5 graphically. Section 7 concludes the paper.

2. Rational Cubic Spline

In this Section, we recall the piecewise rational cubic function used in this paper which was initially developed by Sarfraz [11]. Let (x_i, f_i), $i = 0, 1, 2, \ldots, n$ be given set of data points where $x_0 < x_1 < x_2 < \cdots < x_n$. The piecewise rational cubic function is defined over each interval $I_i = [x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$ as:

$$S(x) = \frac{p_i(\theta)}{q_i(\theta)}, \tag{2. 1}$$

with

$$\begin{aligned} p_i(\theta) &= f_i(1-\theta)^3 + (\alpha_i f_i + h_i d_i)(1-\theta)^2\theta + (\beta_i f_{i+1} - h_i d_{i+1})(1-\theta)\theta^2 \\ &\quad + f_{i+1}\theta^3, \\ q_i(\theta) &= (1-\theta)^3 + \alpha_i(1-\theta)^2\theta + \beta_i(1-\theta)\theta^2 + \theta^3, \end{aligned}$$

$$h_i = x_{i+1} - x_i, \quad \theta = \frac{x - x_i}{h_i}.$$

The rational cubic function (1) has the following properties:

$$S(x_i) = f_i, \; S(x_{i+1}) = f_{i+1}, \; S^{(1)}(x_i) = d_i, \; S^{(1)}(x_{i+1}) = d_{i+1}.$$

$S^{(1)}(x)$ denotes the derivative with respect to x and d_i denotes derivative values (given or estimated by some method) at knot x_i. $S(x) \in C^{(1)}[x_0, x_n]$ has α_i and β_i as free parameters in the interval $[x_i, x_{i+1}]$. We note that in each interval I_i, when we take $\alpha_i = 3$ and $\beta_i = 3$, the piecewise rational cubic function reduces to standard Cubic Hermite.

3. Rational Bicubic Spline

The piecewise rational cubic function (2.1) is extended to rational bicubic function $S(x,y)$ over the rectangular domain $D = [x_0, x_m] \times [y_0, y_n]$. Let $\pi : a = x_0 < x_1 < \cdots < x_m = b$ be partition of $[a,b]$ and $\tilde{\pi} : c = y_0 < y_1 < \cdots < y_n = d$ be partition of $[c,d]$. The rational bicubic function is defined over each rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, where $i = 0,1,2,\ldots,m-1;\ j = 0,1,2,\ldots,n-1$ as:

$$S(x,y) = S_{i,j}(x,y) = A_i(\theta)F(i,j)\hat{A}_j^T(\phi), \tag{3. 1}$$

where

$$F(i,j) = \begin{pmatrix} F_{i,j} & F_{i,j+1} & F^y_{i,j} & F^y_{i,j+1} \\ F_{i+1,j} & F_{i+1,j+1} & F^y_{i+1,j} & F^y_{i+1,j+1} \\ F^x_{i,j} & F^x_{i,j+1} & F^{xy}_{i,j} & F^{xy}_{i,j+1} \\ F^x_{i+1,j} & F^x_{i+1,j+1} & F^{xy}_{i+1,j} & F^{xy}_{i+1,j+1} \end{pmatrix},$$

$$A_i(\theta) = [a_0(\theta)\ a_1(\theta)\ a_2(\theta)\ a_3(\theta)], \qquad \hat{A}_j(\phi) = [\hat{a}_0(\phi)\ \hat{a}_1(\phi)\ \hat{a}_2(\phi)\ \hat{a}_3(\phi)],$$

with

$$a_0(\theta) = \frac{(1-\theta)^3 + \alpha_i(1-\theta)^2\theta}{q_i(\theta)}, \qquad a_1(\theta) = \frac{\theta^3 + \beta_i(1-\theta)\theta^2}{q_i(\theta)},$$

$$a_2(\theta) = \frac{h_i(1-\theta)^2\theta}{q_i(\theta)}, \qquad a_3(\theta) = \frac{-h_i(1-\theta)\theta^2}{q_i(\theta)},$$

$$\hat{a}_0(\phi) = \frac{(1-\phi)^3 + \hat{\alpha}_j(1-\phi)^2\phi}{q_j(\phi)}, \qquad \hat{a}_1(\phi) = \frac{\phi^3 + \hat{\beta}_j(1-\phi)\phi^2}{q_j(\phi)},$$

$$\hat{a}_2(\phi) = \frac{\hat{h}_j(1-\phi)^2\phi}{q_j(\phi)}, \qquad \hat{a}_3(\phi) = \frac{-\hat{h}_j(1-\phi)\phi^2}{q_j(\phi)}.$$

Substituting the values of A, F and $\hat{A}$ in equation (3.1) the rational bicubic function $S(x,y)$ can be expressed as:

$$S(x,y) = \frac{(1-\theta)^3\gamma_{i,j} + (1-\theta)^2\theta\eta_{i,j} + (1-\theta)\theta^2\delta_{i,j} + \theta^3\omega_{i,j}}{(1-\theta)^3 + \alpha_i(1-\theta)^2\theta + \beta_i(1-\theta)\theta^2 + \theta^3}, \tag{3. 2}$$

with

$$\gamma_{i,j} = \frac{\sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i A_i}{q_j(\phi)}, \tag{3. 3}$$

$$\begin{aligned} A_0 &= F_{i,j}, \\ A_1 &= \hat{\alpha}_j F_{i,j} + \hat{h}_j F^y_{i,j}, \\ A_2 &= \hat{\beta}_j F_{i,j+1} - \hat{h}_j F^y_{i,j+1}, \\ A_3 &= F_{i,j+1}, \\ q_j(\phi) &= (1-\phi)^3 + \hat{\alpha}_j(1-\phi)^2\phi + \hat{\beta}_j(1-\phi)\phi^2 + \phi^3. \end{aligned}$$

$$\eta_{i,j} = \frac{\sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i B_i}{q_j(\phi)}, \tag{3. 4}$$

$$
\begin{aligned}
B_0 &= \alpha_i F_{i,j} + h_i F^x_{i,j},\\
B_1 &= \hat{\alpha}_j(\alpha_i F_{i,j} + h_i F^x_{i,j}) + \hat{h}_j(\alpha_i F^y_{i,j} + h_i F^{xy}_{i,j}),\\
B_2 &= \hat{\beta}_j(\alpha_i F_{i,j+1} + h_i F^x_{i,j+1}) - \hat{h}_j(\alpha_i F^y_{i,j+1} + h_i F^{xy}_{i,j+1}),\\
B_3 &= \alpha_i F_{i,j+1} + h_i F^x_{i,j+1},\\
q_j(\phi) &= (1-\phi)^3 + \hat{\alpha}_j(1-\phi)^2\phi + \hat{\beta}_j(1-\phi)\phi^2 + \phi^3.
\end{aligned}
$$

$$
\delta_{i,j} = \frac{\sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i C_i}{q_j(\phi)}, \tag{3. 5}
$$

$$
\begin{aligned}
C_0 &= \beta_i F_{i+1,j} - h_i F^x_{i+1,j},\\
C_1 &= \hat{\alpha}_j(\beta_i F_{i+1,j} - h_i F^x_{i+1,j}) + \hat{h}_j(\beta_i F^y_{i+1,j} - h_i F^{xy}_{i+1,j}),\\
C_2 &= \hat{\beta}_j(\beta_i F_{i+1,j+1} - h_i F^x_{i+1,j+1}) - \hat{h}_j(\beta_i F^y_{i+1,j+1} - h_i F^{xy}_{i+1,j+1}),\\
C_3 &= \beta_i F_{i+1,j+1} - h_i F^x_{i+1,j+1},\\
q_j(\phi) &= (1-\phi)^3 + \hat{\alpha}_j(1-\phi)^2\phi + \hat{\beta}_j(1-\phi)\phi^2 + \phi^3.
\end{aligned}
$$

$$
\omega_{i,j} = \frac{\sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i D_i}{q_j(\phi)}, \tag{3. 6}
$$

$$
\begin{aligned}
D_0 &= F_{i+1,j},\\
D_1 &= \hat{\alpha}_j F_{i+1,j} + \hat{h}_j F^y_{i+1,j},\\
D_2 &= \hat{\beta}_j F_{i+1,j+1} - \hat{h}_j F^y_{i+1,j+1},\\
D_3 &= F_{i+1,j+1},\\
q_j(\phi) &= (1-\phi)^3 + \hat{\alpha}_j(1-\phi)^2\phi + \hat{\beta}_j(1-\phi)\phi^2 + \phi^3.
\end{aligned}
$$

The normalized variables θ and ϕ along x and y axes are defined as:

$$
\theta = \frac{x - x_i}{h_i}, \quad \phi = \frac{y - y_j}{\hat{h}_j},
$$

with

$$
h_i = x_{i+1} - x_i, \quad \hat{h}_j = y_{j+1} - y_j.
$$

Unfortunately, these rational functions are not very useful for surface design as any one of the free parameter α_i, β_i, $\hat{\alpha}_j$ and $\hat{\beta}_j$ applies to the entire network of curves. Thus there is no local control on the surface. This ambiguity is overcome by introducing variable weights and desired local control has been achieved. For this purpose new free parameters $\alpha_{i,j}$, $\beta_{i,j}$, $\hat{\alpha}_{i,j}$ and $\hat{\beta}_{i,j}$ are introduced such that:

$$
\alpha_i(y_j) = \alpha_{i,j},\ \beta_i(y_j) = \beta_{i,j},\ \hat{\alpha}_j(x_i) = \hat{\alpha}_{i,j},\ \hat{\beta}_j(x_i) = \hat{\beta}_{i,j},
$$

$$
i = 0, 1, 2, \ldots, m-1;\ j = 0, 1, 2, \ldots, n-1.
$$

The shape of the surface can be modified by assigning different values to these parameters. This property of free parameters will impose different constraints on $\alpha_{i,j}$, $\beta_{i,j}$, $\hat{\alpha}_{i,j}$ and $\hat{\beta}_{i,j}$.

3.1. **Choice of Derivatives.** In most applications, the derivative parameters d_i, $F^x_{i,j}$, $F^y_{i,j}$ and $F^{xy}_{i,j}$ are not given and hence must be determined either from given data or by some other means. These methods are the approximation based on various mathematical theories. An obvious choice is mentioned here:

$$\begin{aligned}
F^x_{0,j} &= \Delta_{0,j} + (\Delta_{0,j} - \Delta_{1,j})\frac{h_0}{(h_0+h_1)}, \\
F^x_{m,j} &= \Delta_{m-1,j} + (\Delta_{m-1,j} - \Delta_{m-2,j})\frac{h_{m-1}}{(h_{m-1}+h_{m-2})}, \\
F^x_{i,j} &= \frac{\Delta_{i,j} + \Delta_{i-1,j}}{2}, \\
& \quad i = 1,2,3,\ldots,m-1; \quad j = 0,1,2,\ldots,n. \\
F^y_{i,0} &= \hat{\Delta}_{i,0} + (\hat{\Delta}_{i,0} - \hat{\Delta}_{i,1})\frac{\hat{h}_0}{(\hat{h}_0+\hat{h}_1)}, \\
F^y_{i,n} &= \hat{\Delta}_{i,n-1} + (\hat{\Delta}_{i,n-1} - \hat{\Delta}_{i,n-2})\frac{\hat{h}_{n-1}}{(\hat{h}_{n-1}+\hat{h}_{n-2})}, \\
F^y_{i,j} &= \frac{\hat{\Delta}_{i,j} + \hat{\Delta}_{i,j-1}}{2}, \\
& \quad i = 0,1,2,\ldots,m; \quad j = 1,2,3,\ldots,n-1. \\
F^{xy}_{i,j} &= \frac{1}{2}\left\{\frac{F^x_{i,j+1} - F^x_{i,j-1}}{\hat{h}_{j-1}+\hat{h}_j} + \frac{F^y_{i+1,j} - F^y_{i-1,j}}{h_{i-1}+h_i}\right\}, \\
& \quad i = 1,2,\ldots,m-1; \quad j = 1,2,\ldots,n-1.
\end{aligned}$$

Where $\Delta_{i,j} = \frac{F_{i+1,j}-F_{i,j}}{h_i}$ and $\hat{\Delta}_{i,j} = \frac{F_{i,j+1}-F_{i,j}}{\hat{h}_j}$. These arithmetic mean methods are computationally economical and suitable for visualization of shaped data.

4. Visualization of Positive Data

Let $(x_i, y_j, F_{i,j})$, $i = 0,1,2,\ldots,m$; $j = 0,1,2,\ldots,n$ be positive data defined over the rectangular grid $D = [x_0, x_m] \times [y_0, y_n]$ such that

$$F_{i,j} > 0 \quad \forall i,j.$$

The rational bicubic function (3.2) preserves the shape of positive data if

$$S(x,y) > 0, \quad \forall (x,y) \in D.$$

$S(x,y) > 0$ if

$$\begin{aligned}
(1-\theta)^3\gamma_{i,j} + (1-\theta)^2\theta\eta_{i,j} + (1-\theta)\theta^2\delta_{i,j} + \theta^3\omega_{i,j} &> 0, \\
q_i(\theta) = (1-\theta)^3 + \alpha_{i,j}(1-\theta)^2\theta + \beta_{i,j}(1-\theta)\theta^2 + \theta^3 &> 0.
\end{aligned}$$

$q_i(\theta) > 0$ if

$$\alpha_{i,j} > 0, \quad \beta_{i,j} > 0.$$

$(1-\theta)^3\gamma_{i,j} + (1-\theta)^2\theta\eta_{i,j} + (1-\theta)\theta^2\delta_{i,j} + \theta^3\omega_{i,j} > 0$ if

$$\gamma_{i,j} > 0, \quad \eta_{i,j} > 0, \quad \delta_{i,j} > 0, \quad \omega_{i,j} > 0.$$

$\gamma_{i,j} > 0$ if

$$A_i > 0, \; i = 0,1,2,3 \; and \; q_j(\phi) > 0.$$

$q_j(\phi) > 0$ if

$$\hat{\alpha}_{i,j} > 0, \quad \hat{\beta}_{i,j} > 0.$$

$A_i > 0,\ i = 0, 1, 2, 3$ if

$$\hat{\alpha}_{i,j} > max\left\{0, \frac{-\hat{h}_j F^y_{i,j}}{F_{i,j}}\right\}.$$
$$\hat{\beta}_{i,j} > max\left\{0, \frac{\hat{h}_j F^y_{i,j+1}}{F_{i,j+1}}\right\}.$$

$\eta_{i,j} > 0$ if

$$B_i > 0,\ i = 0, 1, 2, 3\ and\ q_j(\phi) > 0.$$

$q_j(\phi) > 0$ if

$$\hat{\alpha}_{i,j} > 0, \quad \hat{\beta}_{i,j} > 0.$$

$B_i > 0,\ i = 0, 1, 2, 3$ if

$$\hat{\alpha}_{i,j} > max\left\{0, \frac{-\hat{h}_j F^y_{i,j}}{F_{i,j}}\right\}.$$
$$\hat{\beta}_{i,j} > max\left\{0, \frac{\hat{h}_j F^y_{i,j+1}}{F_{i,j+1}}\right\}.$$
$$\alpha_{i,j} > max\left\{\frac{-h_i F^x_{i,j}}{F_{i,j}}, \frac{-h_i F^x_{i,j+1}}{F_{i,j+1}}\right\}.$$
$$\alpha_{i,j} > max\left\{\frac{-h_i(\hat{\alpha}_{i,j} F^x_{i,j} + \hat{h}_j F^{xy}_{i,j})}{(\hat{\alpha}_{i,j} F_{i,j} + \hat{h}_j F^y_{i,j})}, \frac{-h_i(\hat{\beta}_{i,j} F^x_{i,j+1} - \hat{h}_j F^{xy}_{i,j+1})}{(\hat{\beta}_{i,j} F_{i,j+1} - \hat{h}_j F^y_{i,j+1})}\right\}.$$

$\delta_{i,j} > 0$ if

$$C_i > 0,\ i = 0, 1, 2, 3\ and\ q_j(\phi) > 0.$$

$q_j(\phi) > 0$ if

$$\hat{\alpha}_{i,j} > 0, \quad \hat{\beta}_{i,j} > 0.$$

$C_i > 0,\ i = 0, 1, 2, 3$ if

$$\hat{\alpha}_{i,j} > max\left\{0, \frac{-\hat{h}_j F^y_{i+1,j}}{F_{i+1,j}}\right\}.$$
$$\hat{\beta}_{i,j} > max\left\{0, \frac{\hat{h}_j F^y_{i+1,j+1}}{F_{i+1,j+1}}\right\}.$$
$$\beta_{i,j} > max\left\{\frac{h_i F^x_{i+1,j}}{F_{i+1,j}}, \frac{h_i F^x_{i+1,j+1}}{F_{i+1,j+1}}\right\}.$$
$$\beta_{i,j} > max\left\{\frac{h_i(\hat{\alpha}_{i,j} F^x_{i+1,j} + \hat{h}_j F^{xy}_{i+1,j})}{(\hat{\alpha}_{i,j} F_{i+1,j} + \hat{h}_j F^y_{i+1,j})}, \frac{h_i(\hat{\beta}_{i,j} F^x_{i+1,j+1} - \hat{h}_j F^{xy}_{i+1,j+1})}{(\hat{\beta}_{i,j} F_{i+1,j+1} - \hat{h}_j F^y_{i+1,j+1})}\right\}.$$

$\omega_{i,j} > 0$ if

$$D_i >,\ i = 0, 1, 2, 3\ and\ q_j(\phi) > 0.$$

$q_j(\phi) > 0$ if

$$\hat{\alpha}_{i,j} > 0, \quad \hat{\beta}_{i,j} > 0.$$

$D_i > 0, \ i = 0, 1, 2, 3$ if

$$\hat{\alpha}_{i,j} > max\left\{0, \frac{-\hat{h}_j F^y_{i+1,j}}{F_{i+1,j}}\right\}.$$

$$\hat{\beta}_{i,j} > max\left\{0, \frac{\hat{h}_j F^y_{i+1,j+1}}{F_{i+1,j+1}}\right\}.$$

All this discussion is summarized in the following theorem:

Theorem 4.1. The rational bicubic function defined in (3.2) visualizes positive data in the view of positive surface if in each rectangular patch $I_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, the free parameters $\alpha_{i,j}$, $\beta_{i,j}$, $\hat{\alpha}_{i,j}$ and $\hat{\beta}_{i,j}$ satisfy the following conditions:

$$\hat{\alpha}_{i,j} = l_{i,j} + max\left\{0, \frac{-\hat{h}_j F^y_{i,j}}{F_{i,j}}, \frac{-\hat{h}_j F^y_{i+1,j}}{F_{i+1,j}}\right\}, \ l_{i,j} > 0.$$

$$\hat{\beta}_{i,j} = q_{i,j} + max\left\{0, \frac{\hat{h}_j F^y_{i,j+1}}{F_{i,j+1}}, \frac{\hat{h}_j F^y_{i+1,j+1}}{F_{i+1,j+1}}\right\}, \ q_{i,j} > 0.$$

$$A_{i,j} = \left\{\frac{-h_i F^x_{i,j}}{F_{i,j}}, \frac{-h_i F^x_{i,j+1}}{F_{i,j+1}}\right\}.$$

$$B_{i,j} = \left\{\frac{-h_i(\hat{\alpha}_{i,j} F^x_{i,j} + \hat{h}_j F^{xy}_{i,j})}{(\hat{\alpha}_{i,j} F_{i,j} + \hat{h}_j F^y_{i,j})}, \frac{-h_i(\hat{\beta}_{i,j} F^x_{i,j+1} - \hat{h}_j F^{xy}_{i,j+1})}{(\hat{\beta}_{i,j} F_{i,j+1} - \hat{h}_j F^y_{i,j+1})}\right\}.$$

$$\alpha_{i,j} = m_{i,j} + max\{0, A_{i,j}, B_{i,j}\}, \ m_{i,j} > 0.$$

$$C_{i,j} = \left\{\frac{h_i F^x_{i+1,j}}{F_{i+1,j}}, \frac{h_i F^x_{i+1,j+1}}{F_{i+1,j+1}}\right\}.$$

$$D_{i,j} = \left\{\frac{h_i(\hat{\alpha}_{i,j} F^x_{i+1,j} + \hat{h}_j F^{xy}_{i+1,j})}{(\hat{\alpha}_{i,j} F_{i+1,j} + \hat{h}_j F^y_{i+1,j})}, \frac{h_i(\hat{\beta}_{i,j} F^x_{i+1,j+1} - \hat{h}_j F^{xy}_{i+1,j+1})}{(\hat{\beta}_{i,j} F_{i+1,j+1} - \hat{h}_j F^y_{i+1,j+1})}\right\}.$$

$$\beta_{i,j} = n_{i,j} + max\{0, C_{i,j}, D_{i,j}\}, \ n_{i,j} > 0.$$

5. Visualization of Constrained Data

Let $(x_i, y_j, F_{i,j})$, $i = 0, 1, 2, \ldots, m$; $j = 0, 1, 2, \ldots, n$ be a given set of data points lying above the plane

$$Z = C\left[1 - \frac{x}{A} - \frac{y}{B}\right]$$

i.e.

$$F_{i,j} > Z_{i,j}, \quad \forall i, j.$$

It is required that the surface generated by the rational bicubic function (3.2) will lie on the same side of the plane as the data. This situation is expressed mathematically as:

$$S(x, y) > C\left[1 - \frac{x}{A} - \frac{y}{B}\right]. \tag{5. 1}$$

The parametric equation of the plane is:

$$Z = Z_{i,j} + \theta(Z_{i+1,j} - Z_{i,j}) + \phi(Z_{i,j+1} - Z_{i,j}), \tag{5. 2}$$

where

$$Z_{i,j} = C\left[1 - \frac{x_i}{A} - \frac{y_j}{B}\right]. \tag{5. 3}$$

A, B and C are x,y and z intercepts respectively. Substituting the values from (3.2) and (5.2) in (5.1), condition (5.1) is expressed as:

$$\frac{(1-\theta)^3\gamma_{i,j} + (1-\theta)^2\theta\eta_{i,j} + (1-\theta)\theta^2\delta_{i,j} + \theta^3\omega_{i,j}}{(1-\theta)^3 + \alpha_{i,j}(1-\theta)^2\theta + \beta_{i,j}(1-\theta)\theta^2 + \theta^3} > Z_{i,j} + \theta(Z_{i+1,j} - Z_{i,j}) + \phi(Z_{i,j+1} - Z_{i,j}).$$

After some rearrangement above relation can be expressed as:

$$U_{i,j}(\theta,\phi) = \lambda_{i,j}(1-\theta)^4 + \mu_{i,j}(1-\theta)^3\theta + \nu_{i,j}(1-\theta)^2\theta^2 + \chi_{i,j}(1-\theta)\theta^3 + \tau_{i,j}\theta^4,$$

where

$$\lambda_{i,j} = \frac{\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j A_j}{q_j(\phi)}, \tag{5. 4}$$

with

$$\begin{aligned}
A_0 &= F_{i,j} - Z_{i,j},\\
A_1 &= \hat{\alpha}_{i,j}(F_{i,j} - Z_{i,j}) + F_{i,j} - Z_{i,j+1} + \hat{h}_j F^y_{i,j},\\
A_2 &= \hat{\alpha}_{i,j}(F_{i,j} - Z_{i,j}) + \hat{\beta}_{i,j}(F_{i,j+1} - Z_{i,j+1}) - \hat{h}_j(F^y_{i,j+1} - F^y_{i,j})\\
&\quad + (\hat{\alpha}_{i,j} - \hat{\beta}_{i,j})(Z_{i,j} - Z_{i,j+1}),\\
A_3 &= \hat{\beta}_{i,j}(F_{i,j+1} - Z_{i,j+1}) + F_{i,j+1} - Z_{i,j} - \hat{h}_j F^y_{i,j+1},\\
A_4 &= F_{i,j+1} - Z_{i,j+1}.
\end{aligned}$$

$$\mu_{i,j} = \frac{\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j B_j}{q_j(\phi)}, \tag{5. 5}$$

with

$$\begin{aligned}
B_0 &= \alpha_{i,j}(F_{i,j} - Z_{i,j}) + F_{i,j} + h_i F^x_{i,j} - Z_{i+1,j},\\
B_1 &= (\hat{\alpha}_{i,j} + 1)(\alpha_{i,j}(F_{i,j} - Z_{i,j}) + F_{i,j} + h_i F^x_{i,j} - Z_{i+1,j}) + h_i\hat{h}_j F^{xy}_{i,j}\\
&\quad + (\alpha_{i,j} + 1)(\hat{h}_j F^y_{i,j} + Z_{i,j} - Z_{i,j+1}),\\
B_2 &= \hat{\alpha}_{i,j}(\alpha_{i,j}(F_{i,j} - Z_{i,j}) + F_{i,j} + h_i F^x_{i,j} - Z_{i+1,j}) + \hat{\beta}_{i,j}((\alpha_{i,j} + 1)\\
&\quad (F_{i,j+1} - Z_{i,j+1}) + h_i F^x_{i,j+1} - Z_{i+1,j} + Z_{i,j}) + (\hat{\alpha}_{i,j} - \hat{\beta}_{i,j})(\alpha_{i,j} + 1)\\
&\quad (Z_{i,j} - Z_{i,j+1}) - (\alpha_{i,j} + 1)\hat{h}_j(F^y_{i,j+1} - F^y_{i,j}) - h_i\hat{h}_j(F^{xy}_{i,j+1} - F^{xy}_{i,j}),\\
B_3 &= (\hat{\beta}_{i,j} + 1)((\alpha_{i,j} + 1)(F_{i,j+1} - Z_{i,j+1}) + h_i F^x_{i,j+1} - Z_{i+1,j} + Z_{i,j})\\
&\quad + (\alpha_{i,j} + 1)(Z_{i,j+1} - Z_{ij} - \hat{h}_j F^y_{i,j+1}) - h_i\hat{h}_j F^{xy}_{i,j+1},\\
B_4 &= (\alpha_{i,j} + 1)(F_{i,j+1} - Z_{i,j+1}) + h_i F^x_{i,j+1} - Z_{i+1,j} + Z_{i,j}.
\end{aligned}$$

$$\nu_{i,j} = \frac{\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j C_j}{q_j(\phi)}, \tag{5. 6}$$

with

$$\begin{aligned}
C_0 &= \alpha_{i,j}(F_{i,j}-Z_{i,j})+\beta_{i,j}(\alpha_{i,j}(F_{i,j}-Z_{i,j})+F_{i,j}+h_iF^x_{i,j}-Z_{i+1,j})\\
&\quad -h_i(F^x_{i+1,j}-F^x_{i,j})+(\alpha_{i,j}-\beta_{i,j})(Z_{i,j}-Z_{i+1,j}),\\
C_1 &= (\hat{\alpha}_{i,j}+1)(\alpha_{i,j}(F_{i,j}-Z_{i,j})+\beta_{i,j}(\alpha_{i,j}(F_{i,j}-Z_{i,j})+F_{i,j}+h_iF^x_{i,j}\\
&\quad -Z_{i+1,j})-h_i(F^x_{i+1,j}-F^x_{i,j})+(\alpha_{i,j}-\beta_{i,j})(Z_{i,j}-Z_{i+1,j}))+\alpha_{i,j}(Z_{i,j}\\
&\quad -Z_{i,j+1}+\hat{h}_jF^y_{i,j})+\beta_{i,j}(Z_{i,j}-Z_{i,j+1}+\hat{h}_jF^y_{i+1,j})-h_i\hat{h}_j(F^{xy}_{i+1,j}-F^{xy}_{i,j}),\\
C_2 &= \hat{\alpha}_{i,j}(\alpha_{i,j}(F_{i,j}-Z_{i,j})+\beta_{i,j}(\alpha_{i,j}(F_{i,j}-Z_{i,j})+F_{i,j}+h_iF^x_{i,j}-Z_{i+1,j})\\
&\quad -h_i(F^x_{i+1,j}-F^x_{i,j})+(\alpha_{i,j}-\beta_{i,j})(Z_{i,j}-Z_{i+1,j}))\\
&\quad +\hat{\beta}_{i,j}(\alpha_{i,j}(F_{i,j+1}-Z_{i,j+1})+\beta_{i,j}((\alpha_{i,j}+1)(F_{i,j+1}-Z_{i,j+1})\\
&\quad +h_iF^x_{i,j+1}-Z_{i+1,j}+Z_{i,j})+(\alpha_{i,j}-\beta_{i,j})(Z_{i,j}-Z_{i+1j})\\
&\quad -h_i(F^x_{i+1,j+1}-F^x_{i,j+1}))+(\hat{\alpha}_{i,j}-\hat{\beta}_{i,j})(\alpha_{i,j}+\beta_{i,j})(Z_{i,j}-Z_{i,j+1})\\
&\quad -\hat{h}_j\{\alpha_{i,j}(F^y_{i,j+1}-F^y_{i,j})+\beta_{i,j}(F^y_{i+1,j+1}-F^y_{i+1,j})\}+h_i\hat{h}_j(F^{xy}_{i+1,j+1}\\
&\quad -F^{xy}_{i,j+1}-F^{xy}_{i+1,j}+F^{xy}_{i,j}),\\
C_3 &= (\hat{\beta}_{i,j}+1)(\alpha_{i,j}(F_{i,j+1}-Z_{i,j+1})+\beta_{i,j}((\alpha_{i,j}+1)(F_{i,j+1}-Z_{i,j+1})\\
&\quad +h_iF^x_{i,j+1}-Z_{i+1,j}+Z_{i,j})+(\alpha_{i,j}-\beta_{i,j})(Z_{i,j}-Z_{i+1j})\\
&\quad -h_i(F^x_{i+1,j+1}-F^x_{i,j+1}))+\alpha_{i,j}(Z_{i,j+1}-Z_{i,j}-\hat{h}_jF^y_{i,j+1})\\
&\quad +\beta_{i,j}(Z_{i,j+1}-Z_{i,j}-\hat{h}_jF^y_{i+1,j+1})+h_i\hat{h}_j(F^{xy}_{i+1,j+1}-F^{xy}_{i,j+1}),\\
C_4 &= \alpha_{i,j}(F_{i,j+1}-Z_{i,j+1})+\beta_{i,j}((\alpha_{i,j}+1)(F_{i,j+1}-Z_{i,j+1})+h_iF^x_{i,j+1}\\
&\quad -Z_{i+1,j}+Z_{i,j})+(\alpha_{i,j}-\beta_{i,j})(Z_{i,j}-Z_{i+1j})-h_i(F^x_{i+1,j+1}-F^x_{i,j+1}).
\end{aligned}$$

$$\chi_{i,j}=\frac{\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^jD_j}{q_j(\phi)}, \tag{5. 7}$$

with

$$\begin{aligned}
D_0 &= \beta_{i,j}(\alpha_{i,j}(F_{i,j}-Z_{i,j})+F_{i,j}+h_iF^x_{i,j}-Z_{i+1,j})+F_{i+1,j}-h_iF^x_{i+1,j}-Z_{i,j},\\
D_1 &= (\hat{\alpha}_{i,j}+1)(\beta_{i,j}(\alpha_{i,j}(F_{i,j}-Z_{i,j})+F_{i,j}+h_iF^x_{i,j}-Z_{i+1,j})+F_{i+1,j}-h_iF^x_{i+1,j}\\
&\quad -Z_{i,j})+(\beta_{i,j}+1)(\hat{h}_jF^y_{i+1,j}-Z_{i,j+1}+Z_{i,j})-h_i\hat{h}_jF^{xy}_{i+1,j},\\
D_2 &= \hat{\alpha}_{i,j}(\beta_{i,j}(\alpha_{i,j}(F_{i,j}-Z_{i,j})+F_{i,j}+h_iF^x_{i,j}-Z_{i+1,j})+F_{i+1,j}-h_iF^x_{i+1,j}-Z_{i,j})\\
&\quad +\hat{\beta}_{i,j}(\beta_{i,j}((\alpha_{i,j}+1)(F_{i,j+1}-Z_{i,j+1})+h_iF^x_{i,j+1}-Z_{i+1,j}+Z_{i,j})+F_{i+1,j+1}\\
&\quad -h_iF^x_{i+1,j+1}-Z_{i,j+1})+(\hat{\alpha}_{i,j}-\hat{\beta}_{i,j})(\beta_{i,j}+1)(Z_{i,j}-Z_{i,j+1})+\hat{h}_j(\beta_{i,j}\\
&\quad +1)(F^y_{i+1,j}-F^y_{i+1,j+1})+h_i\hat{h}_j(F^{xy}_{i+1,j+1}-F^{xy}_{i+1,j}),\\
D_3 &= (\hat{\beta}_{i,j}+1)(\beta_{i,j}((\alpha_{i,j}+1)(F_{i,j+1}-Z_{i,j+1})+h_iF^x_{i,j+1}-Z_{i+1,j}+Z_{i,j})\\
&\quad +F_{i+1,j+1}-h_iF^x_{i+1,j+1}-Z_{i,j+1})+(\beta_{i,j}+1)(Z_{i,j+1}-Z_{i,j}-\hat{h}_jF^y_{i+1,j+1})\\
&\quad +h_i\hat{h}_jF^{xy}_{i+1,j+1},\\
D_4 &= \beta_{i,j}((\alpha_{i,j}+1)(F_{i,j+1}-Z_{i,j+1})+h_iF^x_{i,j+1}-Z_{i+1,j}+Z_{i,j})+F_{i+1,j+1}\\
&\quad -h_iF^x_{i+1,j+1}-Z_{i,j+1}.
\end{aligned}$$

$$\tau_{i,j}=\frac{\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^jE_j}{q_j(\phi)}, \tag{5. 8}$$

with

$$\begin{aligned}
E_0 &= F_{i+1,j} - Z_{i+1,j}, \\
E_1 &= (\hat{\alpha}_{i,j}+1)(F_{i+1,j} - Z_{i+1,j}) + \hat{h}_j F^y_{i+1,j} - Z_{i,j+1} + Z_{i,j}, \\
E_2 &= \hat{\alpha}_{i,j}(F_{i+1,j} - Z_{i+1,j}) + \hat{\beta}_{i,j}(F_{i+1,j+1} - Z_{i+1,j} - Z_{i,j+1} + Z_{i,j}) \\
&\quad -\hat{h}_j(F^y_{i+1,j+1} - F^y_{i+1,j}) + (\hat{\beta}_{i,j} - \hat{\alpha}_{i,j})(Z_{i,j+1} - Z_{i,j}), \\
E_3 &= \hat{\beta}_{i,j}(F_{i+1,j+1} - Z_{i+1,j} - Z_{i,j+1} + Z_{i,j}) - \hat{h}_j F^y_{i+1,j+1} + F_{i+1,j+1} \\
&\quad - Z_{i+1,j}, \\
E_4 &= F_{i+1,j+1} - Z_{i+1,j} - Z_{i,j+1} + Z_{i,j}.
\end{aligned}$$

$U_{i,j}(\theta,\phi) > 0$ if

$$\lambda_{i,j} > 0,\ \mu_{i,j} > 0,\ \nu_{i,j} > 0,\ \chi_{i,j} > 0,\ \tau_{i,j} > 0.$$

$\lambda_{i,j} > 0$ if

$$\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j A_j > 0,\ q_j(\phi) > 0.$$

$q_j(\phi) > 0$ if

$$\hat{\alpha}_{i,j} > 0 \ and\ \hat{\beta}_{i,j} > 0.$$

$\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j A_j > 0$ if

$$\hat{\alpha}_{i,j} = \hat{\beta}_{i,j} > max\left\{\frac{-F_{i,j} - \hat{h}_j F^y_{i,j} + Z_{i,j+1}}{F_{i,j} - Z_{i,j}}, \frac{-F_{i,j+1} + \hat{h}_j F^y_{i,j+1} - Z_{i,j}}{F_{i,j+1} - Z_{i,j+1}}, \frac{\hat{h}_j(F^y_{i,j+1} - F^y_{i,j})}{(F_{i,j} - Z_{i,j})}\right\}.$$

$\mu_{i,j} > 0$ if

$$\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j B_j > 0,\ q_j(\phi) > 0.$$

$q_j(\phi) > 0$ if

$$\hat{\alpha}_{i,j} > 0 \ and\ \hat{\beta}_{i,j} > 0.$$

$\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j B_j > 0$

$$\begin{aligned}
\hat{\alpha}_{i,j} &= \hat{\beta}_{i,j}. \\
\alpha_{i,j} &> max\left\{0, \frac{-F_{i,j} - h_i F^x_{i,j} + Z_{i+1,j}}{F_{i,j} - Z_{i,j}}, \frac{-h_i F^x_{i,j+1} + Z_{i+1,j} - Z_{i,j}}{F_{i,j+1} - Z_{i,j+1}}, \frac{h_i(F^{xy}_{i,j+1} - F^{xy}_{i,j})}{(F^y_{i,j} - F^y_{i,j+1})}\right\}.
\end{aligned}$$

$\nu_{i,j} > 0$ if

$$\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j C_j > 0,\ q_j(\phi) > 0.$$

$q_j(\phi) > 0$ if

$$\hat{\alpha}_{i,j} > 0 \ and\ \hat{\beta}_{i,j} > 0.$$

$\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j C_j > 0$ if

$$\begin{aligned}\hat{\alpha}_{i,j} &= \hat{\beta}_{i,j}.\\ \alpha_{i,j} &= \beta_{i,j}.\\ \alpha_{i,j} &= \beta_{i,j}\\ &> max\left\{0, \frac{h_i(F^x_{i+1,j}-F^x_{i,j})}{(F_{i,j}-Z_{i,j})}, \frac{h_i(F^x_{i+1,j+1}-F^x_{i,j+1})}{(F_{i,j+1}-Z_{i,j+1})},\right.\\ &\left.\frac{-h_i(F^{xy}_{i,j}-F^{xy}_{i+1,j}+F^{xy}_{i+1,j+1}-F^{xy}_{i,j+1})}{(F^y_{i,j}-F^y_{i,j+1}+F^y_{i+1,j}-F^y_{i+1,j+1})}\right\}.\end{aligned}$$

$\chi_{i,j} > 0$ if

$$\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j D_j >,\ q_j(\phi) > 0.$$

$q_j(\phi) > 0$ if

$$\hat{\alpha}_{i,j} > 0 \ and \ \hat{\beta}_{i,j} > 0.$$

$\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j D_j > 0$ if

$$\begin{aligned}\hat{\alpha}_{i,j} &= \hat{\beta}_{i,j}.\\ \beta_{i,j} &> max\left\{0, \frac{-F_{i+1,j}+h_iF^x_{i+1,j}+Z_{i,j}}{F_{i+1,j}-Z_{i+1,j}}, \frac{-F_{i+1,j+1}+h_iF^x_{i+1,j+1}+Z_{i,j+1}}{F_{i+1,j+1}-Z_{i,j+1}-Z_{i+1,j}+Z_{i,j}},\right.\\ &\left.\frac{h_i(F^{xy}_{i+1,j}-F^{xy}_{i+1,j+1})}{(F^y_{i+1,j}-F^y_{i+1,j+1})}\right\}.\end{aligned}$$

$\tau_{i,j} > 0$ if

$$\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j E_j > 0,\ q_j(\phi) > 0.$$

$q_j(\phi) > 0$ if

$$\hat{\alpha}_{i,j} > 0 \ and \ \hat{\beta}_{i,j} > 0.$$

$\sum_{j=0}^{4}(1-\phi)^{4-j}\phi^j E_j > 0$ if

$$\begin{aligned}\hat{\alpha}_{i,j} = \hat{\beta}_{i,j} \quad &> \quad max\left\{\frac{-\hat{h}_jF^y_{i+1,j}+Z_{i,j+1}-Z_{i,j}}{F_{i+1,j}-Z_{i+1,j}}, \frac{-F_{i+1,j+1}+\hat{h}_jF^y_{i+1,j+1}+Z_{i+1,j}}{F_{i+1,j+1}-Z_{i+1,j}-Z_{i,j+1}+Z_{i,j}},\right.\\ &\left.\frac{\hat{h}_j(F^y_{i+1,j+1}-F^y_{i+1,j})}{(F_{i+1,j}-Z_{i+1,j})}\right\}.\end{aligned}$$

All this discussion is summarized in the following theorem.

Theorem 5.1. The rational bicubic function (3.2) generates surface that lie on same side of plane as that of data if free parameters $\alpha_{i,j}$, $\beta_{i,j}$, $\hat{\alpha}_{i,j}$ and $\hat{\beta}_{i,j}$ satisfy the following conditions:

$$\begin{aligned}\hat{\alpha}_{i,j} &= \hat{\beta}_{i,j}\\ &= g_{i,j} + max\{0, \frac{-F_{i,j} - \hat{h}_j F^y_{i,j} + Z_{i,j+1}}{F_{i,j} - Z_{i,j}}, \frac{-F_{i,j+1} + \hat{h}_j F^y_{i,j+1} - Z_{i,j}}{F_{i,j+1} - Z_{i,j+1}},\\ &\frac{\hat{h}_j(F^y_{i,j+1} - F^y_{i,j})}{(F_{i,j} - Z_{i,j})}, \frac{-\hat{h}_j F^y_{i+1,j} + Z_{i,j+1} - Z_{i,j}}{F_{i+1,j} - Z_{i+1,j}}, \frac{\hat{h}_j(F^y_{i+1,j+1} - F^y_{i+1,j})}{(F_{i+1,j} - Z_{i+1,j})},\\ &\frac{-F_{i+1,j+1} + \hat{h}_j F^y_{i+1,j+1} + Z_{i+1,j}}{F_{i+1,j+1} - Z_{i+1,j} - Z_{i,j+1} + Z_{i,j}}\}, \ g_{i,j} > 0.\end{aligned} \tag{5. 9}$$

$$\begin{aligned}\alpha_{i,j} = \beta_{i,j} &= k_{i,j} + max\{0, \frac{-F_{i,j} - h_i F^x_{i,j} + Z_{i+1,j}}{F_{i,j} - Z_{i,j}}, \frac{-h_i F^x_{i,j+1} + Z_{i+1,j} - Z_{i,j}}{F_{i,j+1} - Z_{i,j+1}},\\ &\frac{h_i(F^{xy}_{i,j+1} - F^{xy}_{i,j})}{(F^y_{i,j} - F^y_{i,j+1})}, \frac{-F_{i+1,j} + h_i F^x_{i+1,j} + Z_{i,j}}{F_{i+1,j} - Z_{i+1,j}}, \frac{h_i(F^{xy}_{i+1,j} - F^{xy}_{i+1,j+1})}{(F^y_{i+1,j} - F^y_{i+1,j+1})},\\ &\frac{-F_{i+1,j+1} + h_i F^x_{i+1,j+1} + Z_{i,j+1}}{F_{i+1,j+1} - Z_{i,j+1} - Z_{i+1,j} + Z_{i,j}}, \frac{h_i(F^x_{i+1,j} - F^x_{i,j})}{(F_{i,j} - Z_{i,j})},\\ &\frac{h_i(F^x_{i+1,j+1} - F^x_{i,j+1})}{(F_{i,j+1} - Z_{i,j+1})},\\ &\frac{-h_i(F^{xy}_{i,j} - F^{xy}_{i+1,j} + F^{xy}_{i+1,j+1} - F^{xy}_{i,j+1})}{(F^y_{i,j} - F^y_{i,j+1} + F^y_{i+1,j} - F^y_{i+1,j+1})}\}, \ k_{i,j} > 0.\end{aligned} \tag{5. 10}$$

6. Applications

A set of positive data is considered in Table 1. The data is generated from following function:

$$F = \ln(x^2 + y^2) + 10.$$

Figure 1 is produced by implementing the scheme developed in Theorem 4.1. From

Table 1

y/x	0.01	100	200	300
0.01	1.4828	19.2103	20.5966	21.4076
100	19.2103	19.9035	20.8198	21.5129
200	20.5966	20.8198	21.2898	21.7753
300	21.4076	21.5129	21.7753	22.1007

the figure it is clear that the positive surface is generated through positive data of Table 1. Another set of positive data is considered in Table 2. These data is generated from the following function:

$$F = 0.5(||x| - |y|| - |x| - |y|) + 3.1.$$

Figure 2 is produced by implementing Theorem 4.1 on data set in Table 2. From the figure it is clear that a positivity is assured. The data set of Table 3 is of the plane:

$$Z = \left(1 - \frac{x}{6} - \frac{y}{6}\right).$$

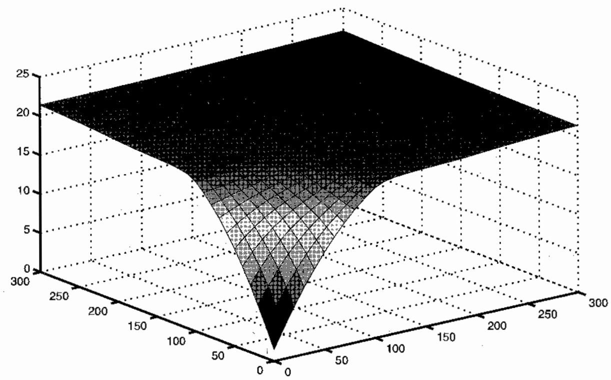

FIGURE 1

TABLE 2

y/x	-3	-2	-1	1	2	3
-3	0.1	1.1	2.1	2.1	1.1	0.1
-2	1.1	1.1	2.1	2.1	1.1	1.1
-1	2.1	2.1	2.1	2.1	2.1	2.1
1	2.1	2.1	2.1	2.1	2.1	2.1
2	1.1	1.1	2.1	2.1	1.1	1.1
3	0.1	1.1	2.1	2.1	1.1	0.1

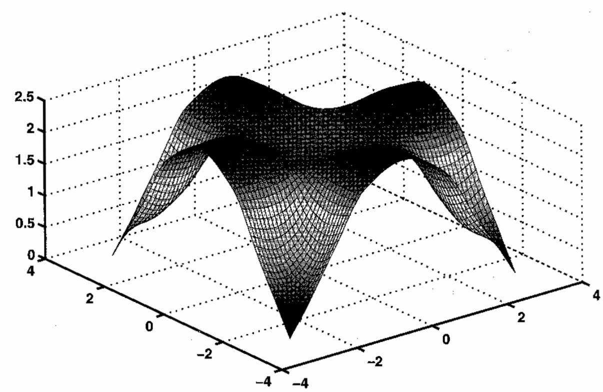

FIGURE 2

TABLE 3

y /x	1	2	3	4	5	6
1	0.6667	0.5000	0.3333	0.1667	0.0000	-0.1667
2	0.5000	0.3333	0.1667	0.0000	-0.1667	-0.3333
3	0.3333	0.1667	0.0000	-0.1667	-0.3333	-0.5000
4	0.1667	0.0000	-0.1667	-0.3333	-0.5000	-0.6667
5	-0.0000	-0.1667	-0.3333	-0.5000	-0.6667	-0.8333
6	-0.1667	-0.3333	-0.5000	-0.6667	-0.8333	-1.0000

TABLE 4

y/x	1	2	3	4	5	6
1	2	5	10	17	26	37
2	5	8	13	20	29	40
3	10	13	18	25	34	45
4	17	20	25	32	41	52
5	26	29	34	41	50	61
6	37	40	45	52	61	72

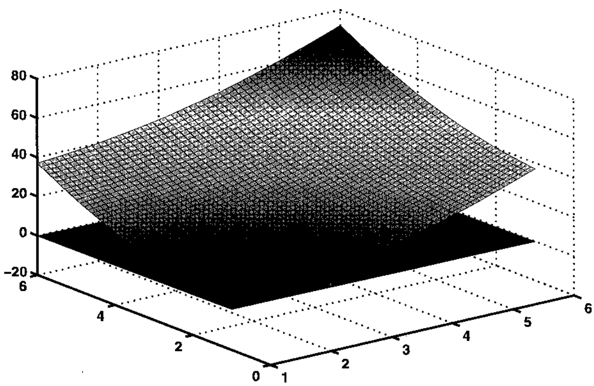

FIGURE 3

The data set in Table 4 is generated from the following function:

$$F = x^2 + y^2.$$

The data set in Table 4 is lying above the plane:

$$Z = \left(1 - \frac{x}{6} - \frac{y}{6}\right).$$

Figure 3 is generated using the scheme developed in Section 5. From the figure it is clear that surface lies above the plane. The data set in Table 5 is generated from the function

$$F = sin(|x| + |y|) + 2.$$

TABLE 5

y/x	1	2	3	4	5	6
1	2.9093	2.1411	1.2432	1.0411	1.7206	2.6570
2	2.1411	1.2432	1.0411	1.7206	2.6570	2.9894
3	1.2432	1.0411	1.7206	2.6570	2.9894	2.4121
4	1.0411	1.7206	2.6570	2.9894	2.4121	1.4560
5	1.7206	2.6570	2.9894	2.4121	1.4560	1.0000
6	2.6570	2.9894	2.4121	1.4560	1.0000	1.4634

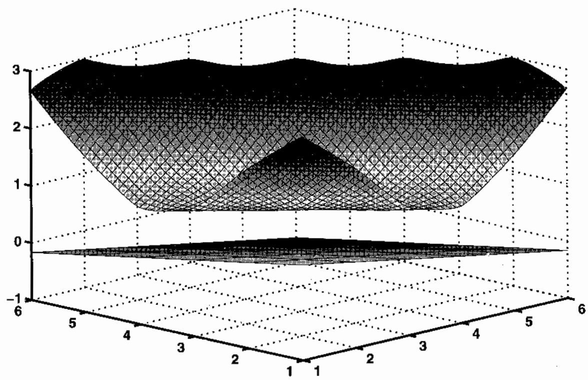

FIGURE 4

This data set is lying above the plane:

$$Z = \left(1 - \frac{x}{6} - \frac{y}{6}\right).$$

Figure 4 is produced by using the scheme developed in Section 5. From the figure it is clear that surface lies above the plane.

7. CONCLUSION

The paper is concerned with two major problems of scientific visualization, shape preservation and shape control. To achieve the goal, the method of variational design is adopted, i.e. introduction of free parameters in the description of rational bicubic function. Four free parameters $\alpha_{i,j}$, $\beta_{i,j}$, $\hat{\alpha}_{i,j}$ and $\hat{\beta}_{i,j}$ are introduced in the definition of a rational bicubic function. The free parameters are turned into the constrained parameters to preserve the shape of data. For shape control additional design elements (free parameters) are introduced in the definition of constrained parameters. Advantageous features of the methods are that they are applicable to both the cases when derivatives values are provided or estimated by some method. It works for equally as well as unequally spaced data. It is easy to implement as compared to previous methods. Same algorithm is applicable to every data. For example in the method of inserting knots we have to implement different knots for different data.

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