Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 38(2006) pp. 9-13

Some Notes On An Integral Inequality Related To G.H. Hardy'S Integral Inequality

K. Rauf
Department of Mathematics
University of Ilorin
Ilorin, Nigeria
E-mail: balk_ r@yahoo.com

J.O. Omolehin
Department of Mathematics
University of Ilorin
Ilorin, Nigeria
E-mail: omolehin_ joseph@yahoo.com

Abstract. We obtain an extension of Hardy inequality for convex functions, which is a special case of Boas's version of the Hardy's integral inequality.

1. Introduction

Hardy in an attempt to simplify Hilbert's integral inequality ([5], Theorem 316) discovered the following result:

Theorem A *If* $p > 1$,$f(x) \geq 0$ *and* $F(x) = \int_0^x f(t)d(t)$,*then*

$$\int_0^\infty (\frac{F}{x})^p dx < (\frac{p}{p-1})^p \int_0^\infty f(x)^p dx \tag{1. 1}$$

holds,unless $f \equiv 0$. The constant $(\frac{p}{p-1})^p$ is the best possible.

This result is called the Hardy's inequality (see, for example [4] and [5], Theorem 327).

Another inequality due to Hardy ([5], Theorem 328) was given by using the converse of Holder's inequality as follows:

Theorem B *If* $p > 1$,$f(x) \geq 0$ *and* $F(x) = \int_x^\infty f(t)d(t)$,*then*

$$\int_0^\infty F^p dx < p^p \int_0^\infty (xf)^p dx \tag{1. 2}$$

holds, unless $f \equiv 0$. The constant p^p is the best possible. Since, the inequality has wide applications in analysis. A number of researchers have developed interest in the results and a lot of effort and time have been expended in the study and extension of the inequality in various directions (see for example [5] ,chapter ix). Of particular interest is the work due to [2]. The main purpose of this paper, therefore, is to study and establish some new inequalities from the Jensen inequality

for convex functions. Indeed, our main result is an extension of Boas's version of the Hardy's inequality where the variable x additionally is a linear function of two further variables, says u and v, and then obtains an estimate of the integral of the double integral of f with respect to the variables u and v. Special cases of our result yield some of the earlier, as well as some recent generalization of Hardy's inequality given in [[1], [6], [7], and [9] - [12]] and on convex function [[3] and [8]]. The result is as follows:

Theorem 1. *Let $g(x)$ be continuous and non-decreasing on$[0,\infty]$ with $g(0)=0$, $g(x)>0$, $g(\infty)=\infty$.$\sum_{i=1}^{n} u_i = u$ and $\sum_{i=1}^{n} v_i = v$ where x, u and v are all positive. Also, let $f:[a,b]\to\Re(a<b)$, be continuous and convex on the real interval $[a,b]$.Assume $\prod_{i=1}^{n}\alpha_i\beta_i \geq 0$ with $\sum_{i=1}^{n}(\alpha+\beta)>0$ for all $i\in\aleph$.*
Then, the following inequality holds:

$$\int_0^\infty g(x)^{-1}[\int_a^b\int_a^b f(\sum_{i=1}^{n}(\alpha_i u_i+\beta_i v_i))u^{\alpha-1}dudv]^p dg(x) \leq L^p\int_0^\infty G(x)dg(x) \tag{1. 3}$$

where, $L=(\alpha^{-1}\beta)(b-a)(b^\alpha-a^\alpha)(k+1)$ and $G(x)=f(x)^p g(x)^{-1}$.

We note, however, that the left sides of (1.1), (1.2) and (1.3) exist when the right sides do.

2. PRELIMINARY LEMMAS

We shall prove the following lemma which is a simple consequence of the Jensen inequality proved by standard methods.

Lemma 2.1 *If Φ is convex and continuous, f is a non-negative and λ is non-decreasing on $[0,\infty]$.Then,*

$$\int_0^\infty g(x)^{-1}\Phi[L^{-1}\int_0^\infty f(v)d\lambda(v)]dg(x) \leq \int_0^\infty g(x)^{-1}\Phi(f(x))dg(x) \tag{2. 1}$$

where, $L=\int_0^\infty d\lambda(v)$ with the inequality reversed when Φ is concave.

Proof. Let Φ be convex, then Jensen's inequality says

$$\Phi(\int_0^\infty f(v)d\lambda(v)) \leq \int_0^\infty \Phi f(v)d\lambda(v)$$

Hence,

$$\begin{aligned}\int_0^\infty g(x)^{-1}\Phi[L^{-1}\int_0^\infty f(v)d\lambda(v)]dg(x) &\leq L^{-1}\int_0^\infty g(x)^{-1}dg(x)\int_0^\infty \Phi(f(v))d\lambda(v)\\ &= L^{-1}\int_0^\infty d\lambda(v)\int_0^\infty g(x)^{-1}\Phi(f(x))dg(x)\\ &= L^{-1}L\int_0^\infty g(x)^{-1}\Phi(f(x))dg(x)\\ &= \int_0^\infty g(x)^{-1}\Phi(f(x))dg(x)\end{aligned}$$

Similar result was obtained by replacing $g(x)$ by x in [2] and this proves the lemma when Φ is convex. □

In order to obtain the next result, we shall make use of lemma 2.1 and our method of proof is simply one step of integration by parts.

Corollary 2.1 *If $p > 1$, $f \geq 0$, g is continuous, non-decreasing on $[0, \infty)$. Let Φ be continuous and convex and suppose Φ has a continuous inverse (which is necessarily concave) on $[0, \infty)$ and $d\lambda(v)$ be defined as $v^{\alpha-1}dv$ on $[0, 1]$ and 0 for $v > 1, \alpha > 0$. Then,*

$$\int_0^\infty g(x)^{-1}\Phi[\int_0^1 f(v)v^{\alpha-1}dv]^p dg(x) \leq \alpha^{-p}\int_0^\infty g(x)^{-1}\Phi f(x)^p dg(x) \qquad (2.\ 2)$$

Proof. Let

$$I = \int_0^\infty g(x)^{-1}\Phi[\int_0^1 f(v)v^{\alpha-1}dv]^p dg(x)$$

Then, integrating by part of the inner integral yields

$$\begin{aligned} & \int_0^\infty g(x)^{-1}\Phi[\left[\frac{f(v)v^\alpha}{\alpha}\right]_0^1 - \int_0^1 \frac{v^\alpha}{\alpha} f^{'}(v)dv]^p dg(x) \\ = & \int_0^\infty g(x)^{-1}\Phi(\alpha^{-1}f(v))^p dg(x) - [Non-negativeterm] \\ \leq & \alpha^{-p}\int_0^\infty g(x)^{-1}\Phi f(v)^p dg(x). \qquad (2.\ 3) \end{aligned}$$

this completes the proof of the corollary. □

Remark 2. If $v^{\alpha-1} = L^{-1}, \alpha = p = 1$ and dv be defined as $d\lambda(v)$ on $[0, x]$, then we get lemma 2.1.
Also, if $\alpha = 1 - \frac{1}{p}$ replace $g(x)$ by x on $[0, \infty]$ and $\Phi = v = 1$, then we get (1.1).
Similarly, we get (1.2) by letting $\alpha = p, g(x) = x, (x^{-1}\Phi) = x^p$ and $v^{\alpha-1} = x^{-1}$ on $[x, \infty]$.

3. Proof of Theorem 1

The method of proof of this theorem is induction by means of partial integration. From inequality (3), we obtain

$$I = \int_0^\infty g(x)^{-1}[\int_a^b \int_a^b f(\sum_{i=1}^n (\alpha_i u_i + \beta_i v_i))u^{\alpha-1}dudv]^p dg(x)$$

Also, we obtain on using integration by part of the inner integral

$$I \leq \alpha^{-p}(b^\alpha - a^\alpha)^p \int_0^\infty g(x)^{-1}[\int_a^b f(\sum_{i=1}^n (\alpha_i(b_i - a_i) + \beta_i v_i))dv]^p dg(x)$$

Let $i = 1$ then,

$$I \leq \alpha^{-p}(b^\alpha - a^\alpha)^p \int_0^\infty g(x)^{-1}[\int_a^b f(\alpha_1(b_1 - a_1) + \beta_1 v_1)dv]^p dg(x)$$

Assume the theorem for $i = k > 1$, we have

$$I \leq \int_0^\infty g(x)^{-1}[\alpha^{-1}(b^\alpha - a^\alpha)\int_a^b f(\sum_{i=1}^{k}(\alpha_i(b_i - a_i) + \beta_i v_i))dv]^p dg(x)$$

Then, for $i = k + 1$, we have

$$I \leq \int_0^\infty g(x)^{-1}[\alpha^{-1}(b^\alpha - a^\alpha)\int_a^b f(\sum_{i=1}^{k+1}(\alpha_i(b_i - a_i) + \beta_i v_i))dv]^p dg(x)$$

To see this, we note that f is a continuous and convex function on a real interval $[a, b]$.
Then,

$$\begin{aligned} f(\sum_{i=1}^{k+1}(\alpha_i(b_i - a_i) + \beta_i v_i)) &= f(\sum_{i=1}^{k}(\alpha_i(b_i - a_i) + \beta_i v_i) + \alpha_{k+1}(b_{k+1} - a_{k+1}) \\ &\quad +\beta_{k+1}v_{k+1}) \\ &\leq (\sum_{i=1}^{k}(\alpha_i(b_i - a_i) + \beta_i f(v_i)) + \alpha_{k+1}(b_{k+1} - a_{k+1}) \\ &\quad +\beta_{k+1}f(v_{k+1})) \end{aligned} \tag{3. 1}$$

Integrating both sides of (3.1) on the $(k + 1)$ rectangles $[a, b]X[a, b]X...X[a, b]$ from a to b
We then have,

$$\begin{aligned} \int_a^b [f(\sum_{i=1}^{k+1}(\alpha_i(b_i - a_i) + \beta_i v_i))]dv &\leq \int_a^b [(\sum_{i=1}^{k}(\alpha_i(b_i - a_i) + \beta_i f(v_i) \\ &\quad +\alpha_{k+1}(b_{k+1} - a_{k+1}) + \beta_{k+1}f(v_{k+1})))]dv \\ &= \int_a^b [\sum_{i=1}^{k}(\alpha_i(b_i - a_i) + \beta_i f(v_i))]dv \\ &\quad + \int_a^b [(\alpha_{k+1}(b_{k+1} - a_{k+1}) + \beta_{k+1}f(v_{k+1}))]dv \\ &= k(b - a)\beta f + (b - a)\beta f \\ &= (b - a)(k + 1)\beta f \end{aligned}$$

Therefore,

$$\begin{aligned} I &\leq [(\alpha^{-1}\beta)(b - a)(b^\alpha - a^\alpha)(k + 1)]^p \int_0^\infty g(x)^{-1}f(x)^p dg(x) \\ &= L^p \int_0^\infty G(x)dg(x). \end{aligned}$$

This completes the proof of the theorem.

Remark 3. If we take $\alpha = 1, v = 0, L = (1 - \frac{1}{p})^{-1}$ and $g(x) = x$ and replace $f(x)^p x^{-1}$ by $f(x)^p$. We then obtain a useful version of Hardy's inequality (1.1). Similarly, we get (1.2) by letting $L = p$.

References

[1] Bicheng Y., Zhuohua Z. and Debnath, L. , *Note on new-generalizations of Hardy's Integral Inequality* , Journal of Mathematical Analysis and applications ; **217**(1998), No.4 , 321-327.

[2] Boas,R.P.Jr. , *Some Integral Inequalities Related to Hardy's Inequality*, Journal d'Analyse Math. ; **23**(1971), 53-63.

[3] Dragomir,S.S. , *On Some Integral Inequalities for Convex Function* , Zbornik Radova Prirodno-matematickog fakulteta u Kragujevcu ; **18**(1996), 21-24.

[4] Hardy G. H. , *Notes on some points in the integral calculus (LXIV)* , Messenger of Math.; **57**(1928), No.3, 12-16.

[5] Hardy, G..H., Littlewood, J.E.and Polya,G. , *Inequalities. Second edition* , Cambridge Univ. Press ; **2**, (1952).

[6] Imoru,C.O. , *On some extensions of Hardy's inequality* , Internat. J. Math and Math. Sc.; **8**(1), (1995), 165-171.

[7] Izumi,M.Izumi,S. and Peterson,G. , *On Hardy's Ineqality and its Generalization* , Tohoku Math .J. ; **21**, (1999), 601-613.

[8] Oguntuase, J. A., , *Remark on some integral inequalities for convex functions* , Zbornik Radova Prirodno-matematickog fakulteta u Kragujevcu; **21**(1999), 49-53.

[9] Rauf, K. , *A Characterization of Weighted Norm Inequalities for the Hardy- Littlewood Maximal Function* , The Nigerian Journal of Pure and Applied Sciences; **19** (2004), 1734 -1740.

[10] Rauf, K. and Imoru, C.O. , *Some generalisation of weighted norm inequalities for a certain class of integral operators*, Kragujevac Journal of Mathematics; **24**(2002), 95-105.

[11] Rauf, K. and Omolehin, J.O. , *On weighted norm integral inequality of G.H. Hardy's type* , Kragujevac Journal of Mathematics; **29**(2006), 165-173.

[12] Stepanov, V. D. , *A weighted inequality of Hardy type for higher order derivatives* , Trud Mathematishe Institute,Steklov ; **187**(1989), 178-190.