Some Notes On An Integral Inequality
Related To G.H. Hardy’S Integral Inequality

K. Rauf
Department of Mathematics
University of Ilorin
Ilorin, Nigeria
E-mail: balk_r@yahoo.com

J.O. Omolehin
Department of Mathematics
University of Ilorin
Ilorin, Nigeria
E-mail: omolehin_joseph@yahoo.com

Abstract. We obtain an extension of Hardy inequality for convex functions, which is a special case of Boas’s version of the Hardy’s integral inequality.

1. Introduction

Hardy in an attempt to simplify Hilbert’s integral inequality ([5], Theorem 316) discovered the following result:

Theorem A If $p > 1, f(x) \geq 0$ and $F(x) = \int_0^x f(t)dt$, then

$$\int_0^\infty \left( \frac{F}{x} \right)^p dx < \left( \frac{p}{p-1} \right)^p \int_0^\infty f(x)^p dx$$

(1. 1)

holds, unless $f \equiv 0$. The constant $\left( \frac{p}{p-1} \right)^p$ is the best possible.

This result is called the Hardy’s inequality (see, for example [4] and [5], Theorem 327).

Another inequality due to Hardy ([5], Theorem 328) was given by using the converse of Holder’s inequality as follows:

Theorem B If $p > 1, f(x) \geq 0$ and $F(x) = \int_x^\infty f(t)dt$, then

$$\int_0^\infty F^p dx < p^p \int_0^\infty (xf)^p dx$$

(1. 2)

holds, unless $f \equiv 0$. The constant $p^p$ is the best possible. Since, the inequality has wide applications in analysis. A number of researchers have developed interest in the results and a lot of effort and time have been expended in the study and extension of the inequality in various directions (see for example [5], chapter ix). Of particular interest is the work due to [2]. The main purpose of this paper, therefore, is to study and establish some new inequalities from the Jensen inequality.
Hence, \( L \) where, interval \([0, \infty)\) is decreasing on \([0, \infty)\). Assume \( \prod_{i=1}^{n} \alpha_i \beta_i \geq 0 \) with \( \sum_{i=1}^{n} (\alpha + \beta) > 0 \) for all \( i \in \mathbb{R} \). Then, the following inequality holds:

\[
\int_{0}^{\infty} g(x)^{-1} \left[ \int_{a}^{b} f(\sum_{i=1}^{n} (\alpha_i u_i + \beta_i v_i)) u^{\alpha-1} dudv \right] dg(x) \leq L^{p} \int_{0}^{\infty} G(x) dg(x)
\]

(1.3)

where, \( L = (\alpha^{-1} \beta)(b-a)(b^{\alpha} - a^{\alpha})(k+1) \) and \( G(x) = f(x)^{p} g(x)^{-1} \).

We note, however, that the left sides of (1.1), (1.2) and (1.3) exist when the right sides do.

2. PRELIMINARY LEMMAS

We shall prove the following lemma which is a simple consequence of the Jensen inequality proved by standard methods.

**Lemma 2.1** If \( \Phi \) is convex and continuous, \( f \) is a non-negative and \( \lambda \) is non-decreasing on \([0, \infty)\). Then,

\[
\int_{0}^{\infty} g(x)^{-1} \Phi[L^{-1} \int_{0}^{\infty} f(v) d\lambda(v)] dg(x) \leq \int_{0}^{\infty} g(x)^{-1} \Phi(f(x)) dg(x)
\]

(2.1)

where, \( L = \int_{0}^{\infty} d\lambda(v) \) with the inequality reversed when \( \Phi \) is concave.

**Proof.** Let \( \Phi \) be convex, then Jensen’s inequality says

\[
\Phi\left( \int_{0}^{\infty} f(v) d\lambda(v) \right) \leq \int_{0}^{\infty} \Phi(f(v)) d\lambda(v)
\]

Hence,

\[
\int_{0}^{\infty} g(x)^{-1} \Phi[L^{-1} \int_{0}^{\infty} f(v) d\lambda(v)] dg(x) \leq L^{-1} \int_{0}^{\infty} g(x)^{-1} dg(x) \int_{0}^{\infty} \Phi(f(v)) d\lambda(v)
\]

\[
= L^{-1} \int_{0}^{\infty} d\lambda(v) \int_{0}^{\infty} g(x)^{-1} \Phi(f(x)) dg(x)
\]

\[
= L^{-1} L \int_{0}^{\infty} g(x)^{-1} \Phi(f(x)) dg(x)
\]

\[
= \int_{0}^{\infty} g(x)^{-1} \Phi(f(x)) dg(x)
\]
Similar result was obtained by replacing $g(x)$ by $x$ in [2] and this proves the lemma when $\Phi$ is convex. □

In order to obtain the next result, we shall make use of lemma 2.1 and our method of proof is simply one step of integration by parts.

**Corollary 2.1** If $p > 1$, $f \geq 0$, $g$ is continuous, non-decreasing on $[0, \infty)$.

Let $\Phi$ be continuous and convex and suppose $\Phi$ has a continuous inverse (which is necessarily concave) on $[0, \infty)$ and $d\lambda(v)$ be defined as $v^{\alpha - 1}dv$ on $[0,1]$ and 0 for $v > 1, \alpha > 0$. Then,

$$
\int_0^\infty g(x)^{-1}\Phi\left[\int_0^1 f(v)v^{\alpha - 1}dv\right]^pdg(x) \leq \alpha^{-p}\int_0^\infty g(x)^{-1}\Phi f(x)^pdg(x) \quad (2.2)
$$

**Proof.** Let

$$
I = \int_0^\infty g(x)^{-1}\Phi\left[\int_0^1 f(v)v^{\alpha - 1}dv\right]^pdg(x)
$$

Then, integrating by part of the inner integral yields

$$
\int_0^\infty g(x)^{-1}\Phi\left[\frac{f(v)v^{\alpha - 1}}{\alpha}\right]_0^1 - \int_0^\infty \frac{v^\alpha f'(v)}{\alpha}dvdg(x)
$$

$$
= \int_0^\infty g(x)^{-1}\Phi(\alpha^{-1}f(v))^pdg(x) - [Non - negative term]
$$

$$
\leq \alpha^{-p}\int_0^\infty g(x)^{-1}\Phi f(v)^pdg(x). \quad (2.3)
$$

this completes the proof of the corollary. □

**Remark 2.** If $v^{\alpha - 1} = L^{-1}, \alpha = p = 1$ and $dv$ be defined as $d\lambda(v)$ on $[0, x]$, then we get lemma 2.1.

Also, if $\alpha = 1 - \frac{1}{p}$ replace $g(x)$ by $x$ on $[0, \infty]$ and $\Phi = v = 1$, then we get (1.1).

Similarly, we get (1.2) by letting $\alpha = p, g(x) = x, (x^{-1}\Phi) = x^p$ and $v^{\alpha - 1} = x^{-1}$ on $[x, \infty]$.

### 3. Proof of Theorem 1

The method of proof of this theorem is induction by means of partial integration. From inequality (3), we obtain

$$
I = \int_0^\infty g(x)^{-1}\int_a^b f\left(\sum_{i=1}^n (\alpha_i u_i + \beta_i v_i)\right)u^{\alpha - 1}du\,dg(x)
$$

Also, we obtain on using integration by part of the inner integral

$$
I \leq \alpha^{-p}(b^\alpha - a^\alpha)^p\int_0^\infty g(x)^{-1}\int_a^b f\left(\sum_{i=1}^n (\alpha_i (b_i - a_i) + \beta_i v_i)\right)dv\,dg(x)
$$

Let $i = 1$ then,

$$
I \leq \alpha^{-p}(b^\alpha - a^\alpha)^p\int_0^\infty g(x)^{-1}\left[\int_a^b f(\alpha_1 (b_1 - a_1) + \beta_1 v_1)dv\right]^pdg(x)
$$
Assume the theorem for \( i = k > 1 \), we have

\[
I \leq \int_0^\infty g(x)^{-1}[\alpha^{-1}(b^\alpha - a^\alpha) \int_a^b f\left(\sum_{i=1}^k (\alpha_i(b_i - a_i) + \beta_i v_i)\right)dv]dg(x)
\]

Then, for \( i = k + 1 \), we have

\[
I \leq \int_0^\infty g(x)^{-1}[\alpha^{-1}(b^\alpha - a^\alpha) \int_a^b f\left(\sum_{i=1}^{k+1} (\alpha_i(b_i - a_i) + \beta_i v_i)\right)dv]dg(x)
\]

To see this, we note that \( f \) is a continuous and convex function on a real interval \( [a, b] \).

Then,

\[
f\left(\sum_{i=1}^{k+1} (\alpha_i(b_i - a_i) + \beta_i v_i)\right) = f\left(\sum_{i=1}^k (\alpha_i(b_i - a_i) + \beta_i v_i) + \alpha_{k+1}(b_{k+1} - a_{k+1}) + \beta_{k+1}v_{k+1}\right)
\]

\[
\leq \left(\sum_{i=1}^k (\alpha_i(b_i - a_i) + \beta_i f(v_i)) + \alpha_{k+1}(b_{k+1} - a_{k+1}) + \beta_{k+1}f(v_{k+1})\right)
\]

Integrating both sides of (3.1) on the \((k + 1)\) rectangles \([a, b]X[a, b]...X[a, b]\) from \( a \) to \( b \).

We then have,

\[
\int_a^b f\left(\sum_{i=1}^{k+1} (\alpha_i(b_i - a_i) + \beta_i v_i)\right)dv \leq \int_a^b f\left(\sum_{i=1}^k (\alpha_i(b_i - a_i) + \beta_i f(v_i)) + \alpha_{k+1}(b_{k+1} - a_{k+1}) + \beta_{k+1}f(v_{k+1})\right)dv
\]

\[
= \int_a^b f\left(\sum_{i=1}^k (\alpha_i(b_i - a_i) + \beta_i f(v_i))\right)dv
\]

\[
+ \int_a^b (\alpha_{k+1}(b_{k+1} - a_{k+1}) + \beta_{k+1}f(v_{k+1}))dv
\]

\[
= k(b - a)\beta f + (b - a)\beta f
\]

\[
= (b - a)(k + 1)\beta f
\]

Therefore,

\[
I \leq \frac{[(\alpha^{-1}\beta)(b - a)(a^\alpha - b^\alpha)(k + 1)]^p \int_0^\infty g(x)^{-1}f(x)^pdg(x)}{L^p \int_0^\infty G(x)dg(x)}
\]

This completes the proof of the theorem.

Remark 3. If we take \( \alpha = 1, v = 0, L = (1 - \frac{1}{p})^{-1} \) and \( g(x) = x \) and replace \( f(x)^p x^{-1} \) by \( f(x)^p \). We then obtain a useful version of Hardy’s inequality (1.1). Similarly, we get (1.2) by letting \( L = p \).
REFERENCES


