FIXED COEFFICIENTS FOR CERTAIN SUBCLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

In this paper we consider the class $T_c(n, \lambda, \alpha)$ consisting of analytic and univalent functions with negative coefficients and fixed second coefficient. The object of the present paper is to show coefficient estimates, convex linear combinations, some distortion theorems and radii of starlikeness and convexity for $f(z)$ in the class $T_c(n, \lambda, \alpha)$. The results are generalized to families with finitely many fixed coefficients.

1. INTRODUCTION

Let $S$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. Given two
functions \( f, g \in S \), where \( f(z) \) is given by (1.1) and \( g(z) \) is given by

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k,
\]

(1.2)

Their Hadamard product (or convolution) \( f \ast g(z) \), is defined by

\[
f \ast g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad (z \in U)
\]

(1.3)

By using Hadamard product, Ruscheweyh [4] defined

\[
D^\beta f(z) = \frac{z}{(1 - z)^{\beta + 1}} \ast f(z) (\beta \geq -1)
\]

(1.4)

and observed that

\[
D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}
\]

(1.5)

where \( \beta = n \in N_0 = NU\{0\}; \) and \( N = \{1, 2, \cdots\} \). This symbol \( D^n f(z) (n \in N_0) \) was called teh \( n \)-th order Ruscheweyh derivative of \( f(z) \) by Al-Amiri [1]. We note that \( D^0 f(z) = f(z) \) and \( D^1 f(z) = zf'(z) \). It is easy to see that

\[
D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k,
\]

(1.6)

where

\[
\delta(n,k) = \binom{n+k-1}{n}
\]

(1.7)

Note that

\[
z(D^n f(z))' = (n + 1)D^{n+1} f(z) - nD^n f(z)
\]

(1.8)

Let \( T \) denote the subclass of \( S \) consisting of functions of the form:

\[
f(z) = z - \sum_{k=2}^{\infty} a_k z^k (a_k \geq 0)
\]

(1.9)
Further, we say that a function \( f(z) \) belonging to \( T \) is in the class \( T(n, \lambda, \alpha) \) if and only if
\[
\Re \left\{ \frac{D^{n+1}f(z)}{D^nf(z)} \right\} > \alpha, \quad (n \in \mathbb{N}_0) \tag{1.10}
\]
for some \( 0 \leq \alpha < 1 \), \( 0 \leq \lambda < 1 \), and for all \( z \in U \). The class \( T(n, \lambda, \alpha) \) was introduced by Aouf and Chen [3]. We note that by specializing the parameters \( n, \lambda \) and \( \alpha \), we obtain the following subclasses studied by various authors:

1. \( T(0, \lambda, \alpha) = T(\lambda, \alpha) \) (Altintas and Owa [2]);
2. \( T(0, 0, \alpha) = T^{*}(\alpha) \) (Silverman [5]);
3. \( T(n, 0, \alpha) \) represents the class of functions \( f(z) \in T \) satisfying the condition
\[
\Re \left\{ \frac{D^{n+1}f(z)}{D^nf(z)} \right\} > \alpha(0 \leq \alpha < 1; \ n \in \mathbb{N}_0). \tag{1.11}
\]

For the class \( T(n, \lambda, \alpha) \) Aouf and Chen [3] showed the following Lemma:

**Lemma 1**

Let the function \( f(z) \) be defined by (1.9). Then \( f(z) \in T(n, \lambda, \alpha) \) if and only if
\[
\sum_{k=2}^{\infty} C_k(n, \lambda, \alpha) \delta(n,k)a_k \leq (1 - \alpha)(n + 1) \tag{1.12}
\]
where
\[
C_k(n, \lambda, \alpha) = n + k - \alpha[\lambda(k - 1) + n + 1] \tag{1.13}
\]
The result is sharp.

In view of Lemma 1, we can see that the coefficient \( a_2 \) of the function \( f(z) \) defined by (1.9) and belonging to the class \( T(n, \lambda, \alpha) \) satisfies the inequality:
Thus we let $T_c(n, \lambda, \alpha)$ denote the class of functions $f(z)$ in $T(n, \lambda, \alpha)$ which are of the form:

$$f(z) = z - \frac{c(1 - \alpha)}{C_2(n, \lambda, \alpha)} z^2 - \sum_{k=3}^{\infty} a_k z^k \quad (a_k \geq 0; \ 0 \leq c \leq 1) \quad (1.15)$$

For the class $T_c(n, \lambda, \alpha)$ of analytic functions with negative coefficients and fixed second coefficients, defined above, we shall derive a number of interesting properties and characteristics (including, for example, coefficient estimates, closure properties involving convex linear combinations, growth and distortion theorems and radii of starlikeness and convexity). We also extend many of these results to hold true for analogous classes of functions with finitely many fixed coefficients.

2. COEFFICIENT ESTIMATES FOR THE CLASS $T_c(n, \lambda, \alpha)$

Theorem 1

Let the function $f(z)$ be defined by (1.15). Then $f(z) \in T_c(n, \lambda, \alpha)$ if and only if

$$\sum_{k=3}^{\infty} C_k(n, \lambda, \alpha) \delta(n, k) a_k \leq (1 - c)(1 - \alpha)(n + 1) \quad (2.1)$$

where $C_k(n, \lambda, \alpha)$ is defined by (1.13) and $\delta(n, k)$ is defined by (1.7). The result is sharp.

Proof

Putting

$$a_2 = \frac{(1 - \alpha)c}{C_2(n, \lambda, \alpha)} \quad (0 \leq c \leq 1) \quad (2.2)$$

in Lemma 1 and simplifying the resulting inequality, we readily arrive at the assertion (2.1) of Theorem 1. The result is sharp for the function $f(z)$ given
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by

\[ f(z) = z - \frac{c(1 - \alpha)}{C_2(n, \lambda, \alpha)} z^2 - \frac{(1 - c)(1 - \alpha)(n + 1)}{C_k(n, \lambda, \alpha) \delta(n, k)} z^k \quad (k = 3, 4, 5, \ldots) \] (2.3)

**Corollary 1**

Let the function \( f(z) \) defined by (1.15) be in the class \( T_c(n, \lambda, \alpha) \). Then

\[ a_k \leq \frac{(1 - c)(1 - \alpha)(n + 1)}{C_k(n, \lambda, \alpha) \delta(n, k)} \quad (k = 3, 4, \ldots) \] (2.4)

The result is sharp for the function \( f(z) \) given by (2.3)

3. **Closure Theorems for the Class \( T_c(n, \lambda, \alpha) \)**

**Theorem 2**

The class \( T_c(n, \lambda, \alpha) \) is closed under convex linear combination.

**Proof**

Let the function \( f(z) \) be defined by (1.15). Define the function \( g(z) \) by

\[ g(z) = z - \frac{c(1 - \alpha)}{C_2(n, \lambda, \alpha)} z^2 - \sum_{k=3}^{\infty} b_k z^k \quad (b_k \geq 0) \] (3.1)

Assuming that \( f(z) \) and \( g(z) \) are in the class \( T_c(n, \lambda, \alpha) \), it is sufficient to prove that the function \( h(z) \) defined by

\[ h(z) = \mu f(z) + (1 - \mu) g(z) \quad (0 \leq \mu \leq 1) \] (3.2)

is also in the class \( T_c(n, \lambda, \alpha) \). Since

\[ h(z) = z - \frac{c(1 - \alpha)}{C_2(n, \lambda, \alpha)} z^2 - \sum_{k=3}^{\infty} \{\mu a_k + (1 - \mu) b_k\} z^k, \] (3.3)

we observe that
with the aid of Theorem 1. Hence \( h(z) \in T_c(n, \lambda, \alpha) \). This completes the proof of Theorem 2.

**Theorem 3**

Let the functions

\[ f_j(z) = z - \frac{c(1 - \alpha)}{C_2(n, \lambda, \alpha)} z^2 - \sum_{k=3}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0) \]  

be in the class \( T_c(n, \lambda, \alpha) \) for every \( j = 1, \ldots, m \). Then the function \( F(z) \) defined by

\[ F(z) = \sum_{j=1}^{\infty} \mu_j f_j(z) \quad (u_j \geq 0) \]

is also in the class \( T_c(n, \lambda, \alpha) \), where

\[ \sum_{j=1}^{m} \mu_j = 1 \]

**Proof**

Combining the definitions (3.5) and (3.6), we have

\[ F(z) = z - \frac{c(1 - \alpha)}{C_2(n, \lambda, \alpha)} z^2 - \sum_{k=3}^{\infty} \left( \sum_{j=1}^{m} \mu_j a_{k,j} \right) z^k, \]

where we have also used the relationship (3.7). Since \( f_j(z) \in T_c(n, \lambda, \alpha) \) for every \( j = 1, 2, \ldots, m \), Theorem 1 yields.

\[ \sum_{k=3}^{\infty} C_k(n, \lambda, \alpha) \delta(n, k) a_{k,j} \leq (1 - c)(1 - \alpha)(n + 1) \quad (j = 1, 2, \ldots, m) \]  

Thus we obtain
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\[ \sum_{k=3}^{\infty} C_k(n, \lambda, \alpha) \delta(n, k) \left( \sum_{j=1}^{m} \mu_j a_{k,j} \right) = \sum_{j=1}^{m} \mu_j \left[ \sum_{k=3}^{\infty} C_k(n, \lambda, \alpha) \delta(n, k) a_{k,j} \right] \leq (1 - c)(1 - \alpha)(n + 1) \]

which (in view of Theorem 1) implies that \( F(z) \in T_c(n, \lambda, \alpha) \).

**Theorem 4**

Let

\[ f_2(z) = z - \frac{c(1 - \alpha)}{C_2(n, \lambda, \alpha)} z^2 \]  \hspace{1cm} (3.10)

and

\[ f_k(z) = z - \frac{c(1 - \alpha)}{C_2(n, \lambda, \alpha)} z^2 - \frac{(1 - c)(1 - \alpha)(n + 1)}{C_k(n, \lambda, \alpha) \delta(n, k)} z^k \quad (k = 3, 4, 5, \cdots), \]  \hspace{1cm} (3.11)

Then \( f(z) \) is in the class \( T_c(n, \lambda, \alpha) \) if and only if it can be expressed in the form:

\[ f(z) = \sum_{k=2}^{\infty} \mu_k f_k(z) \]  \hspace{1cm} (3.12)

where

\[ \mu_k \geq 0 \quad \text{and} \quad \sum_{k=2}^{\infty} \mu_k = 1 \]  \hspace{1cm} (3.13)

**Proof**

We suppose that \( f(z) \) can be expressed in the form (3.12). Then we have

\[ f(z) = z - \frac{c(1 - \alpha)}{C_2(n, \lambda, \alpha)} z^2 - \frac{(1 - c)(1 - \alpha)(n + 1) \mu_k}{C_k(n, \lambda, \alpha) \delta(n, k)} z^k, \]  \hspace{1cm} (3.14)

Since

\[ \sum_{k=3}^{\infty} \frac{(1 - c)(1 - \alpha)(n + 1) \mu_k}{C_k(n, \lambda, \alpha) \delta(n, k)} \cdot \frac{C_k(n, \lambda, \alpha) \delta(n, k)}{(1 - \alpha)(n + 1)} = (1 - c)(1 - \lambda_2) \leq (1 - c), \]  \hspace{1cm} (3.15)
it follows from (2.1) that \( f(z) \) is in the class \( T_c(n, \lambda, \alpha) \).

Conversely, we suppose that \( f(z) \) defined by (1.15) is in the class \( T_c(n, \lambda, \alpha) \). Then, by making use of (2.4), we get

\[
a_k \leq \frac{(1 - c)(1 - \alpha)(n + 1)}{C_k(n, \lambda, \alpha)\delta(n,k)} \quad (k \geq 3) \tag{3.16}
\]

Setting

\[
\mu_k = \frac{C_k(n, \lambda, \alpha)\delta(n,k)}{(1 - c)(1 - \alpha)(n + 1)} a_k \quad (k \geq 3) \tag{3.17}
\]

and

\[
\mu_2 = 1 - \sum_{k=3}^{\infty} \mu_k, \tag{3.18}
\]

we have (3.12). This completes the proof of Theorem 4.

**Corollary 2**

The extreme points of the class \( T_c(n, \lambda, \alpha) \) are the functions \( f_k(z)(k \in \mathbb{N}/\{1\}) \) given by Theorem 4.

**4. GROWTH AND DISTORTION THEOREMS FOR THE CLASS \( T_c(n, \lambda, \alpha) \)**

Lemma 2, 3, and 4 below will be required in our investigation of the growth and distortion properties of the general class \( T_c(n, \lambda, \alpha) \).

**Lemma 2**

Let the function \( f_3(z) \) be defined by

\[
f_3(z) = z - \frac{c(1 - \alpha)}{[n + 2 - \alpha(\lambda + n + 1)]} z^2 - \frac{2(1 - c)(1 - \alpha)}{[n + 3 - \alpha(2\lambda + n + 1)](n + 2)} z^3 \tag{4.1}
\]
Then, for $0 \leq r < 1$ and $0 \leq c \leq 1$,

$$|f_3(re^{i\theta})| \geq r - \frac{c(1 - \alpha)}{[n + 2 - \alpha(\lambda + n + 1)]}r^2 - \frac{2(1 - c)(1 - \alpha)}{[n + 3 - \alpha(2\lambda + n + 1)](n + 2)}r^3 \quad (4.2)$$

with equality for $\theta = 0$. For either $0 \leq c \leq c_0$ and $0 \leq r \leq r_0$, or $c_0 \leq c \leq 1$,

$$|f_3(re^{i\theta})| \geq r + \frac{c(1 - \alpha)}{[n + 2 - \alpha(\lambda + n + 1)]}r^2 - \frac{2(1 - c)(1 - \alpha)}{[n + 3 - \alpha(2\lambda + n + 1)](n + 2)}r^3 \quad (4.3)$$

with equality for $\theta = \pi$. Furthermore, for $0 \leq c \leq c_0$ and $r_0 \leq r < 1$.

$$|f_3(re^{i\theta})| \leq r \left\{ 1 + \frac{c^2(1 - \alpha)[n + 3 - \alpha(2\lambda + n + 1)](n + 2)}{8(1 - c)[n + 2 - \alpha(\lambda + n + 1)]^2} + \left[ \frac{c^2(1 - \alpha)^2}{2[n + 2 - \alpha(\lambda + n + 1)]^2} + \frac{4(1 - c)(1 - \alpha)}{[n + 3 - \alpha(2\lambda + n + 1)](n + 2)} \right]r^2 + \left[ \frac{4(1 - c)^2(1 - \alpha)^2}{[n + 3 - \alpha(2\lambda + n + 1)]^2(n + 1)^2} + \frac{c^2(1 - \alpha)(1 - \alpha)^3}{2[n + 2 - \alpha(\lambda + n + 1)]^2[n + 3 - \alpha(2\lambda + n + 1)](n + 2)} \right]r^4 \right\}^{1/2} \quad (4.4)$$

with equality for

$$\theta = \cos^{-1} \left( \frac{2c(1 - c)(1 - \alpha)r^2 - c(n + 2)[n + 3 - \alpha(2\lambda + n + 1)]}{8(1 - c)[n + 2 - \alpha(\lambda + n + 1)]r} \right), \quad (4.5)$$

where

$$c_0 = \frac{1}{4(1 - \alpha)} \{-8[n + 2 - \alpha(\lambda + n + 1)] +$$
\[(n + 2)[n + 3 - \alpha(2\lambda + n + 1)] - 2(1 - \alpha)] + \\
\{[8[n + 2 - \alpha(\lambda + n + 1)] + (n + 2)[n + 3 - \alpha(2\lambda + n + 1)] \\
-2(1 - \alpha)]^2 + 64(1 - \alpha)[n + 2 - \alpha(\lambda + n + 1)]^{1/2}\}
\end{equation}

and
\[r_0 = \frac{1}{2c(1-c)(1-\alpha)}\{-4(1-c)[n + 2 - \alpha(\lambda + n + 1)] \\
+\{16(1-c)^2[n + 2 - \alpha(\lambda + n + 1)]^2 \\
+2c^2(1-c)(1-\alpha)(n+2)[n + 3 - \alpha(2\lambda + n + 1)]\}^{1/2}\}.
\end{equation}

**Proof**

We employ the same technique as used by Silverman and Silvia [6]. Since
\[\frac{\partial |f_3(re^{i\theta})|^2}{\partial \theta} = \frac{2(1 - \alpha)}{[n + 2 - \alpha(\lambda + n + 1)]}r^3 \sin \theta[c + \\
8(1-c)[n + 2 - \alpha(\lambda + n + 1)]r \cos \theta - \\
2c(1-c)(1-\alpha)[n + 3 - \alpha(2\lambda + n + 1)](n+2)r^2],\]
we can see that
\[\frac{\partial |f_3(re^{i\theta})|^2}{\partial \theta} = 0\]
\end{equation}
for \(\theta_1 = 0, \ \theta_2 = \pi\) and
\[\theta_3 = \cos^{-1}\left(\frac{2c(1-c)(1-\alpha)r^2 - c(n+2)[n + 3 - \alpha(2\lambda + n + 1)]}{8(1-c)[n + 2 - \alpha(\lambda + n + 1)]r}\right).
\end{equation}

Since \(\theta_3\) is a valid root only when \(-1 \leq \cos \theta_3 \leq 1\), we have a third root if and only if \(r_0 \leq r < 1\) and \(0 \leq c < c_0\). Thus the results of Lemma 2 follow
upon comparing the extremal values $f_3(re^{i\theta_k})(k = 1, 2, 3)$ on the appropriate intervals.

**Lemma 3**

Let the function $f_k(z)$ be defined by (3.11) and $k \geq 4$. Then

$$|f_k(re^{i\theta})| \leq |f_4(-r)| \quad (k \geq 4). \quad (4.11)$$

**Proof**

$$f_k(z) = z - \frac{c(1 - \alpha)}{[n + 2 - \alpha(\lambda + n + 1)]} z^2 - \frac{(1 - c)(1 - \alpha)(n + 1)}{\{n + k - \alpha[\lambda(k - 1) + n + 1]\} \delta(n, k)} z^k$$

and

$$\frac{(1 - c)(1 - \alpha)(n + 1)r^k}{\{n + k - \alpha[\lambda(k - 1) + n + 1]\} \delta(n, k)}$$

is a decreasing function of $k$, we have

$$|f_k(re^{i\theta})| \leq r + \frac{c(1 - \alpha)}{[n + 2 - \alpha(\lambda + n + 1)]} r^2 + \frac{6(1 - c)(1 - \alpha)}{(n + 2)(n + 3)\{n + 4 - \alpha[3\lambda + n + 1]\}} r^4$$

$$= -f_4(-r),$$

which proves (4.11).

**Theorem 5**

Let the function $f(z)$ defined by (1.15) belong to class $T_c(n, \lambda, \alpha)$. Then, for $0 \leq r < 1$,

$$|f(re^{i\theta})| \geq r - \frac{c(1 - \alpha)}{[n + 2 - \alpha(\lambda + n + 1)]} r^2 -$$
\[
\frac{2(1-c)(1-\alpha)}{(n+2)[n+3-\alpha(2\lambda+n+1)]} f_3^3 \tag{4.12}
\]

will equality for \(f_3(z)\) at \(z = r\), and

\[
|f(re^{i\theta})| \leq \max\{\max_{\theta} |f_3(re^{i\theta})|, -f_4(-r)\}, \tag{4.13}
\]

where

\[
\max_{\theta} |f_3(re^{i\theta})|
\]

is given by Lemma 2. The proof of Theorem 5 is obtained by comparing the bounds given by Lemma 2 and Lemma 3.

**Remark 1**

Putting \(c = 1\) and \(n = 0\) in Theorem 5 we obtain the following result obtained by Altintas and Owa [2].

**Corollary 3**

Let the function \(f(z)\) defined by (1.9) be in the class \(T_1(0, \lambda, \alpha) = T(\lambda, \alpha)\). Then for \(|z| = r < 1\), we have

\[
r - \frac{(1-\alpha)}{[2-\alpha(1+\lambda)]} r^2 \leq |f(z)| \leq r + \frac{(1-\alpha)}{[2-\alpha(1+\lambda)]} r^2. \tag{4.14}
\]

The result is sharp.

**Lemma 4**

Let the function \(f_3(z)\) be defined by (4.1). Then, for \(0 \leq r < 1\) and \(0 \leq c \leq 1\),

\[
|f_3'(re^{i\theta})| \geq 1 - \frac{2c(1-\alpha)}{[n + 2 - \alpha(\lambda + n + 1)]} r - \frac{6(1-c)(1-\alpha)}{[n + 3 - \alpha(2\lambda + n + 1)][n+2]} r^2 \tag{4.15}
\]
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with equality for $\theta = 0$. For either $0 \leq c \leq c_1$ and $0 \leq r \leq r_1$ or $c_1 \leq c \leq 1$,

$$|f_3'(r e^{i\theta})| \leq 1 + \frac{2c(1 - \alpha)}{[n + 2 - \alpha(\lambda + n + 1)]} r - \frac{6(1 - c)(1 - \alpha)}{[n + 3 - \alpha(2\lambda + n + 1)](n + 2)} r^2$$

(4.16)

with equality for $\theta = \pi$. Furthermore, for $0 \leq c < c_1$ and $r_1 \leq r < 1$,

$$|f_3'(r e^{i\theta})| \leq \left\{1 + \frac{c^2(1 - \alpha)[n + 3 - \alpha(2\lambda + n + 1)](n + 2)}{6(1 - c)[n + 2 - \alpha(\lambda + n + 1)]^2} \right\}$$

$$+ \left[\frac{2c^2(1 - \alpha)^2}{[n + 2 - \alpha(\lambda + n + 1)]^2 + (n + 2)[n + 3 - \alpha(2\lambda + n + 1)]}\right] r^2$$

$$+ \left[\frac{6c^2(1 - c)(1 - \alpha)^3}{(n + 2)^2[n + 3 - \alpha(2\lambda + n + 1)]}\right] r^4\right\}^{1/2}$$

(4.17)

with equality for

$$\theta = \cos^{-1}\left(\frac{6c(1 - c)(1 - \alpha) r^2 - c(n + 2)[n + 3 - \alpha(2\lambda + n + 1)]}{12[n + 2 - \alpha(\lambda + n + 1)](1 - c) r}\right)$$

(4.18)

where

$$c_1 = \frac{1}{12(1 - \alpha)} \{ -12[n + 2 - \alpha(\lambda + n + 1)] -$$

$$(n + 2)[n + 3 - \alpha(2\lambda + n + 1)] + 6(1 - \alpha) +$$

$$\{12[n + 2 - \alpha(\lambda + n + 1)] + (n + 2)[n + 3 - \alpha(2\lambda + n + 1)]$$

$$- 6(1 - \alpha)\}^2 + 288(1 - \alpha)[n + 2 - \alpha(\lambda + n + 1)]^{1/2}\}\}

(4.19)

and

$$r_1 = \frac{1}{6c(1 - c)(1 - \alpha)} \{ -6(1 - c)[n + 2 - \alpha(\lambda + n + 1)] +$$

...
\[
[36(1-c)^2[n + 2 - \alpha(\lambda + n + 1)]^2 + \\
6c^2(1-c)(1-\alpha)(n+2)[n + 3 - \alpha(2\lambda + n + 1)]^{1/2}].
\]

(4.20)

The proof of Lemma 4 is given in much the same way as Lemma 2.

**Theorem 6**

Let the function \( f(z) \) defined by (1.15) be in the class \( T_c(n, \lambda, \alpha) \). Then, for \( 0 \leq r < 1 \),

\[
|f'(re^{i\theta})| \geq 1 - \frac{2c(1-\alpha)}{[n + 2 - \alpha(\lambda + n + 1)]}r - \\
\frac{6(1-c)(1-\alpha)}{(n + 2)[n + 3 - \alpha(2\lambda + n + 1)]}r^2
\]

with equality for \( f_3'(z) \) at \( z = r \), and

\[
|f'(re^{i\theta})| \leq \max\{\max_{\theta} |f_3'(re^{i\theta})|, f_4'(-r)\},
\]

(4.22)

where

\[
\max_{\theta} |f_3'(re^{i\theta})|
\]

is given by Lemma 4.

**Remark 2**

Putting \( c = 1 \) and \( n = 0 \) in Theorem 6 we obtain the following result obtained by Altintas and Owa [2].

**Corollary 4**

Let the function \( f(z) \) defined by (1.9) be in the class \( T_1(0, \lambda, \alpha) = T(\lambda, \alpha) \). Then, for \( |z| = r < 1 \),

\[
1 - \frac{2(1-\alpha)}{[2 - \alpha(\lambda + 1)]}r \leq |f'(z)| \leq 1 + \frac{2(1-\alpha)}{[2 - \alpha(\lambda + 1)]}r.
\]

(4.23)
The result is sharp.

5. RADII OF STARLIKENESS AND CONVEXITY

Theorem 7

Let the function $f(z)$ defined by (1.15) be in the class $T_c(n, \lambda, \alpha)$. Then $f(z)$ is starlike of order $\rho (0 \leq \rho < 1)$ in the disc $|z| < r_1(n, \lambda, \alpha, c, \rho)$, where $r_1(n, \lambda, \alpha, c, \rho)$ is the largest value for which the following inequality holds true.

\[
\frac{c(1 - \alpha)(2 - \rho)r}{n + 2 - \alpha(\lambda + n + 1)} + \frac{(1 - c)(1 - \alpha)(n + 1)(k - \rho)}{\{n + k - \alpha[\lambda(k - 1) + n + 1]\}\delta(n,k) r^{k-1}} 
\leq 1 - \rho (k = 3, 4, \ldots).
\]

(5.1)

The result is sharp, the external function being given by

\[
f_k(z) = z - \frac{c(1 - \alpha)}{n + 2 - \alpha(n + \lambda + 1)} z^2 - \frac{(1 - c)(1 - \alpha)(n + 1)}{\{n + k - \alpha[\lambda(k - 1) + n + 1]\}\delta(n,k) z^k}
\]

(5.2)

for some $k$.

Proof

It suffices to show that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \ (0 \leq \rho < 1)
\]

$|z| < r_1(n, \lambda, \alpha, c, \rho)$. We note that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{c(1 - \alpha)}{n + 2 - \alpha(\lambda + n + 1)} r + \sum_{k=3}^{\infty} (k - 1)a_k r^{k-1} - \sum_{k=3}^{\infty} a_k r^{k-1} 
\leq 1 - \rho (|z| \leq r)
\]

(5.3)
if and only if
\[
\frac{c(1 - \alpha)(2 - \rho)}{[n + 2 - \alpha(\lambda + n + 1)]} r + \sum_{k=3}^{\infty} (k - \rho)a_k r^{k-1} \leq 1 - \rho. \tag{5.4}
\]

Since \( f(z) \) is in \( T_c(n, \lambda, \alpha) \), from (2.1) we may take
\[
a_k = \frac{(1 - c)(1 - \alpha)(n + 1)\mu_k}{\{n + k - \alpha[\lambda(k - 1) + n + 1]\} \delta(n, k)} \quad (k = 3, 4, 5, \ldots), \tag{5.5}
\]
where
\[
\mu_k \geq 0 (k = 3, 4, \ldots) \quad \text{and} \quad \sum_{k=3}^{\infty} \mu_k \leq 1. \tag{5.6}
\]

For each fixed \( r \), we choose the positive integer \( k_0 = k_0(r) \) for which
\[
\frac{(k_0 - \rho)}{\{n + k_0 - \alpha[\lambda(k_0 - 1) + n + 1]\} \delta(n, k_0)} r^{k_0-1}
\]
is maximal. Then it follows that
\[
\sum_{k=3}^{\infty} (k - \rho)a_k r^{k-1} \leq \frac{(1 - c)(1 - \alpha)(n + 1)(k_0 - \rho)}{\{n + k_0 - \alpha[\lambda(k_0 - 1) + n + 1]\} \delta(n, k_0)} r^{k_0-1}. \tag{5.7}
\]

Hence \( f(z) \) is starlike of order \( \rho \) in \( |z| < r_1(n, \lambda, \alpha, c, \rho) \) provided that
\[
\frac{c(1 - \alpha)(2 - \rho)r}{[n + 2 - \alpha(\lambda + n + 1)]} + \frac{(1 - c)(1 - \alpha)(n + 1)(k_0 - \rho)}{\{n + k_0 - \alpha[\lambda(k_0 - 1) + n + 1]\} \delta(n, k_0)} r^{k_0-1} \leq 1 - \rho. \tag{5.8}
\]

We find the value \( r_0 = r_0(n, \lambda, \alpha, c, \rho) \) and the corresponding integer \( k_0(r_0) \) so that
\[
\frac{c(1 - \alpha)(2 - \rho)r_0}{[n + 2 - \alpha(\lambda + n + 1)]} + \frac{(1 - c)(1 - \alpha)(n + 1)(k_0 - \rho)}{\{n + k_0 - \alpha[\lambda(k_0 - 1) + n + 1]\} \delta(n, k_0)} r^{k_0-1} \leq 1 - \rho.
\]
Fixed coefficients for certain subclass of ...

\[
\frac{(1 - c)(1 - \alpha)(n + 1)(k_0 - \rho)}{\{n + k_0 - \alpha[\lambda(k_0 - 1) + n + 1]\}\delta(n, k_0)} r_0^{k_0 - 1} = 1 - \rho
\]

(5.9)

Then this value \(r_0\) is the radius of starlikeness of order \(\rho\) for functions \(f(z)\) belonging to the class \(T_c(n, \lambda, \alpha)\).

In a similar manner, we can prove the following theorem concerning the radius of convexity of order \(\rho\) for functions in the class \(T_c(n, \lambda, \alpha)\).

**Theorem 8**

Let the function \(f(z)\) defined by (1.15) be in the class \(T_c(n, \lambda, \alpha)\).

Then \(f(z)\) is convex of \(\rho(0 \leq \rho < 1)\) in the disc \(|z| < r_2(n, \lambda, \alpha, c, \rho)\), where \(r_2(n, \lambda, \alpha, c, \rho)\) is the largest value for which the following inequality holds true:

\[
\frac{2c(1 - \alpha)(2 - \rho)r}{n + 2 - \alpha(\lambda + n + 1)} + \frac{(1 - c)(1 - \alpha)(n + 1)k(k - \rho)}{\{n + k - \alpha[\lambda(k - 1) + n + 1]\}\delta(n, k)} r^{k-1} \leq 1 - \rho \quad (k = 3, 4, \ldots).
\]

(5.10)

The result is sharp for the function \(f(z)\) given by (5.2).

6. **THE GENERAL CLASS** \(T_{cK,N}(n, \lambda, \alpha)\)

Instead of fixing only the second coefficient, we can fix finitely many coefficients. Let \(T_{cK,N}(n, \lambda, \alpha)\) denote the class of functions \(f(z)\) in \(T_c(n, \lambda, \alpha)\) of the form:

\[
f(z) = z - \sum_{k=2}^{N} \frac{c_k(1 - \alpha)}{[n + 2 - \alpha(\lambda + n + 1)]} z^k - \sum_{k=N+1}^{\infty} a_k z^k \left( 0 \leq \sum_{k=2}^{\infty} c_k = c \leq 1 \right).
\]

(6.1)
Observe that $T_{ck,2}(n, \lambda, \alpha) = T_c(n, \lambda, \alpha)$.

**Theorem 9**

The extreme points of the class $T_{ck,N}(n, \lambda, \alpha)$ are

$$z - \sum_{k=2}^{N} \frac{c_k(1-\alpha)(n+1)}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)} z^k$$

and

$$z - \sum_{k=2}^{N} \frac{c_k(1-\alpha)(n+1)}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)} z^k - \frac{(1-c)(1+\alpha)(n+1)}{\{n+k-\alpha[\lambda(k-1)+n+1]\}\delta(n,k)} z^k$$

$$(k = N + 1, N + 2, N + 3, \ldots).$$

The details of the proof of Theorem 9 are omitted.

**Remark 3**

The characterization of the extreme points for the general class $T_{ck,N}(n, \lambda, \alpha)$ enables us to solve the standard extremal problems in the same manner as was done for the special class $T_c(n, \lambda, \alpha)$. The details involved may be left as an exercise for the interested reader.

**References**


